Time-like Form Factors
Analyticity and dispersion relations
Unified space-like and time-like description

Dinamica del Modello Standard

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Agenda

Basic knowledge on electromagnetic form factors
- The spin 0 and spin 1/2 cases
- Analyticity
- Nucleon form factors
- Dispersion relations

Form Factor Analyses
- Höhler procedure
- VMD models
- A study of the ratio $G_E^p / G_M^p$
- Two integral equation techniques for $G_M^p$
Form Factors
definitions, formulae
and other facts
Electromagnetic Form Factors

Scattering of a particle by an external electromagnetic field (other particle) at the first order in such a field.

All orders in other fields are included as sum of all Feynman diagrams with two on-shell particle lines and a photon line.

This sum takes the general form of a matrix element of the electromagnetic current $J^\mu(x)$.

### Spacetime translation invariance

$$\langle p', \lambda' | J^\mu(x) | p, \lambda \rangle = e^{i(p-p')x} \langle p', \lambda' | J^\mu(0) | p, \lambda \rangle$$

### Current conservation

$$\partial^\mu J_\mu = 0$$

$$\langle p' - p' \rangle_\mu \langle p', \lambda' | J^\mu(0) | p, \lambda \rangle = 0$$

### Gauss's Law

$$\int d^3 \vec{x} [\mu = 0], \quad \text{with: } \hat{Q} = \int d^3 \vec{x} J^0(x)$$

$$\langle p', \lambda' | \hat{Q} | p, \lambda \rangle = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \langle p', \lambda' | J^0(0) | p, \lambda \rangle$$

$e$ is the particle charge

$$\langle p, \lambda' | J^0(0) | p, \lambda \rangle = (2\pi)^{-3} \delta_{\lambda \lambda'} e$$
The spin-zero case (pion)

\[ \langle p' | J^\mu (0) | p \rangle = (2\pi)^{-3} (4p_0p'_0) - \frac{1}{2} F^\mu (p, p') \]

- Lorentz invariance and spin constraint the current matrix elements.
- In case of spin \( = 0 \) (pion)

\[ \langle p' | J^\mu (0) | p \rangle = \frac{1}{(2\pi)^3} \frac{1}{4p_0p'_0} \frac{1}{2} F^\mu (p, p') \]

\[ F^\mu (p, p') \] is a four-vector function of \( p^\mu, p'^\mu \).

The most general form of \( F^\mu (p, p') \) as linear combination of \( (p' + p)^\mu \) and \( (p' - p)^\mu \) is

\[ F^\mu (p, p') = e \left[ (p' + p)^\mu F(q^2) + i(p' - p)^\mu H(q^2) \right], \quad \text{with} \quad q = p' - p \]

the only non-trivial (\( \neq m^2 \)) scalar is \( q^2 = (p' - p)^2 \), in fact: \( p \cdot p' = m^2 - q^2/2 \).

\[ J^\mu (0) \text{ Hermitianity} \quad \Rightarrow \quad F^\mu (p, p')^* = F^\mu (p', p) \quad \Rightarrow \quad F(q^2) \text{ and } H(q^2) \text{ are real.} \]

Current and total charge conservation:

\[ \begin{cases} \quad q_\mu F^\mu (p, p') = 0 \quad \Rightarrow \quad H(q^2) = 0 \\ \quad F^0 (p, p) = 2p_0 \quad \Rightarrow \quad F(0) = 1 \end{cases} \]

Pion Form Factor

\[ F^\mu (p, p') = e F_\pi (q^2) (p + p')^\mu \]
The spin 1/2 case (baryon)

\[ \langle p', \lambda' | J^\mu(0) | p, \lambda \rangle = ie (2\pi)^{-3} \bar{u}_{\lambda'}(p') \Gamma^\mu(p, p') u_{\lambda}(p) \]

The general Lorentz-invariant matrix element is

Using the Dirac equations for spinors \( u \) and \( \bar{u} \) the four-vector \( \Gamma^\mu \) can be written in terms of only \( p^\mu \), \( p'^\mu \), and \( \gamma^\mu \).

The most general form of \( \Gamma^\mu(p, p') \) in terms of \((p' + p)^\mu\), \((p' - p)^\mu\), and \(\gamma^\mu\) is

\[
\Gamma^\mu(p, p') = \gamma^\mu F(q^2) - \frac{i(p + p')^\mu}{2M} G(q^2) + \frac{(p - p')^\mu}{2M} H(q^2)
\]

the only non-trivial (\( \neq m^2 \)) scalar is \( q^2 = (p' - p)^2 \), in fact: \( p \cdot p' = m^2 - q^2/2 \).

\( J^\mu(0) \) Hermitianity \( \Rightarrow \) \( \gamma^0 \Gamma^{\mu\dagger}(p', p) \gamma^0 = -\Gamma^\mu(p, p') \Rightarrow F, G, \) and \( H \) are real.

Current and total charge conservation:

\[
\begin{align*}
q_\mu \Gamma^\mu(p, p') &= 0 \Rightarrow H(q^2) = 0 \\
\text{charge normalization} &\Rightarrow F(0) + G(0) = 1
\end{align*}
\]

\[
\Gamma^\mu(p, p') = \gamma^\mu \underbrace{F_1(q^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu} q_\nu}{2M} \underbrace{F_2(q^2)}_{\text{Pauli}}
\]

\[
\begin{align*}
F_1 &= F + G \\
F_2 &= -G
\end{align*}
\]
Spin 1/2: $q^2 = 0$ normalization

**Normalization of the Dirac form factor $F_1$ at $p = p'$**

$$\langle p, \lambda' | J^\mu (0) | p, \lambda \rangle = \frac{e}{(2\pi)^3} \frac{p^\mu}{p^0} \delta_{\lambda', \lambda} F_1(0) \Rightarrow F_1(0) = 1$$

**Normalization of the Pauli form factor $F_2$ at $p = p'$ with $|\vec{p}|, |\vec{p}'| \to 0$**

- The contraction of $\sigma^{\mu\nu}$ is written in terms of the spin 1/2 matrix $\vec{S} = \frac{1}{2} \vec{\sigma}$

$$\overline{\psi}_{\lambda'} (p') i \sigma^{ij} \psi_{\lambda}(p) = -i \epsilon_{ijk} (S_k) \lambda' \lambda$$

$$\overline{\psi}_{\lambda'} (p') i \sigma^0 \psi_{\lambda}(p) = 0$$

- In the limit $|\vec{p}|, |\vec{p}'| \to 0$

$$\overline{u}_{\lambda'} (p') \vec{r} \psi_{\lambda}(p) \to \frac{-i(\vec{p} + \vec{p}')}{2M} \delta_{\lambda', \lambda} + \frac{[(\vec{p} + \vec{p}') \times \vec{S}] \lambda' \lambda}{M} [F_1(0) + F_2(0)]$$

- With a potential $A(\vec{x})$ the interaction Hamiltonian is $H_A = -\int d^3 \vec{x} \vec{J} \cdot \vec{A}$, hence

$$\langle p, \lambda' | H_A | p, \lambda \rangle = \frac{e [F_1(0) + F_2(0)]}{M(2\pi)^3} \int d^3 \vec{x} e^{i(\vec{p} - \vec{p}') \cdot \vec{x}} \vec{S}_{\lambda' \lambda} \cdot \vec{B}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

- For almost constant magnetic field

$$\langle p, \lambda' | H_A | p, \lambda \rangle = \frac{e [F_1(0) + F_2(0)]}{M} \vec{S}_{\lambda' \lambda} \cdot \vec{B} \delta^3 (\vec{p} - \vec{p}')$$

$$F_2(0) = \mu - \frac{e}{2M} \Rightarrow F_2(0) = [\text{anomalous magnetic moment}]$$
The scattering amplitude of spin 0 particles $A(p_1, p_2, m_A)$ and $B(k_1, k_2, m_B)$, described by the same field $\phi$ is

$$S_{fi} = -\int d^4x \, d^4y \, e^{i(p_2 y - p_1 x)} (\Box x + m_A^2)(\Box y + m_A^2) \langle k_2 \vert T \phi^\dagger(y) \phi(x) \vert k_1 \rangle$$

The time-ordered product in terms of retarded commutator is

$$T \phi^\dagger(y) \phi(x) \rightarrow \theta(y^0 - x^0)[\phi^\dagger(x), \phi(y)]$$

Being $j(x)$ the source of $\phi(x)$: $(\Box x + m_A^2) \phi(x) = j(x)$ and using translation invariance

$$S_{fi} = -(2\pi)^4 \delta^4(p_2 + k_2 - p_1 - k_1) \mathcal{F}(q), \quad q = \frac{1}{2}(p_1 + p_2)$$

$$\mathcal{F}(q) = \int d^4z \, e^{iqz} \langle k_2 \vert \theta(z^0) [j^\dagger(z/2), j(-z/2)] \vert k_1 \rangle$$

Micro-causality: $[j^\dagger(z/2), j(-z/2)] = 0$ if $z^2 \leq 0$

$\mathcal{F}(q)$ is analytic for $\text{Im}q$ positive time-like vector
Analyticity: a contribution to $F_\pi(q^2)$

Amplitude of the nucleon loop

$$(p + p')^\mu F_\pi(q^2) \sim \int d^4p_1 \text{Tr} \left[ \gamma_5 \frac{1}{p_1 - M} \gamma_5 \frac{1}{p_2 - M} \gamma^\mu \frac{1}{p_3 - M} \right]$$

$$\sim \int d^4p_1 \left[ \prod_{i=1}^3 \int_0^1 d\alpha_i \right] \frac{\text{Tr}[\gamma_5(p_1 - M)\gamma_5(p_2 - M)\gamma^\mu(p_3 - M)]}{\left[ \sum_{i=1}^3 (p_i^2 - M^2) \alpha_i \right]^3} \delta \left(1 - \sum_{i=1}^3 \alpha_i \right)$$

Analytic properties of $F_\pi(q^2)$ are given by

$$f(q^2) = \left[ \prod_{i=1}^3 \int_0^1 d\alpha_i \right] \frac{\delta(1 - \sum_{i=1}^3 \alpha_i)}{q^2\alpha_2\alpha_3 + m_\pi^2 \alpha_1(1 - \alpha_1) - M^2 + i\epsilon}$$

whose singularities are connected to the zero structure in the $\alpha_i$-variables domain of the denominator.

- No poles for $\text{Im} q^2 > 0 \Rightarrow \ldots F_\pi(q^2)$ analytic in the upper-half $q^2$-complex plane.
- $m_\pi^2 \alpha_1(1 - \alpha_1) - M^2 < 0$ and $\alpha_2\alpha_3 \geq 0 \Rightarrow \ldots F_\pi(q^2)$ is real for $q^2 \leq 0$.
- Schwarz principle $\Rightarrow \ldots$ analytic continuation in $\text{Im} q^2 < 0$: $F_\pi^*(q^2) = F_\pi(q^{2*})$.
- The integrand has real poles for each $q^2 \geq 4m_\pi^2 \Rightarrow \ldots$ cut in $(4m_\pi^2, \infty)$. 

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Form Factors and Dispersion Relations
The analytic structure of a matrix element at an arbitrary order of perturbation theory is given by

\[ I(q_i) = \int_0^1 \cdots \int_0^1 \frac{\delta(1 - \sum_{i=1}^{n} \alpha_i) d \alpha_1 \cdots d \alpha_n}{\left[ \sum_{i=1}^{3} c_i(\alpha_j) q_i^2 - \sum_{i=1}^{n} m_i^2 \alpha_i + i \epsilon \right]^{n-2k}} \]

- \( n \) internal lines of masses \( m_i \).
- \( k \) loops.
- three external lines: \( q_i, i = 1, 2, 3 \).
Analyticity term by term

The analytic structure of a matrix element at an arbitrary order of perturbation theory is given by

\[ I(q_i) = \int_0^1 \cdots \int_0^1 \frac{\delta(1 - \sum_{i=1}^n \alpha_i) d\alpha_1 \cdots d\alpha_n}{\left[ \sum_{i=1}^3 c_i(\alpha_j)q_i^2 - \sum_{i=1}^n m_i^2 \alpha_i + i\epsilon \right]^{n-2k}} \]

- \(n\) internal lines of masses \(m_i\).
- \(k\) loops.
- three external lines: \(q_i, i = 1, 2, 3\).

Nucleon form factors

- \(c_i = 0\) for on-shell lines \(\Rightarrow I\) depends only on photon \(q^2\).
- \(I(q^2)\) is analytic in \(\text{Im}q^2 > 0\) and real for \(q^2 \leq q_{th}^2 \Rightarrow I(q^{2*}) = I^*(q^2)\).
- The integrand has poles in the \(\alpha_i\)-domain from \(q_{th}^2\) up to \(\infty\)

\[ q_{th}^2 = \text{[mass of the lightest hadronic state coupled to } \gamma] = (2m_\pi)^2 \]
Nucleon Form Factors definition

Space-like region \((q^2 < 0)\)

**Electromagnetic current** \((q = p' - p)\)

\[
j^\mu = \langle N(p')|J^\mu(0)|N(p)\rangle = e\bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\right]u(p)
\]

- **Dirac and Pauli form factors** \(F_1\) and \(F_2\) are real

- **In the Breit frame**

\[
\begin{align*}
p &= (E, -\vec{q}/2) \\
p' &= (E, \vec{q}/2) \\
q &= (0, \vec{q})
\end{align*}
\]

\[
\rho_q = j^0 = e\left[F_1 + \frac{q^2}{4M^2}F_2\right] \\
j_q = e\bar{u}(p')\gamma^\mu u(p) [F_1 + F_2]
\]

**Sachs form factors**

\[
\begin{align*}
G_E &= F_1 + \frac{q^2}{4M^2}F_2 \\
G_M &= F_1 + F_2
\end{align*}
\]

**Normalizations**

\[
\begin{align*}
F_1(0) &= Q_N \\
F_2(0) &= \kappa_N \\
G_M(0) &= \mu_N \\
G_E(0) &= Q_N
\end{align*}
\]
pQCD asymptotic behavior
Space-like region

\( pQCD: \text{ as } q^2 \to -\infty, F_1(q^2) \text{ and } F_2(q^2) \)

must follow counting rules.

Quarks exchange gluons to distribute momentum.

**Dirac form factor** \( F_1 \)
- Non-spin flip.
- Two gluon propagators.
- \( F_1(q^2) \sim (-q^2)^{-2} \quad (q^2 \to -\infty) \)

**Pauli form factor** \( F_2 \)
- Spin flip.
- Two gluon propagators.
- \( F_2(q^2) \sim (-q^2)^{-3} \quad (q^2 \to -\infty) \)

**Sachs form factors** \( G_E \) and \( G_M \)
- \( G_{E,M}(q^2) \sim (-q^2)^{-2} \quad (q^2 \to -\infty) \)
- Ratio: \( \frac{G_E}{G_M} \sim \text{ constant} \quad (q^2 \to -\infty) \)
Unitarity and Cutkosky rule

Scattering matrix:

\[ S = 1 + iT \]

Unitarity (\( S^\dagger S = 1 \)):

\[ -i(T - T^\dagger) = T^\dagger T \]

Four-momentum conservation:

\[
\begin{align*}
\langle f | T | i \rangle &= (2\pi)^4 \delta^4(p_f - p_i) T_{fi} \\
\langle f | T^\dagger | i \rangle &= (2\pi)^4 \delta^4(p_f - p_i) T_{if}^* 
\end{align*}
\]

Using \( \sum_n |n\rangle \langle n| = 1 \):

\[
\langle f | T^\dagger T | i \rangle = \sum_n \langle f | T^\dagger | n \rangle \langle n | T | i \rangle
\]

\[
\langle f | T^\dagger T | i \rangle = (2\pi)^4 \delta^4(p_f - p_i) \sum_n (2\pi)^4 \delta^4(p_n - p_i) T_{nf}^* T_{ni}
\]

\[ 2 \text{Im} [T_{fi}] = -i(T_{fi} - T_{if}^*) = \sum_n (2\pi)^4 \delta^4(p_n - p_i) T_{nf}^* T_{ni} \]
Nucleon form factors
Time-like region ($q^2 > 0$)

Crossing symmetry:
$$\langle N(p')|J^\mu|N(p)\rangle \rightarrow \langle \bar{N}(p')N(p)|J^\mu|0\rangle$$

Form factors are complex functions of $q^2$

Cutkosky rule for nucleons
$$\text{Im} \langle \bar{N}(p')N(p)|J^\mu(0)|0\rangle \sim \sum_n \langle \bar{N}(p')N(p)|J^\mu(0)|n\rangle \langle n|J^\mu(0)|0\rangle \Rightarrow \begin{cases} \text{Im}F_{1,2} \neq 0 \\ \text{for } q^2 > 4m_\pi^2 \end{cases}$$

$|n\rangle$ are on-shell intermediate states: $2\pi, 3\pi, 4\pi, \ldots$

Time-like asymptotic behavior

Phragmèn Lindelöf theorem:
If a function $f(z) \rightarrow a$ as $z \rightarrow \infty$ along a straight line, and $f(z) \rightarrow b$ as $z \rightarrow \infty$ along another straight line, and $f(z)$ is regular and bounded in the angle between, then $a = b$ and $f(z) \rightarrow a$ uniformly in this angle.

$$\lim_{q^2 \rightarrow -\infty} G_{E,M}(q^2) = \lim_{q^2 \rightarrow +\infty} G_{E,M}(q^2)$$

$G_{E,M} \sim (q^2)^{-2}$, real
Cross sections and analyticity

**q²-complex plane**

- **Space-like region**: $eN \rightarrow eN$
  - FF’s are real
- **Time-like region**: $e^+ e^- \leftrightarrow N\bar{N}$
  - FF’s are complex
- **Data region**: No data

**Im(q²)**

- **Re(q²)**

**Crossing: tot. helicity**

$\begin{align*}
1 &\Rightarrow G_E \\
0 &\Rightarrow G_M
\end{align*}$

$G_E(4M_N^2) = G_M(4M_N^2)$

**Elastic scattering**

$$
\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E'_e \cos^2 \frac{\theta}{2}}{4E_e^3 \sin^4 \frac{\theta}{2}} \left[ G_E^2 - \tau \left( 1 + 2(1 - \tau) \tan^2 \frac{\theta}{2} \right) G_M^2 \right] \frac{1}{1 - \tau}
$$

$\tau = \frac{q^2}{4M_N^2}$

**Annihilation**

$$
\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \beta C}{4q^2} \left[ (1 + \cos^2 \theta) |G_M|^2 + \frac{1}{\tau} \sin^2 \theta |G_E|^2 \right]
$$

$\beta = \sqrt{1 - \frac{1}{\tau}}$
A form factor $f(q^2)$ is an analytic function on the $q^2$ complex plane with the cut: $(s_{th} = 4m_{\pi}^2, \infty)$

$$f(q^2) = |f(q^2)| e^{i\delta(q^2)}$$

Dispersion relation for the imaginary part

$$f(q^2) = \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{C} \frac{f(z)dz}{z - q^2} = \frac{1}{\pi} \int_{s_{th}}^{\infty} \frac{\text{Im}f(s)ds}{s - q^2}$$

Dispersion relation for the logarithm

Assuming no zeros on the physical sheet and using the function

$$\Phi(z) = \frac{\ln[f(z)]}{\sqrt{s_{th} - z}}$$

- $q^2 < s_{th}$: \( \ln[f(q^2)] = \frac{s_{th} - q^2}{\pi} \int_{s_{th}}^{\infty} \frac{\ln|f(s)|ds}{(s - q^2)\sqrt{s - s_{th}}} \)
- $q^2 \geq s_{th}$: $\delta(q^2) = -\frac{\sqrt{q^2 - s_{th}}}{\pi} \text{Pr} \int_{s_{th}}^{\infty} \frac{\ln|f(s)|ds}{(s - q^2)\sqrt{s - s_{th}}}$
Form factors in three regions

1. **Analytic region**
   - Dispersion relations with pQCD, data and resonances

2. **Resonance region**
   - Intermediate vector meson contributions

3. **Asymptotic region**
   - $q^2$-power laws from perturbative QCD

$|G(q^2)|$
Dispersive approach: advantages and drawbacks

**Advantages**

- DR’s are based on unitarity and analyticity ⇒ model-independent approach
- DR’s relate data from different processes in different energy regions
  \[
  \left[ \begin{array}{c}
  \text{space-like} \\
  \text{form factor}
  \end{array} \right]_{eN \rightarrow eN} = \int \left[ \begin{array}{c}
  \text{Im(form factor)} \text{ or } \ln|\text{form factor}|
  \\
  \text{over the time-like cut (} s_{th}, \infty \text{)}
  \end{array} \right]_{e^+ e^- \rightarrow B\bar{B} + \text{theory}}
  \]
- Normalizations and theoretical constraints can be directly implemented
- Form factors can be computed in the whole $q^2$-complex plane

**Drawbacks**

- Very long-range integration
- **Remedy #1** pQCD power laws
- **Remedy #2** Subtracted DR’s
- No data in the unphysical region, crucial in dispersive analyses
The proton Form Factors data
Data obtained assuming $|G_M^p| = |G_E^p| = |G_{\text{eff}}^p|$ (true only at threshold)

$$|G_{\text{eff}}^p|^2 = \frac{\sigma_{p\bar{p}}(q^2)}{16\pi\alpha^2 c_\varepsilon \sqrt{1-1/\tau}} \frac{\sqrt{1-1/\tau}}{4q^2} \left(1 + \frac{1}{2\tau}\right)$$
Data on $R = \mu_p G_E^p / G_M^p$

**Space-like region**
- Old Rosenbluth data in agreement with space-like scaling $G_E^p \sim G_M^p / \mu_p$
- New data from polarization techniques show unexplained increasing behavior
- Only polarization data have been used in the dispersive analysis

**Time-like region**
- Only two sets of data from \textit{Babar} and LEAR obtained studying angular distributions
- \textbf{Unique attempts} to perform a time-like $|G_E^p| - |G_M^p|$ separation
- Only \textit{Babar} data have been used in the dispersive analysis

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**Graphs**
- **Left Graph**: $\mu_p G_M^p / G_E^p$ vs. $q^2$ (GeV$^2$)
  - MIT-JLab (Polarization) [PRL88,092301-PRL84,1398]
  - SLAC (Rosenbluth) [PRD50,5491]
  - Normalization

- **Right Graph**: $|\mu_p G_M^p / G_E^p|$ vs. $q^2$ (GeV$^2$)
  - \textit{Babar} (ISR + ang. dist.) [PRD73,012005]
  - FENICE+DM2-E835 (ang. dist.) [EPJC46,421]
  - LEAR (ang. dist.) [NPB411,3]
Asymptotic behavior

\[ |G_{\text{eff}}^p| \sim q^2 \rightarrow \infty \]

\[ G_{\text{eff}}^p(q^2) \sim G_M^p(q^2) \]

Phragmèn Lindelöf

\[ \lim_{q^2 \rightarrow \infty} \frac{G_{\text{eff}}^p(q^2)}{G_M^p(-q^2)} = 1 \]
Höhler, Mergell, Meissner, Hammer procedure

- Optical theorem
- Dispersion relation for the imaginary part
- No time-like $|G_E| - |G_M|$ separation

$\Rightarrow \; G_{E,n}^p$ and $G_{M,n}^p$ in space and time-like region
Spectral decomposition

\[ \text{Im} \langle \mathcal{N}(p') \mathcal{N}(p) | j^\mu | 0 \rangle \sim \sum_n \langle \mathcal{N}(p') \mathcal{N}(p) | j^\mu | n \rangle \langle n | j^\mu | 0 \rangle \implies \begin{cases} \text{Im} F_{1,2} \neq 0 \\ \text{for } q^2 > 4m^2_\pi \end{cases} \]
Spectral decomposition

\[
\text{Im} \langle \overline{N}(p')N(p) | j^{\mu} | 0 \rangle \sim \sum_n \langle \overline{N}(p')N(p) | j^{\mu} | n \rangle \langle n | j^{\mu} | 0 \rangle \\
\implies \begin{cases}
\text{Im} F^V_{1,2} \neq 0 \\
\text{for } q^2 > 4m^2_\pi
\end{cases}
\]

2\pi continuum is known for 
\[q^2 \in [4m^2_\pi, \sim 40m^2_\pi]\]

- The singularity on the second Riemann sheet in \(\pi N \rightarrow \pi N\) amplitude gives the strong shoulder at threshold
- Poles for higher mass states

\(KK\) continuum from analytic continuation of \(KN\) scattering amplitude
- Further contribution in the \(\phi\)-region is due to \(\pi\rho\) exchange
- Anomalous threshold behavior is masked because the pole in the second Riemann sheet is not close to \((3m_\pi)^2\)
- Poles for higher mass states
Asymptotic behaviors from perturbative QCD

Superconvergence relations:

\[ \int_{4m^2_{\pi}}^{\infty} \text{Im} F_{1,2}(q^2) dq^2 = \int_{4m^2_{\pi}}^{\infty} q^2 \text{Im} F_2(q^2) dq^2 = 0 \]

\[ G_M^p/\mu \rho_G \]

**Space-like**

\[ F(q_{SL}^2) = \frac{1}{\pi} \int_{4m^2_{\pi}}^{\infty} \frac{\text{Im}F_{TL}(q^2) dq^2}{q_{TL}^2 - q_{SL}^2} \]

**Time-like**
VMD Models

- VMD + quark form factors
- DRs $\rightarrow$ analytic VM propagators
  - Time-like $|G_E| - |G_M|$ separation
  $\Rightarrow$ $G_{E}^{p,n}$ and $G_{M}^{p,n}$ in space and time-like region
The **Lomon** and **Iachello** parameterizations for nucleon FF’s are based on VMD, and include:

- coupling to the photons through vector meson exchange [VMD in terms of propagators \( F_M(q^2) \), \( M = \rho, \omega, \phi, \rho', \omega' \)]
- **hadron/quark form factors** \( A_M(q^2) \) at vector meson-nucleon (quark) vertices to control transition to perturbative QCD at high momentum transfers

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### Analytic extension: space-like \( \rightarrow \) time-like

- **\( F_M \) for broad mesons:**
  - simple poles \( \rightarrow \) poles with finite energy-dependent widths
- **Dispersion relations:**
  - rigorous analytic continuation of \( F_M \) from time-like to space-like region
Space-like fits

\[ \mu_p \frac{G_E}{G_M} \]

\(-q^2 (\text{GeV}^2)\)

\[ G^p_M / \mu_p G_D \]

\(-q^2 (\text{GeV}^2)\)

\[ G^n_E \]

\(-q^2 (\text{GeV}^2)\)

\[ G^p_M / \mu_n G_D \]

\(-q^2 (\text{GeV}^2)\)
Time-like fits

\[ q^2 \text{ (GeV}^2) \]

\[ |G^p_{\text{eff}}| \]

\[ |G^n_{\text{eff}}| \]

IJL

Lomon

Form Factors and Dispersion Relations
The ratio \( R = \mu_p G_E^p / G_M^p \)

- Dispersion relation for the imaginary part
- Model-independent approach
  - First time-like \(|G_E| - |G_M|\) separation
  - Ratio in the whole \( q^2 \) complex plane
The dispersive approach for $R = \mu_p G^p_E / G^p_M$

We start from the imaginary part of the ratio $R(q^2)$, written in the most general and model-independent way as

$\text{Im}[R(q^2)] = \text{series of orthogonal polynomials}$

Theoretical constraints can be applied directly on the imaginary part.

Dispersion Relations

The function $R(q^2)$ is reconstructed in time and space-like regions.

Additional theoretical conditions as well as experimental constraints are finally imposed on the obtained analytic expression of $R(q^2)$.
Parameterization and constraints

$\text{Im} R$ is parameterized by two series of orthogonal polynomials $T_i(x)$

$$
\text{Im} R(q^2) \equiv I(q^2) = \begin{cases} 
\sum_i C_i T_i(x) & x = \frac{2q^2 - s_{\text{phy}} - s_{\text{th}}}{s_{\text{phy}} - s_{\text{th}}} \quad s_{\text{th}} \leq q^2 \leq s_{\text{phy}} \\
\sum_j D_j T_j(x') & x' = \frac{2s_{\text{phy}}}{q^2} - 1 \quad q^2 > s_{\text{phy}}
\end{cases}
$$

$S_{\text{th}} = 4m_{\pi}^2$
$S_{\text{phy}} = 4M_N^2$

Theoretical conditions on $\text{Im} R(q^2)$
- $R(4m_{\pi}^2)$ is real $\implies I(4m_{\pi}^2) = 0$
- $R(4M_N^2)$ is real $\implies I(4M_N^2) = 0$
- $R(\infty)$ is real $\implies I(\infty) = 0$

Theoretical conditions on $R(q^2)$
- Continuity at $q^2 = 4m_{\pi}^2$
- $R(4M_N^2)$ is real and $\text{Re} R(4M_N^2) = \mu_p$

Experimental conditions on $R(q^2)$ and $|R(q^2)|$
- Space-like region ($q^2 < 0$) data on $R$ from JLab and MIT-Bates
- Time-like region ($q^2 \geq 4M_N^2$) data on $|R|$ from FENICE+DM2, BABAR, E835 and Lear
$R(q^2)$ in the complex plane

$G_E$, $G_M$ and also $R$, if $G_M$ has no zeros, are analytic on the $q^2$ plane with a cut ($s_{th} = 4m^2_\pi$, $\infty$)
$R(q^2)$ in the complex plane

Dispersion relation for the imaginary part ($q^2 \leq s_{th}$)

$$G(q^2) = \lim_{\mathcal{R} \to \infty} \frac{1}{2\pi i} \oint_{C} \frac{G(z)dz}{z - q^2} = \frac{1}{\pi} \int_{s_{th}}^{\infty} \frac{\text{Im}G(s)ds}{s - q^2}$$
$R(q^2)$ in the complex plane

Dispersion relation for $R$ with subtraction at $q^2 = 0$

$$R(q^2) = R(0) + \frac{q^2}{\pi} \int_{s_{th}}^{\infty} \frac{\text{Im} R(s)}{s(s - q^2)} ds$$
\[ R(q^2) = R(0) + \frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} R(s)}{s(s - q^2)} \, ds \]

- \( R(q^2) \) \text{ space-like}
- \(|R(q^2)| \) \text{ time-like}

\[ q^2 \text{ (GeV}^2) \]

JLab+MIT-Bates

BABAR+DM2/FENICE+E835

Form Factors and Dispersion Relations
\[ R(q^2) = R(0) + \frac{q^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im} R(s)}{s(s - q^2)} ds \]
\[ R(q^2) = R(0) + \frac{q^2}{\pi} \int_{4m^2_\pi}^{\infty} \frac{\text{Im} R(s)}{s(s - q^2)} ds \]

\[ R(q^2) \text{ space-like} \]

\[ |R(q^2)| \text{ time-like} \]
\[ R(q^2) = R(0) + \frac{q^2}{\pi} \int_{4m^2_\pi}^{\infty} \frac{\text{Im} R(s)}{s(s-q^2)} ds \]
$R(q^2)$: space-like zero and phase

**Phase from DR**

$$\phi(q^2) = -\frac{\sqrt{q^2 - s_{th}}}{\pi} \text{Pr} \int_{s_{th}}^{\infty} \frac{\ln |R(s)| ds}{\sqrt{s - s_{th}(s-q^2)}}$$

**Space-like zero**

$$t_0^{BABAR} = (-10 \pm 1) \text{ GeV}^2$$

Phragmèn Lindelöf phase limit ↔ zeros
Asymptotic $G_E^P(q^2)/G_M^P(q^2)$

**Real asymptotic values for $G_E^P / G_M^P$**

$$\frac{G_E^P}{G_M^P} \mid q^2 \rightarrow \infty \rightarrow -1.0 \pm 0.2$$

**Asymptotic behavior of $F_2/F_1$**

$$\frac{q^2}{4M_N^2} \left| \frac{F_2}{F_1} \right| \mid q^2 \rightarrow \infty \rightarrow \left| \frac{G_E^P}{G_M^P} - 1 \right| = 2.0 \pm 0.2$$

**pQCD prediction**

$$\left| \frac{G_E^P(q^2)}{G_M^P(q^2)} \right| \mid q^2 \rightarrow \infty \rightarrow 1$$
\[ |G_{\text{eff}}^p(q^2)| \cdot q^4 \]

- \(|G_E| = |G_M|\)

\[ |G_{\text{eff}}^p(q^2)|^2 = \frac{\sigma_{p\bar{p}}(q^2)}{4\pi\alpha^2\beta C} \left( 1 + \frac{1}{2\tau} \right)^{-1} \]

Usually what is extracted from the cross section \(\sigma(e^+e^-\rightarrow p\bar{p})\) is the effective time-like form factor \(|G_{\text{eff}}^p|\) obtained assuming \(|G_E^p| = |G_M^p|\) i.e. \(|R| = \mu p\)

Using our parametrization for \(R\) and the \(BABAR\) data on \(\sigma(e^+e^-\rightarrow p\bar{p})\), \(|G_E^p|\) and \(|G_M^p|\) may be disentangled
\[ |G^p_E(q^2)| \text{ and } |G^p_M(q^2)| \text{ from } \sigma_{p\bar{p}} \text{ and DR} \]

\[ |G^p_E(q^2)| \cdot q^4 \]

\[ |G^p_M(q^2)|^2 = \frac{\sigma_{p\bar{p}}(q^2)}{4\pi \alpha^2 \beta C} \left( 1 + \frac{|R(q^2)|}{2 \mu_p \tau} \right)^{-1} \]

- Usually what is extracted from the cross section \( \sigma(e^+e^- \rightarrow p\bar{p}) \) is the effective time-like form factor \( |G^p_{\text{eff}}| \) obtained assuming \( |G^p_E| = |G^p_M| \), i.e. \( |R| = \mu_p \).

- Using our parametrization for \( R \) and the \( BABAR \) data on \( \sigma(e^+e^- \rightarrow p\bar{p}) \), \( |G^p_E| \) and \( |G^p_M| \) may be disentangled.
Polarization formulae in the time-like region

The ratio $R(q^2)$ is complex for $q^2 \geq s_{th}$

$$R(q^2) = \mu_p \frac{G_E^p(q^2)}{G_M^p(q^2)} = |R(q^2)| e^{i\rho(q^2)}$$

The polarization depends on the phase $\rho$

$[A.Z.\ Dubnickova,\ S.\ Dubnicka,\ M.P.\ Rekalo,\ NCA109,241(96)]$

$$\mathcal{P}_y = -\frac{\sin(2\theta)|R|\sin(\rho)}{D\sqrt{\tau}} = \frac{d\sigma^\uparrow - d\sigma^\downarrow}{d\sigma^\uparrow + d\sigma^\downarrow} \equiv \mathcal{A}_y \right \} \quad \text{Does not depend on } P_e$$

$$\mathcal{P}_x = -P_e \frac{2\sin(2\theta)|R|\cos(\rho)}{D\sqrt{\tau}}$$

$$\mathcal{P}_z = P_e \frac{2\cos(\theta)}{D} \right \} \quad \text{Does not depend on } \rho$$

$$D = \frac{1 + \cos^2 \theta + \frac{1}{\tau}|R|^2 \sin^2 \theta}{\mu_p}, \quad \tau = \frac{q^2}{4M_N^2}, \quad P_e = \text{electron polarization}$$

Gennaio 2017 - Perugia
Single Polarization

\[ \mathcal{P}_x(P_e=1, \theta=45^\circ) \]

\[ \mathcal{P}_y(\theta=45^\circ) \]

\[ \mathcal{P}_z(P_e=1, \theta=45^\circ) \]

Form Factors and Dispersion Relations
A sum rule for $G^p_M$

- Dispersion relation for the logarithm
- Unphysical region suppression
  - Low-energy data $\rightarrow$ asymptotic behavior
  $\Rightarrow$ Check for the asymptotic power law
Dispersion relations and sum rules

- DR’s connect space and time values of a form factor $G(q^2)$

\[ G(q^2) = \frac{1}{\pi} \int_{s_{th}}^{\infty} \frac{\text{Im}G(s)\,ds}{s - q^2} \]

- The imaginary part is not experimentally accessible
- There are no data in the unphysical region $[s_{th}, s_{phy}]$
- We need to know the asymptotic behavior

- They applied the DR for the imaginary part to the function

\[ \phi(z) = f(z) \frac{\ln G(z)}{z\sqrt{s_{th} - z}} \quad \text{with} \quad \int_{0}^{s_{phy}} f^2(z)\,dz \ll 1 \]

- The DR integral contains the modulus $|G(s)|$
- The unphysical region contribution is suppressed

- Zeros of $G(z)$ are poles for $\phi(z)$
Assuming $G(q^2) \neq 0$ and using the Cauchy theorem, we have the new DR

$$\oint_C \phi(z)dz = 0$$

\[\Rightarrow - \int_{-\infty}^{0} \frac{\text{Im}[f(t)] \ln G(t)}{t \sqrt{s_{\text{th}} - t}} \, dt = \int_{s_{\text{th}}}^{\infty} \frac{f(s) \ln |G(s)|}{s \sqrt{s - s_{\text{th}}}} \, ds\]

<table>
<thead>
<tr>
<th>Space-like</th>
<th>Time-like</th>
</tr>
</thead>
<tbody>
<tr>
<td>$- \int_{-\infty}^{0} \frac{\text{Im}[f(t)] \ln G(t)}{t \sqrt{s_{\text{th}} - t}} , dt$</td>
<td>$\int_{s_{\text{th}}}^{\infty} \frac{f(s) \ln</td>
</tr>
</tbody>
</table>

**Convergence relation to find the asymptotic power-law behavior of $G_M^P$**

$$- \int_{-\infty}^{0} \frac{\text{Im}[f(t)] \ln G(t)}{t \sqrt{s_{\text{th}} - t}} \, dt = \int_{s_{\text{th}}}^{\infty} \frac{f(s) \ln |G(s)|}{s \sqrt{s - s_{\text{th}}}} \, ds \approx \int_{s_{\text{phy}}}^{\infty} \frac{f(s) \ln |G(s)|}{s \sqrt{s - s_{\text{th}}}} \, ds$$

**Space-like data + $(-t)^{-n}$**

**Time-like data + $s^{-n}$**

$n$ is the only free parameter
Sum rule: result for $G^p_M$

$$G^p_M(q^2) \propto \left| q^2 \right|^{-2.27 \pm 0.36}$$
Integral equation

- Dispersion relation for the logarithm
- Regularization to stabilize solutions
- Model-independent approach
- No time-like $|G_E| - |G_M|$ separation

$\Rightarrow |G^p_M|$ and $|G^n_M|$ in the unphysical region
The integral equation for $G_M$

\[
\ln G(t) = \frac{t\sqrt{s_{th} - t}}{\pi} \int_{s_{th}}^{\infty} \frac{\ln |G(s)| ds}{s\sqrt{s - s_{th}(s - t)}}
\]

- Less dependent on the asymptotic behavior of the FF
- $\ln G(0) = 0 \Rightarrow$ no further terms have to be considered

Splitting the integral $\int_{s_{th}}^{\infty}$ into $\int_{s_{th}}^{s'_{phy}} + \int_{s'_{phy}}^{\infty}$ we obtain the integral equation

\[
\text{Unknown:} \quad \ln G(t) - I_{\text{phy}}(t) = \frac{t\sqrt{s_{th} - t}}{\pi} \int_{s_{th}}^{s'_{phy}} \frac{\ln |G(s)| ds}{s\sqrt{s - s_{th}(s - t)}}
\]

- To avoid instabilities around $s_{phy} = 4M_N^2$, the upper boundary has been shifted to $s'_{phy} = s_{phy} + \Delta$, with $\Delta \simeq 0.5 \text{ GeV}^2$
- We impose continuity of the FF at $s'_{phy}$ and $s_{th}$, in addition, at the upper boundary $s'_{phy}$, continuity of the first derivative is also required
- A regularization, depending on a free parameter $\tau$, is introduced by requiring the FF total curvature in the unphysical region to be limited
Solving procedure and test

**Solving procedure**

Minimize: \[ \chi^2 = \chi^2_{\text{data}} + \chi^2_{\text{theory}} + \tau^6 \cdot \chi^2_{\text{regu}} \]

\[ \chi^2_{\text{regu}} = \int_{s_{\text{th}}}^{s'_{\text{phy}}} \left[ \frac{d^2 \ln |G(s)|}{ds^2} \right]^2 ds \propto \left[ \text{total curvature in } [s_{\text{th}}, s'_{\text{phy}}] \right] \]

Pion FF to fix the regularization parameter \( \tau \)

Space-like (DR) and time-like data (yellow bands) have been used as input in the integral equation to retrieve the time-like FF in the nucleon unphysical region.
Solving procedure

Minimize: \( \chi^2 = \chi^2_{\text{data}} + \chi^2_{\text{theory}} + \tau^6 \cdot \chi^2_{\text{regu}} \)

\[
\chi^2_{\text{regu}} = \int_{s_{\text{th}}}^{s'_{\text{phy}}} \left[ \frac{d^2 \ln |G(s)|}{ds^2} \right]^2 ds \propto \left[ \text{total curvature in } [s_{\text{th}}, s'_{\text{phy}}] \right]
\]

Pion FF to fix the regularization parameter \( \tau \)

Space-like (DR) and time-like data (yellow bands) have been used as input in the integral equation to retrieve the time-like FF in the nucleon unphysical region (gray band).
Nucleon magnetic form factors

\[ \frac{G^p_M}{\mu_p}(q^2) \quad \frac{G^n_M}{\mu_n}(q^2) \]

Steep behavior near by the threshold

\[ M_1 \sim 770 \text{ MeV} \quad \Gamma_1 \sim 350 \text{ MeV} \]

\[ M_2 \sim 1600 \text{ MeV} \quad \Gamma_2 \sim 350 \text{ MeV} \]
**“To do” list**

Time-like $|G_E| - |G_M|$ Separation: DR and data

Understand threshold effect(s) (next talk by R. Baldini):

- Dispersive analyses: integral equation, sum rule, ...

  Experimental observation in $p\bar{p} \rightarrow \pi^0 l^+ l^-$


Asymptotic behavior: DR and data for the phase

Zeros $\leftrightarrow$ phases: DR and data

Unphysical region, VMD contributions:

integral equation, sum rule, data on $p\bar{p} \rightarrow \pi^0 l^+ l^-$