

Particle Detectors

Lecture 6

21/03/18

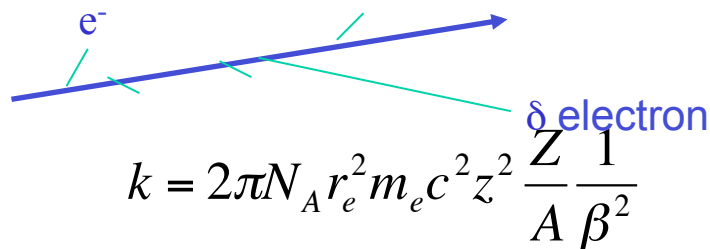
a.a. 2017-2018

Emanuele Fiandrini

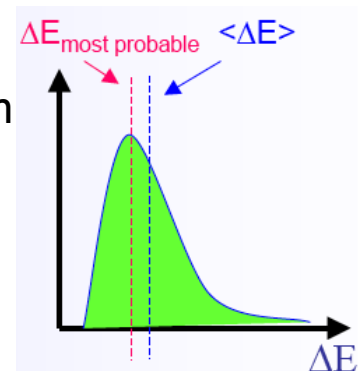
Energy loss probability (2)

■ For thin layers or low density materials:

- Few collisions, some with high energy transfer.
- ⇒ Energy loss distributions show large fluctuation towards high losses
- ⇒ Long Landau tails



$$P(E)dE = k \frac{X}{E^2} dE \quad \text{con}$$



La probabilita' e' piccata a piccoli valori di E , cioe' per valori $\sim E_{\text{min}}$: la maggior parte delle collisioni comporta piccole perdite di E , ma c'e' una probabilita' finita che in una singola collisione avvenga una perdita "grande", vicina a E_{max}

Se N non e' sufficientemente elevato, singole collisioni con grande E non sono compensate da tante collisioni con piccola E : la distribuzione delle perdite non e' simmetrica intorno al valore medio. Le singole collisioni spostano il valore medio verso alti valori di E , mentre la maggior parte delle perdite e' con piccola E , cioe' il valore piu' probabile rimane piccolo

Energy loss probability

Energy loss distribution differs a lot from a gaussian. Only around the peak the E loss distribution is approximated by a gaussian: the width is large and asymmetric, does not fall as $1/\sqrt{X}$.

This has important consequences from a practical point of view: it makes the average E loss, $\langle dE/dX \rangle$, useless for measuring the energy deposit in the medium: it will fluctuate in a non-gaussian way due high energy deposit fluctuations

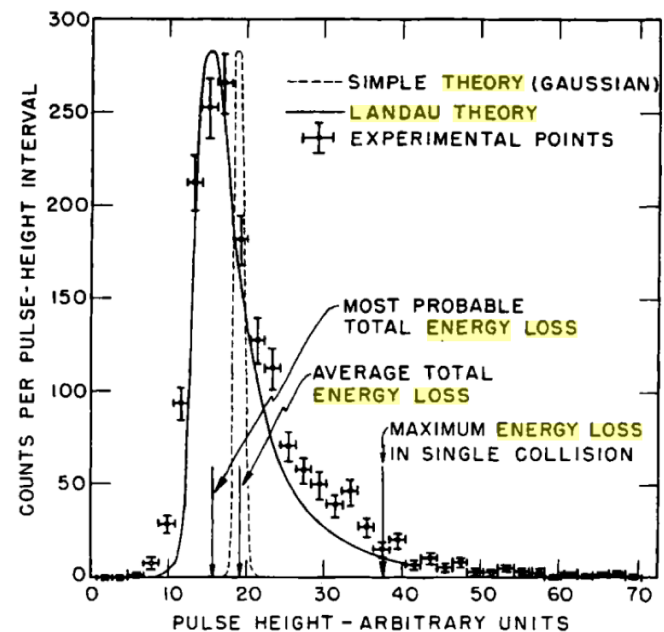
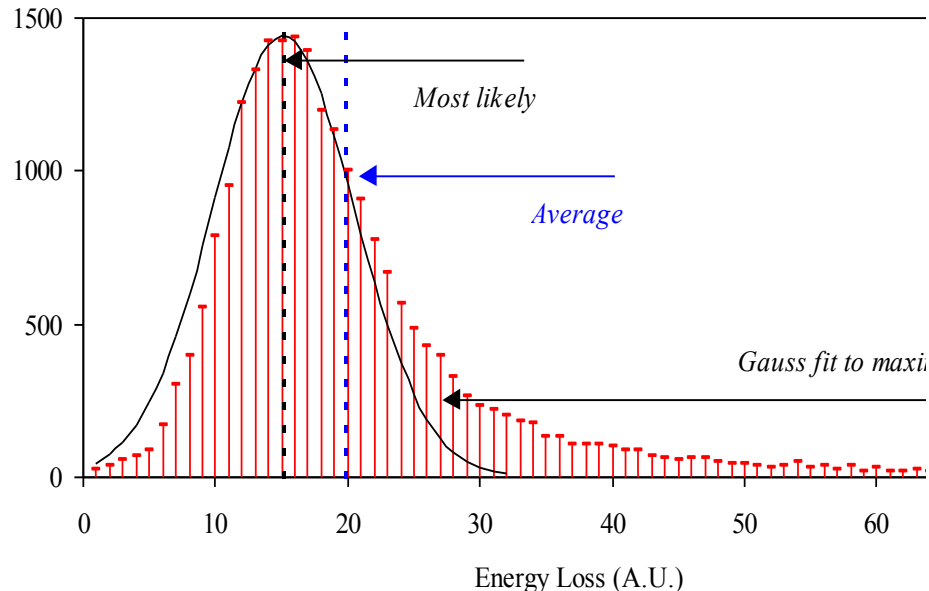


Fig. 2. Frequency distribution of energy losses of 31.5 MeV protons traversing a proportional counter filled with 96% Ar and 4% CO₂. The histogram of experimental points is compared to the theoretical Landau distribution and a gaussian distribution based on ion-electron pair statistics. From Igo *et al.* [3]

Delta rays, most probably energy loss

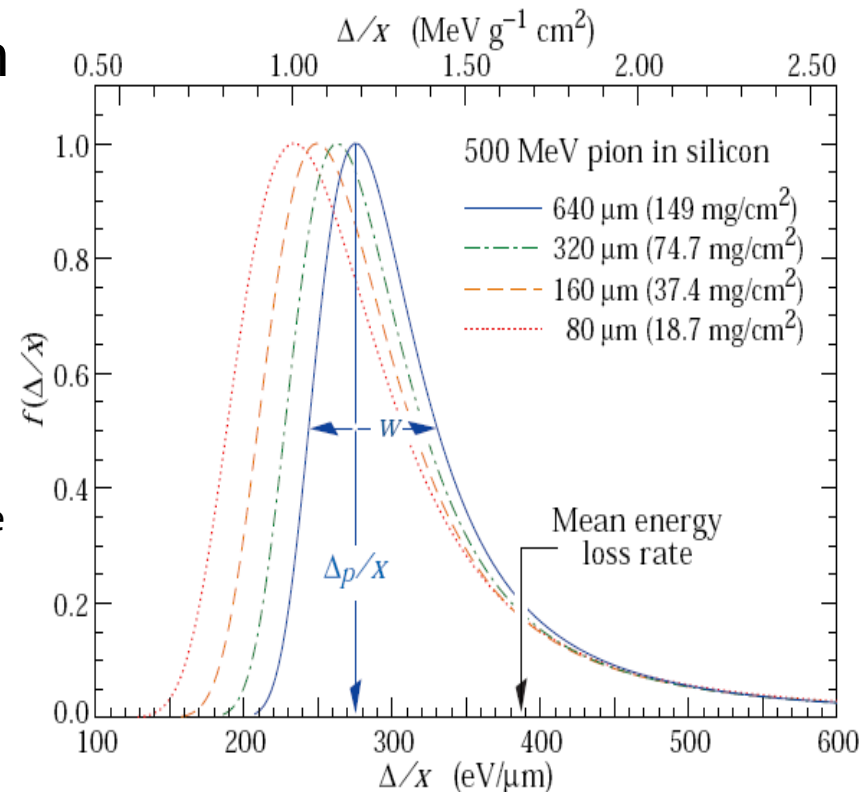
While $\langle dE/dX \rangle$ is affected by fluctuations, peak position (most probable energy loss) is not affected; therefore it is used when sampling dE/dx loss.

- It is important to keep in mind that the m.p. value and the average value do not coincide

Straggling functions in silicon for 500 MeV pions, normalized to unity at the most probable value Δ/x . The width w is the FWHM.

Bibliografia Fernow (Introduction to experimental particle physics)

<http://pdg.lbl.gov>



Landau Tails

■ Restricted dE/dx : mean of truncated distribution excluding energy transfers above some threshold T_{cut}

- Useful for thin detectors in which δ -electrons can escape the detector (energy loss not measured)

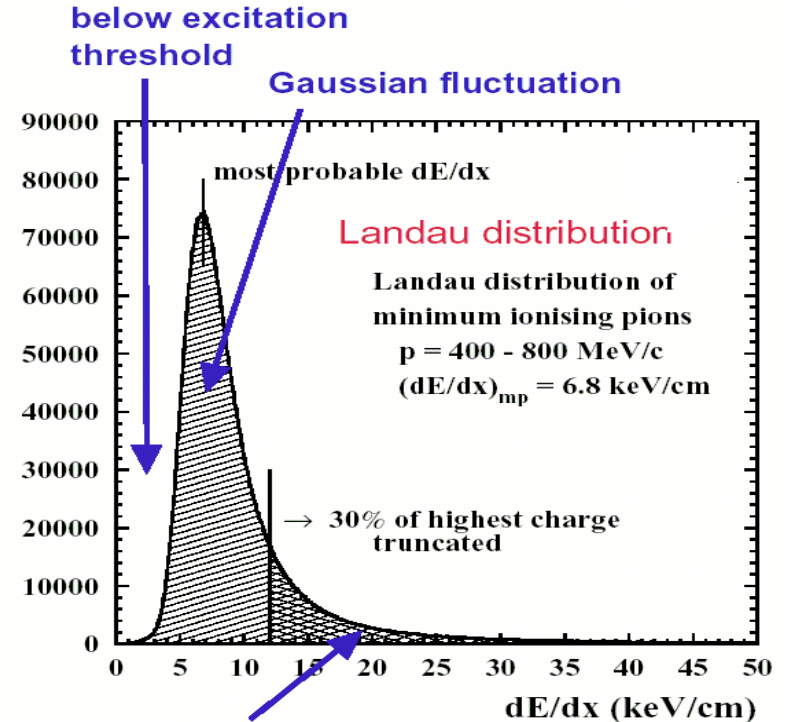
$$-\left. \frac{dE}{dx} \right|_{T < T_{\text{cut}}} = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 T_{\text{cut}}}{I^2} \right.$$

$$\left. - \frac{\beta^2}{2} \left(1 + \frac{T_{\text{cut}}}{T_{\text{max}}} \right) - \frac{\delta}{2} \right].$$

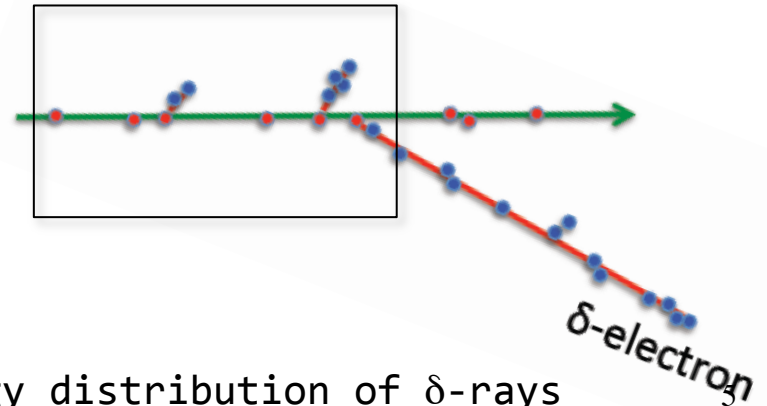
This form approaches the normal Bethe-Bloch function as $T_{\text{cut}} \rightarrow T_{\text{max}}$. It can be verified that the difference between restricted and full loss is equal to

$$\int_{T_{\text{cut}}}^{T_{\text{max}}} T \frac{dn}{dT dx} dT$$

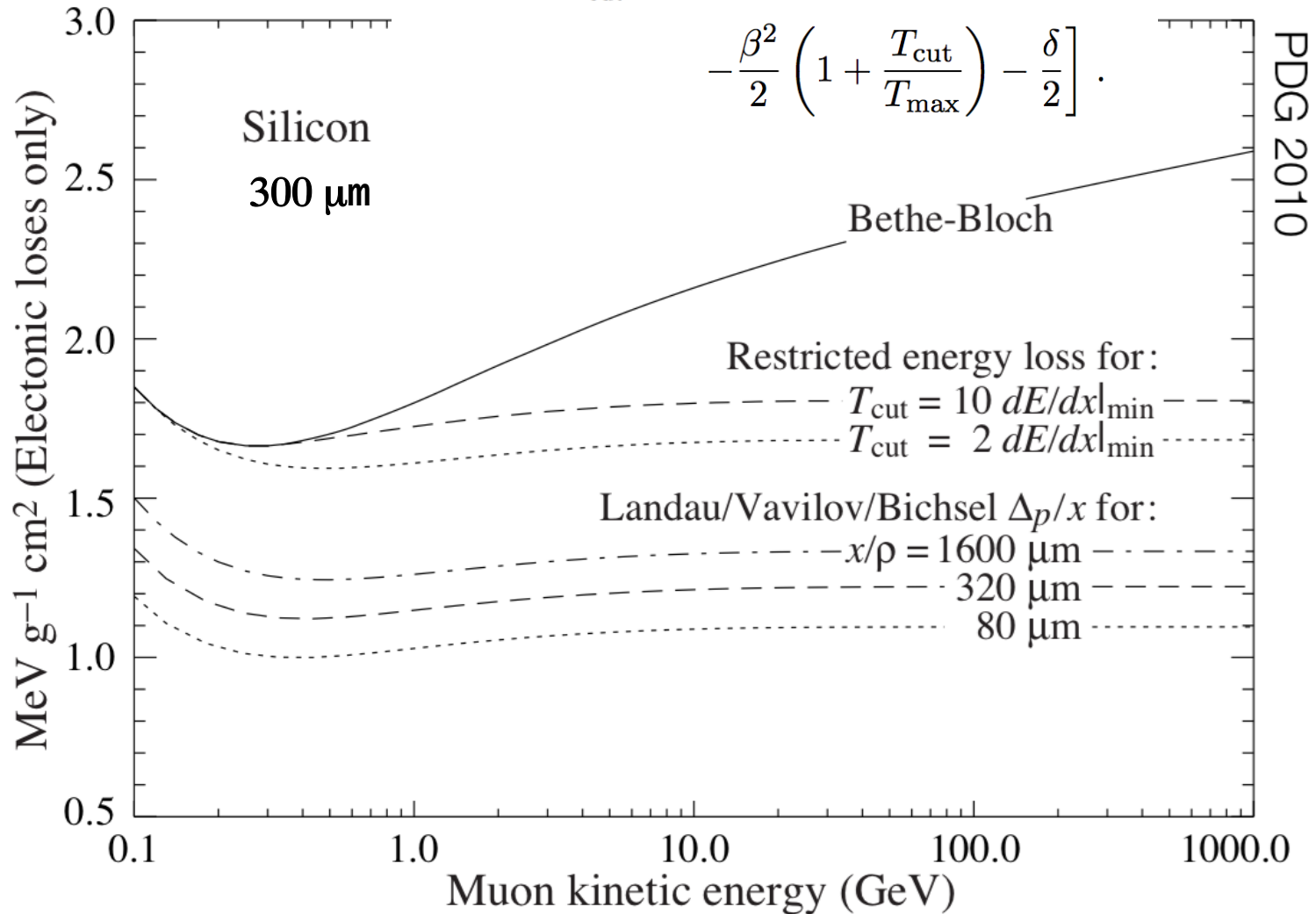
Where $dn/dT dx$ is energy distribution of δ -rays



ionisation by close collisions
producing δ -electrons

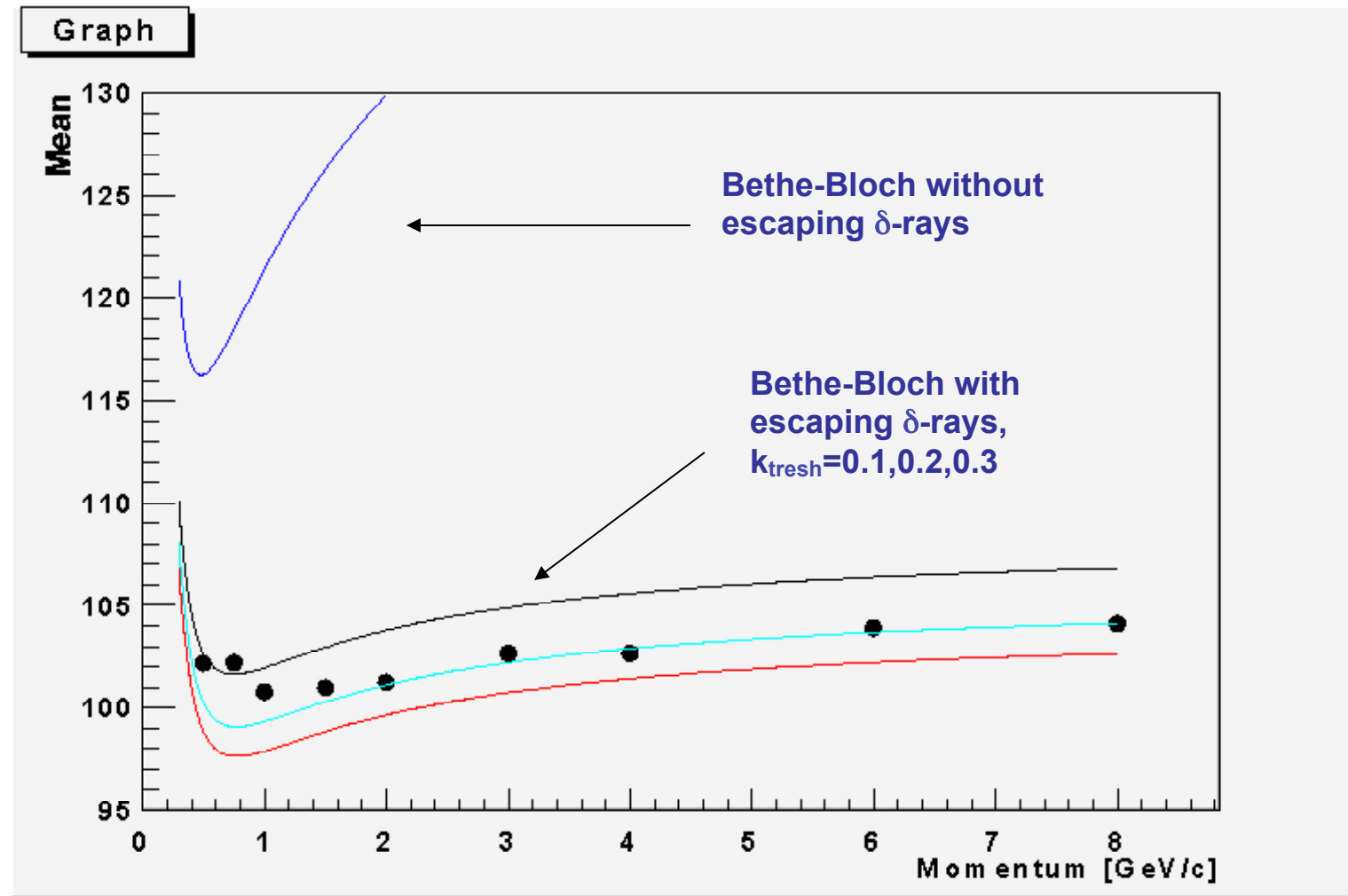


$$-\frac{dE}{dx}\bigg|_{T < T_{\text{cut}}} = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 T_{\text{cut}}}{I^2} - \frac{\beta^2}{2} \left(1 + \frac{T_{\text{cut}}}{T_{\text{max}}} \right) - \frac{\delta}{2} \right].$$



Bethe-Bloch dE/dx , two examples of restricted energy loss, and the Landau most probable energy per unit thickness in silicon. The change of Δ_p/x with thickness x illustrates its $\ln x + b$ dependence. Minimum ionization ($dE/dx|_{\text{min}}$) is $1.664 \text{ MeV g}^{-1} \text{ cm}^2$. Radiative losses are excluded. The incident particles are muons.

Data + Bethe-Bloch - Silicon



Continuous curves are MonteCarlo simulations with different max δ -rays energy cuts

Assorbitori sottili infinitesimi

La probabilità che una particella incidente di energia E perda energia compresa fra W e $W+dW$ attraversando un dx infinitesimo è:

$$\Phi(W)dWdx = n_a \frac{d\sigma(W)}{dW} dWdx$$

Dove $n_a = N_A \rho / A =$ numero di atomi per unità di volume, $d\sigma/dW =$ sezione d'urto differenziale per la particella incidente di perdere energia W in una singola collisione con un atomo.

La probabilità totale di una collisione di perdere qualunque W nell'infinitesimo dx sarà:

$$qdx = \left(n_a \int_{W_{\min}}^{W_{\max}} \frac{d\sigma}{dW} dW \right) dx$$

q si chiama rate di ionizzazione primaria.

Assorbitori sottili

Semplice se dx è infinitesimo, ma complicato per x finito.

Consideriamo un fascio di N particelle di energia E . Sia $\chi(W,x)dW$ la probabilità che una particella perda un'energia fra W e $W+dW$ dopo avere attraversato uno spessore x .

La forma di χ può essere determinata considerando come varia il # di particelle che hanno perso W a x , quando le particelle attraversano un altro spessore dx :

variazione del # di part che subiscono perdite tra W , $W+dW$ a x , $x+dx$ =
part "in" (W , $W+dW$) a (x , $x+dx$) - part "out" da (W , $W+dW$) a (x , $x+dx$)

- Il numero di particelle con perdita di energia fra W e $W+dW$ cresce perché qualcuna che ad x aveva perso meno energia di W colliderà e perderà un'energia fra W e $W+dW$ in dx = $N \times$ [prob di perdere $W - \varepsilon$ fino a x] $\times$ [Prob di perdere ε in dx], sommate su tutti i possibili valori di ε .
- Il numero di particelle con perdita fra W e $W+dW$ diminuisce perché alcune particelle che avevano già perso l'energia giusta prima del tratto dx ne perderanno ancora e quindi ne perdono di più di $W+dW$ = $N \times$ [prob di perdere W fino a x] $\times$ [Prob di perdere un'energia qualunque in dx]

Assorbitori sottili

Se assumiamo che le collisioni che avvengono successivamente siano statisticamente indipendenti, che il mezzo assorbitore sia omogeneo e che la perdita totale di energia sia piccola rispetto all'energia della particella incidente:

$$\begin{aligned} N\chi(W, x+dx)dW - N\chi(W, x)dW = \\ = N \int_{W_{\min}}^{W_{\max}} \chi(W-\varepsilon, x)\Phi(\varepsilon)d\varepsilon dW dx - N\chi(W, x)dW q dx \end{aligned}$$

Cioè:

$$\frac{\partial \chi(W, x)}{\partial x} = \int_{W_{\min}}^{W_{\max}} \Phi(\varepsilon)\chi(W-\varepsilon, x)d\varepsilon - q\chi(W, x)$$

Equazione integro-differenziale molto difficile da risolvere. Le differenze nelle soluzioni derivano essenzialmente dalle assunzioni fatte sulla probabilità $\Phi(W)$ cioè dal trasferimento di energia per collisione singola.

Assorbitori sottili

- The description of ionisation fluctuations is characterised by the significance parameter k , which is proportional to the ratio of mean energy loss to the maximum allowed energy transfer in a single collision with an atomic electron, $k = \xi/E_{\max}$
- ξ represents the energy above which at least a δ -ray will be emitted in the thickness X

$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z \rho \delta x}{m_e \beta^2 c^2 A} = 153.4 \frac{z^2}{\beta^2} \frac{Z}{A} \rho \delta x \quad \text{keV},$$

NB: emission prob of δ with energy between $E, E+dE$ is $P(E)dE = (\xi/E^2)dE$

κ measures the contribution of the collisions with energy transfer close to $E_{\max}$. For a given absorber, κ tends towards large values if δx is large and/or if β is small. Likewise, κ tends towards zero if δx is small and/or if β approaches 1. The value of κ distinguishes two regimes which occur in the description of ionisation fluctuations :

Assorbitori sottili: teoria di Landau (cenni)

1. A large number of collisions involving the loss of all or most of the incident particle energy during the traversal of an absorber.

As the total energy transfer is composed of a multitude of small energy losses, we can apply the central limit theorem and describe the fluctuations by a Gaussian distribution. This case is applicable to non-relativistic particles and is described by the inequality $\kappa > 10$ (i.e. when the mean energy loss in the absorber is greater than the maximum energy transfer in a single collision). **Already seen**

2. Particles traversing thin counters and incident electrons under any conditions.

The relevant inequalities and distributions are $0.01 < \kappa < 10$, Vavilov distribution, and $\kappa < 0.01$, Landau distribution.

An additional regime is defined by the contribution of the collisions with low energy transfer which can be estimated with the relation ξ/I_0 , where I_0 is the mean ionisation potential of the atom. Landau theory assumes that the number of these collisions is high, and consequently, it has a restriction $\xi/I_0 \gg 1$.

Assorbitori sottili: teoria di Landau (cenni)

Valida per $k = \xi/E_{\max} < 0.01$

$$\xi = \rho k z^2 (Z/A) (1/\beta^2) x$$

Assunzioni:

- Perdita di energia piccola rispetto al massimo possibile in una singola collisione ($\xi/E_{\max} \ll 1$), in pratica si assume $E_{\max} \rightarrow \infty$
- Perdita di energia grande se paragonata all'energia di legame degli elettroni (elettrone libero). Si trascurano quindi le piccole perdite di energia dovute alle collisioni lontane ($\xi/E_{\text{bind}} \gg 1$).
- E sottinteso $\xi/E_{\text{inc}} \ll 1$, così che $E_{\text{inc}} \approx \text{cost.}$
- Per i gas, le perdite tipiche sono di qualche keV, comparabili alle energie di legame degli elettroni più interni. In questo caso è necessario tenere conto delle energie di legame e occorre più accurato

Assorbitori sottili: teoria di Landau (cenni)

Con queste assunzioni χ può essere fattorizzata come segue:

$$\chi(W, x) = \frac{1}{\xi} f_L(\lambda)$$

$$\text{con } \lambda = \frac{1}{\xi} \left[W - \bar{W} - \xi \ln \frac{\xi}{\varepsilon'} + 1 - c_E \right];$$

$$\ln \varepsilon' = \ln \frac{(1 - \beta^2) I^2}{2mv^2} + \beta^2;$$

$$c_E = 0.577 \quad (\text{costante di Eulero})$$

ε' è il taglio sulla minima energia trasferibile nel singolo urto, in accordo all'assunzione $\xi/E_{\text{bind}} \gg 1$ (e in modo tale che la perdita media sia quella della bethe-bloch).

Assorbitori sottili: teoria di Landau (cenni)

La funzione universale $f_L(\lambda)$ può essere espressa come segue:

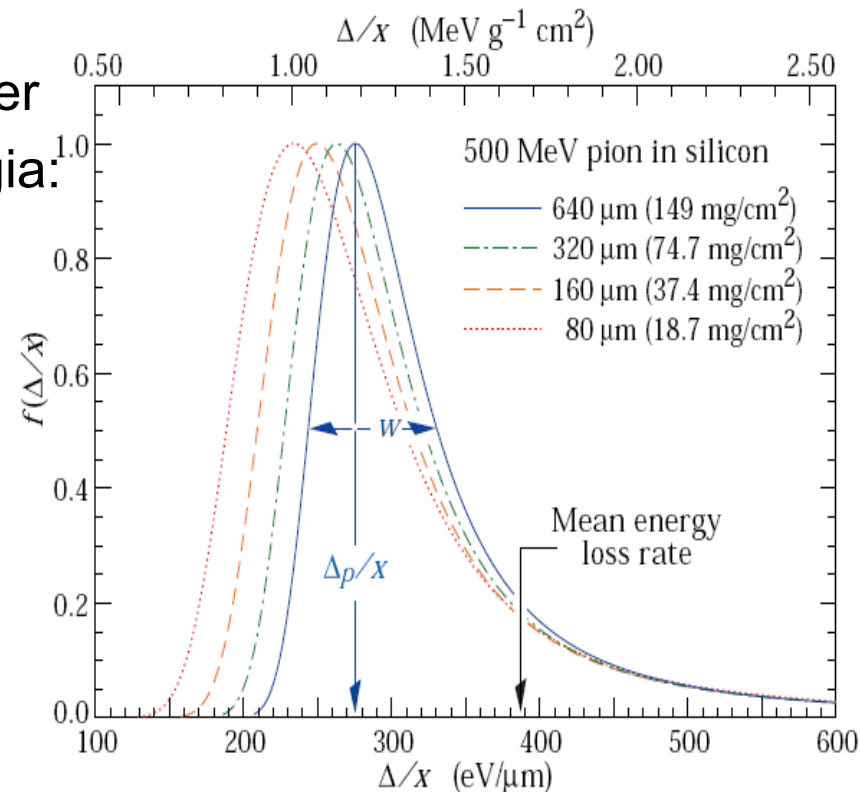
$$f_L(\lambda) = \frac{1}{\pi} \int_0^{\infty} e^{-u(\ln u + \lambda)} \sin(\pi u) du$$

Valutando numericamente $f_L(\lambda)$ si ottiene per il valore più probabile per la perdita di energia:

$$W_{mp} = \xi \left[\ln \frac{\xi}{\epsilon} + 0.198 - \delta \right]$$

δ = correzione per effetto densità
e $\text{FWHM} = 4.02\xi$

$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z \rho \delta x}{m_e \beta^2 c^2 A} = 153.4 \frac{z^2}{\beta^2} \frac{Z}{A} \rho \delta x \quad \text{keV},$$



Assorbitori sottili: teoria di Vavilov (cenni)

Il modello di Landau non funziona bene in gas (ex. TPC) e silicio (tracciatori) dato che in questi materiali le perdite tipiche di energia sono paragonabili alle energia di legame, per cui l'assunzione che $\xi/E_{\text{bind}} \gg 1$ non è più valida

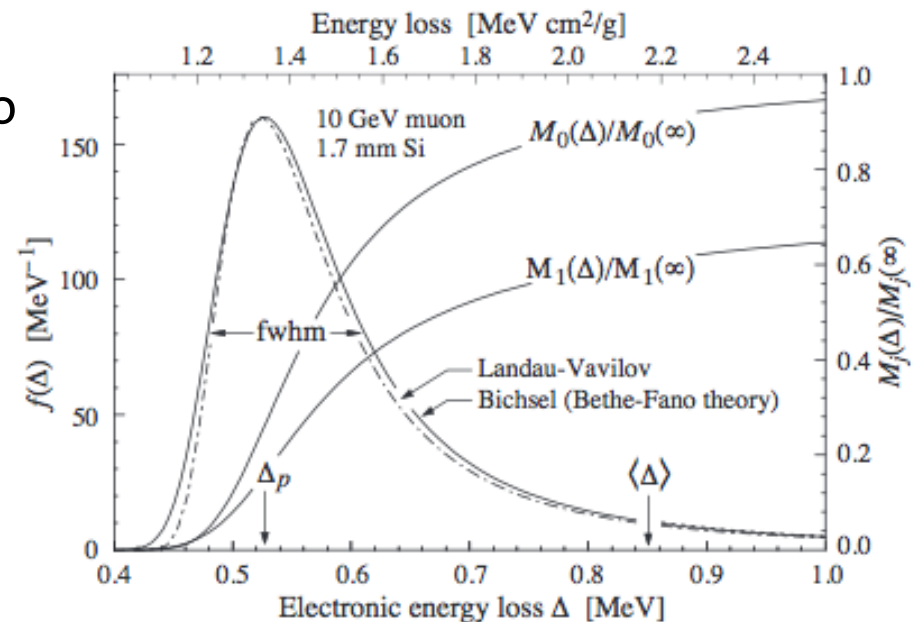
Valida per $k > 0.01$, tiene conto che E_{max} è finito

Caratterizzata da code un po' meno asimmetriche.

Osserviamo:

Anche se il limite gaussiano si ha per $k \geq 10$ già per $k \geq 1$ la distribuzione assomiglia ad una gaussiana.

Vavilov $\rightarrow$ Landau per $k \rightarrow 0$ ed ad una gaussiana per $k \rightarrow \infty$.



Assorbitori sottili: teoria di Vavilov (cenni)

Vavilov[5] derived a more accurate straggling distribution by introducing the kinematic limit on the maximum transferable energy in a single collision, rather than using $E_{\max} = \infty$. Now we can write[2]:

$$f(\epsilon, \delta s) = \frac{1}{\xi} \phi_v(\lambda_v, \kappa, \beta^2)$$

where

$$\phi_v(\lambda_v, \kappa, \beta^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) e^{\lambda s} ds \quad c \geq 0$$

$$\phi(s) = \exp[\kappa(1 + \beta^2 \gamma)] \exp[\psi(s)],$$

$$\psi(s) = s \ln \kappa + (s + \beta^2 \kappa) [\ln(s/\kappa) + E_1(s/\kappa)] - \kappa e^{-s/\kappa},$$

and

$$E_1(z) = \int_{\infty}^z t^{-1} e^{-t} dt \quad (\text{the exponential integral})$$

Assorbitori sottili: teoria di Vavilov (cenni)

$$\lambda_v = \kappa \left[\frac{\epsilon - \bar{\epsilon}}{\xi} - \gamma' - \beta^2 \right]$$

The Vavilov parameters are simply related to the Landau parameter by $\lambda_L = \lambda_v/\kappa - \ln \kappa$. It can be shown that as $\kappa \rightarrow 0$, the distribution of the variable λ_L approaches that of Landau. For $\kappa \leq 0.01$ the two distributions are already practically identical. Contrary to what many textbooks report, the Vavilov distribution *does not* approximate the Landau distribution for small κ , but rather the distribution of λ_L defined above tends to the distribution of the true λ from the Landau density function. Thus the routine **GVAVIV**

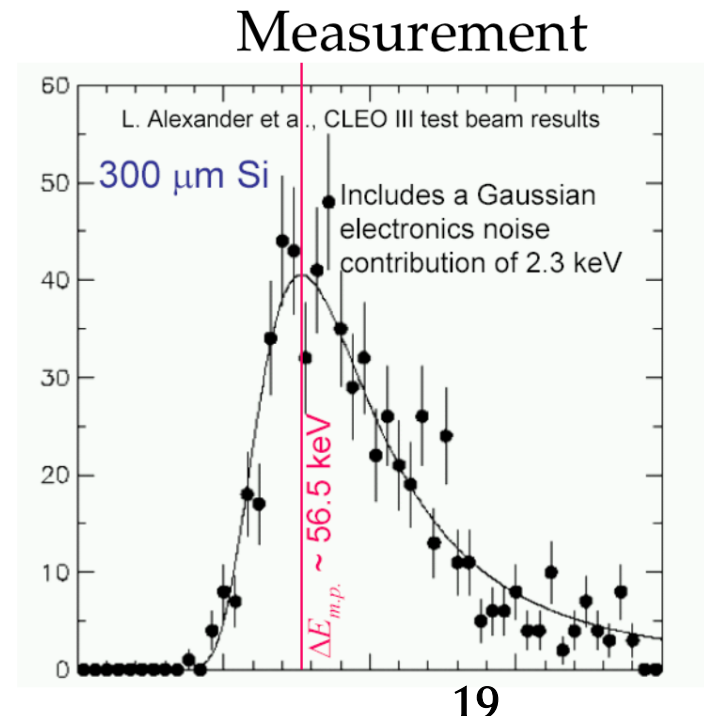
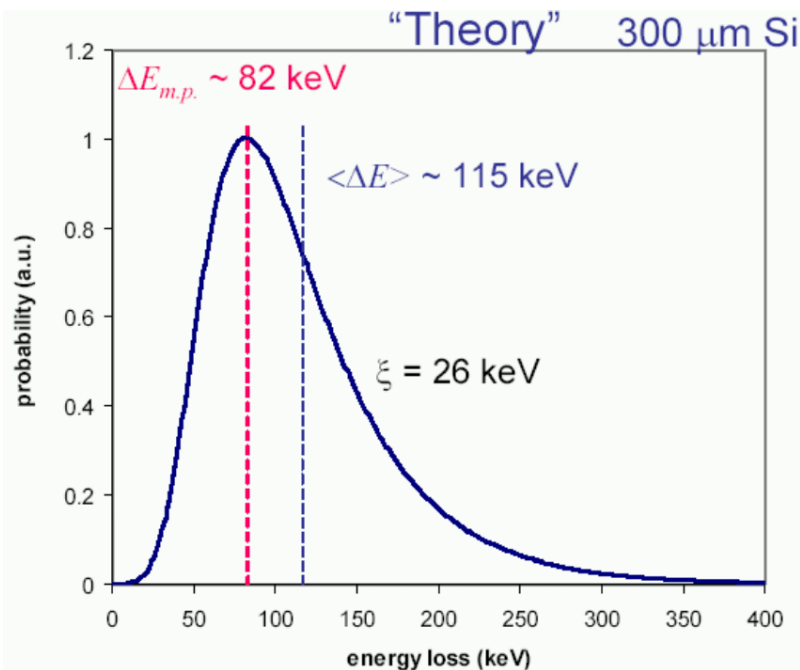
The Landau/Vavilov distribution depends on two parameters:

- ❑ the most probable energy loss W_{mp} and
- ❑ the FWHM of the distribution $\sim \xi$

Typical application of the Landau distribution is the CHARGE Identification of the incident particle, since $W_{mp} = W_{mp}(Z_{inc}, \beta)$ and $\langle W \rangle = W(Z_{inc}, \beta)$

The particle crosses N active layers in which it deposits dE_i , $i=1, \dots, N$ according to the Landau distribution.

Then by a numerical fit, one finds the best parameters W_{mp} , FWHM that describes the measured set of dE_i . Knowing W_{mp} , and the width it is possible to get Z_{inc}

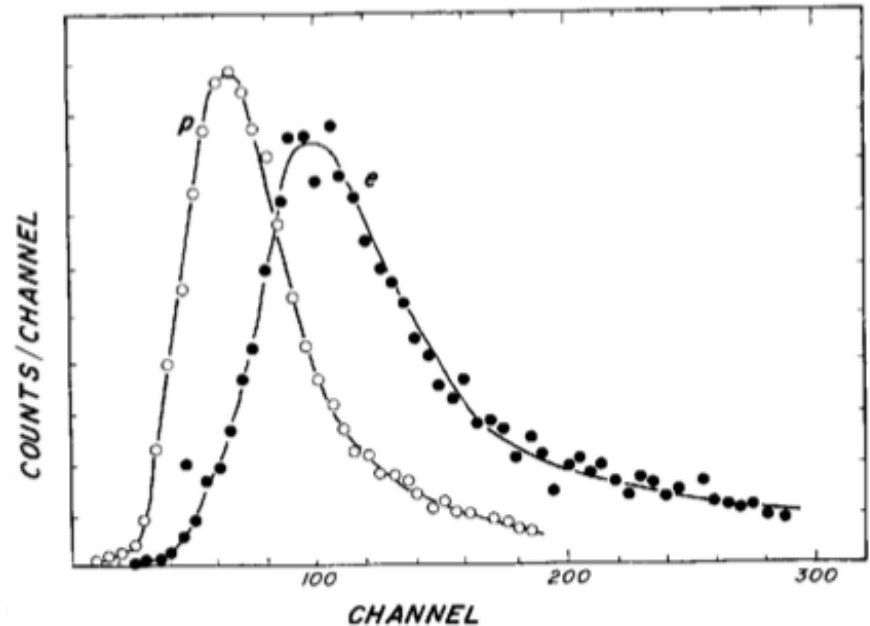


Charge ID

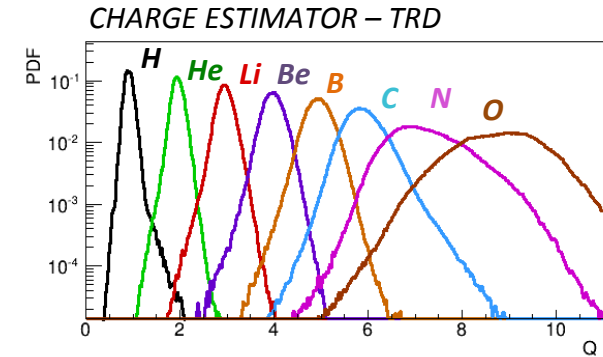
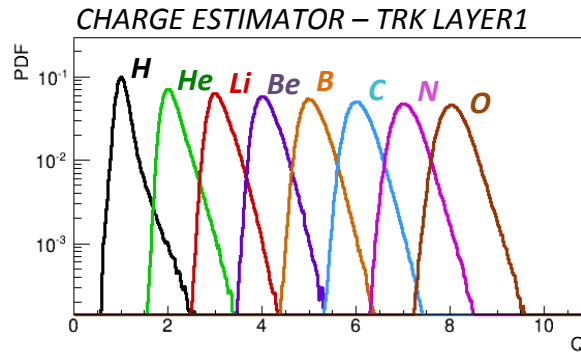
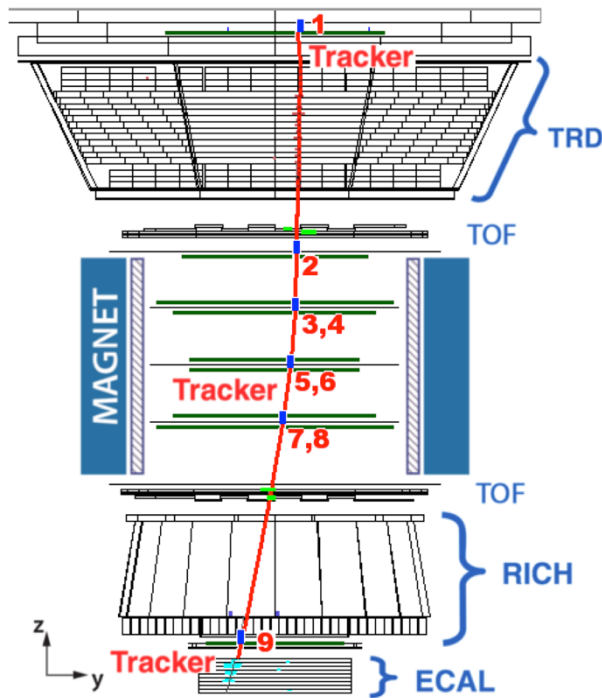
- Conseguenze sperimentali:

- ID di carica fatta tramite la misura del deposito di energia (noto β e/o p): necessita' di avere piu' misure di dE/dX per campionare la distribuzione di Landau/Vavilov da cui ricavare W_{mp} e quindi il valore assoluto della carica dato che $W_{mp} \propto Z_{inc}^2$

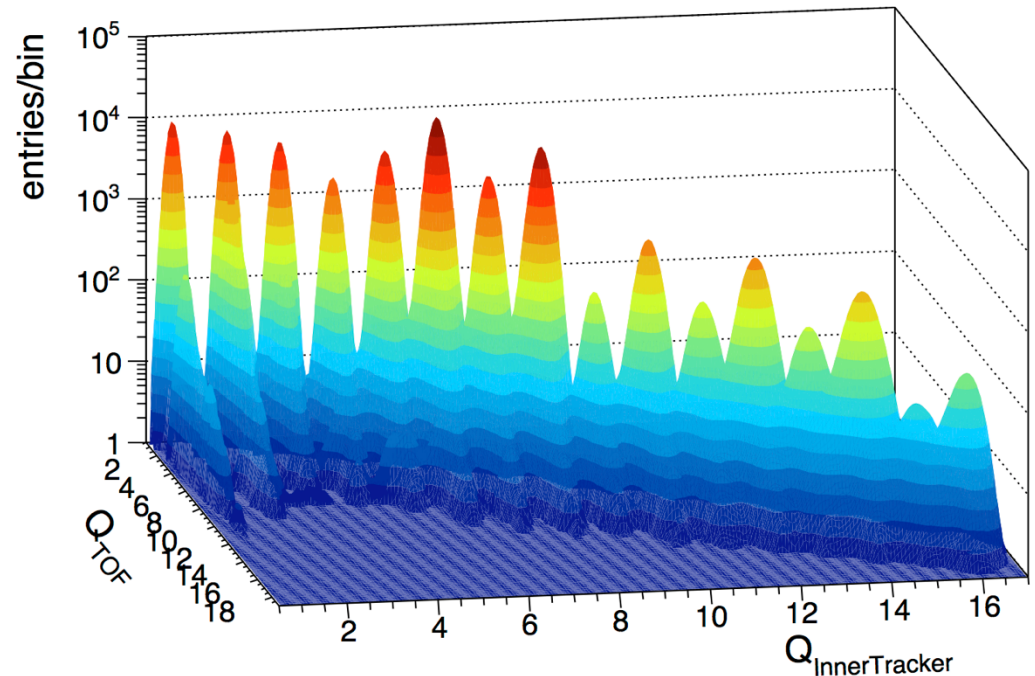
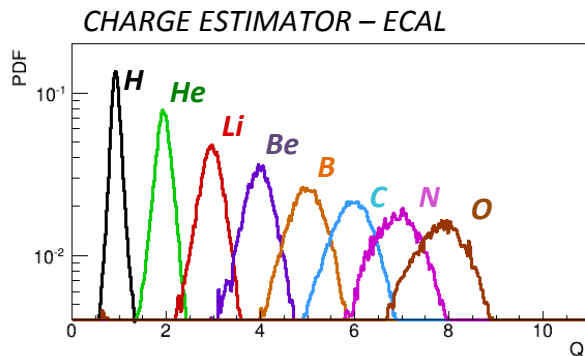
Figure 2.7 Measured pulse height distributions for 3-GeV/c protons and 2-GeV/c electrons in a 90% Ar + 10% CH₄ gas mixture. (After A. Walenta, J. Fischer, H. Okuno, and C. Wang, Nuc. Instr. Meth. 161: 45, 1979.)



Charge ID: Multiple measurements of charge



TOF & TRK charge estimators

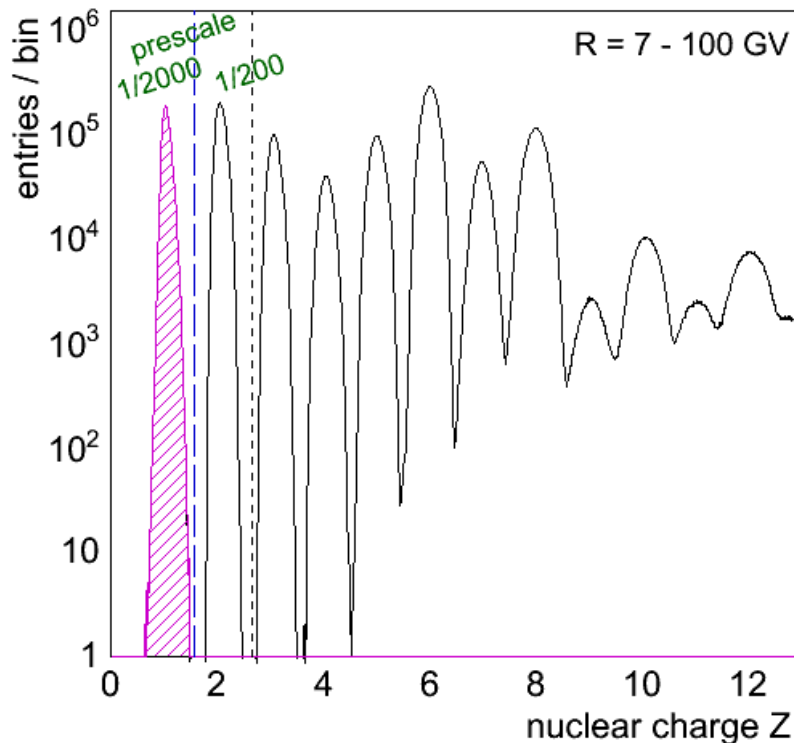


Charge measurement using Tracker

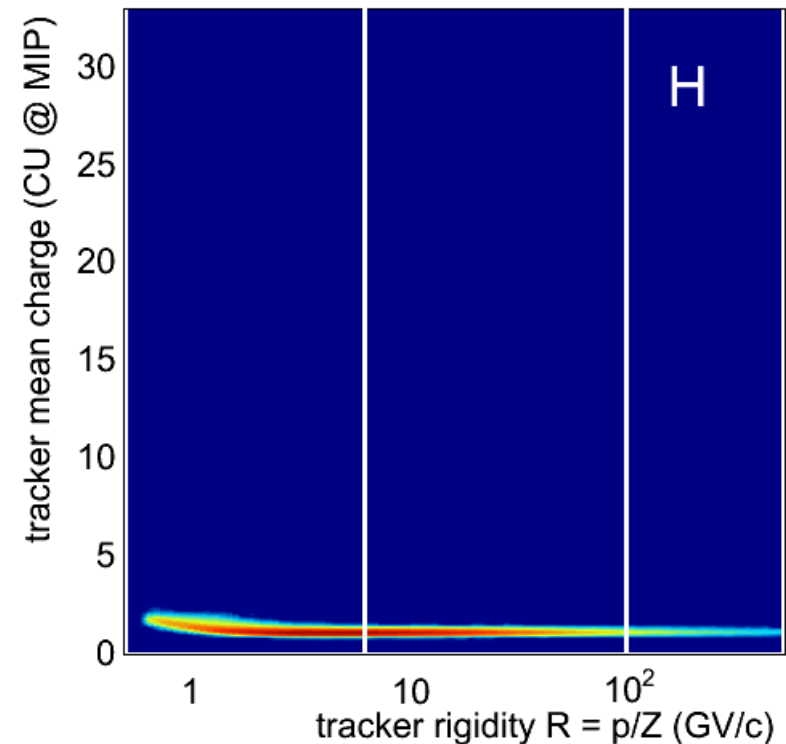
Identification of Light Cosmic Ray Nuclei

with the AMS Silicon Tracker

Selected charge: $Z = 1$ [Hydrogen]

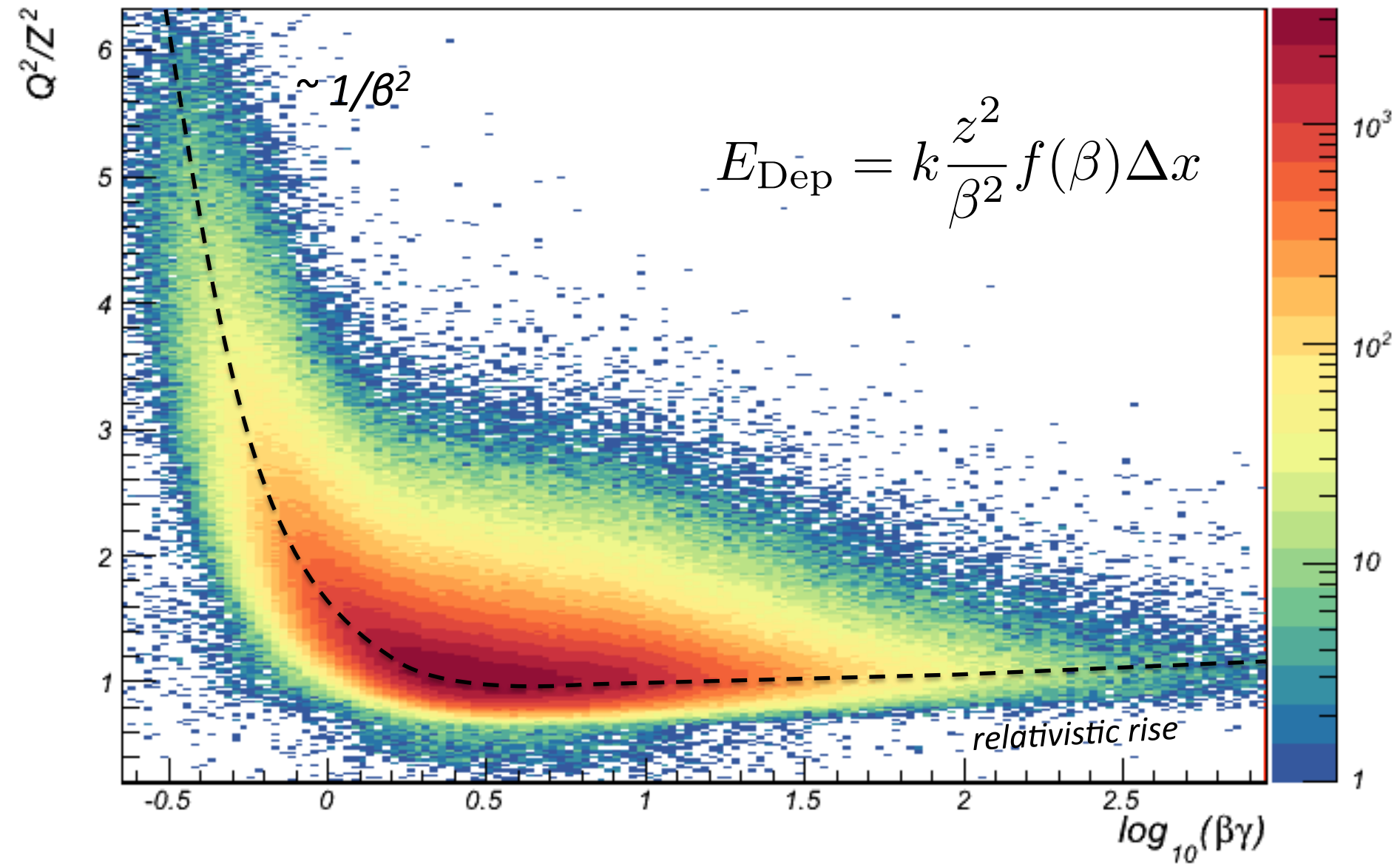


Tracker $\langle dE/dX \rangle$ VS Rigidity



animated slide – set full screen

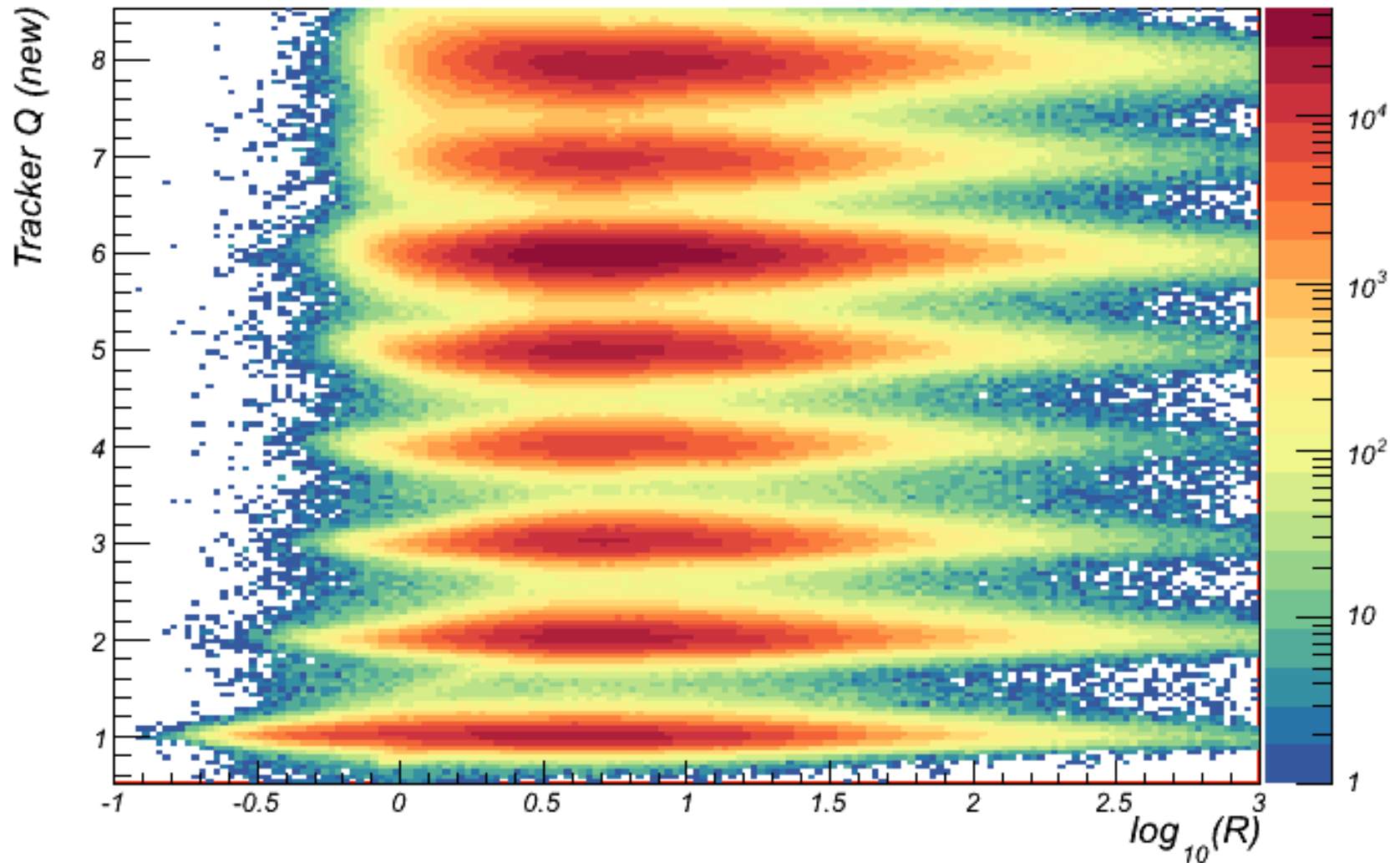
Bethe-Bloch for Protons



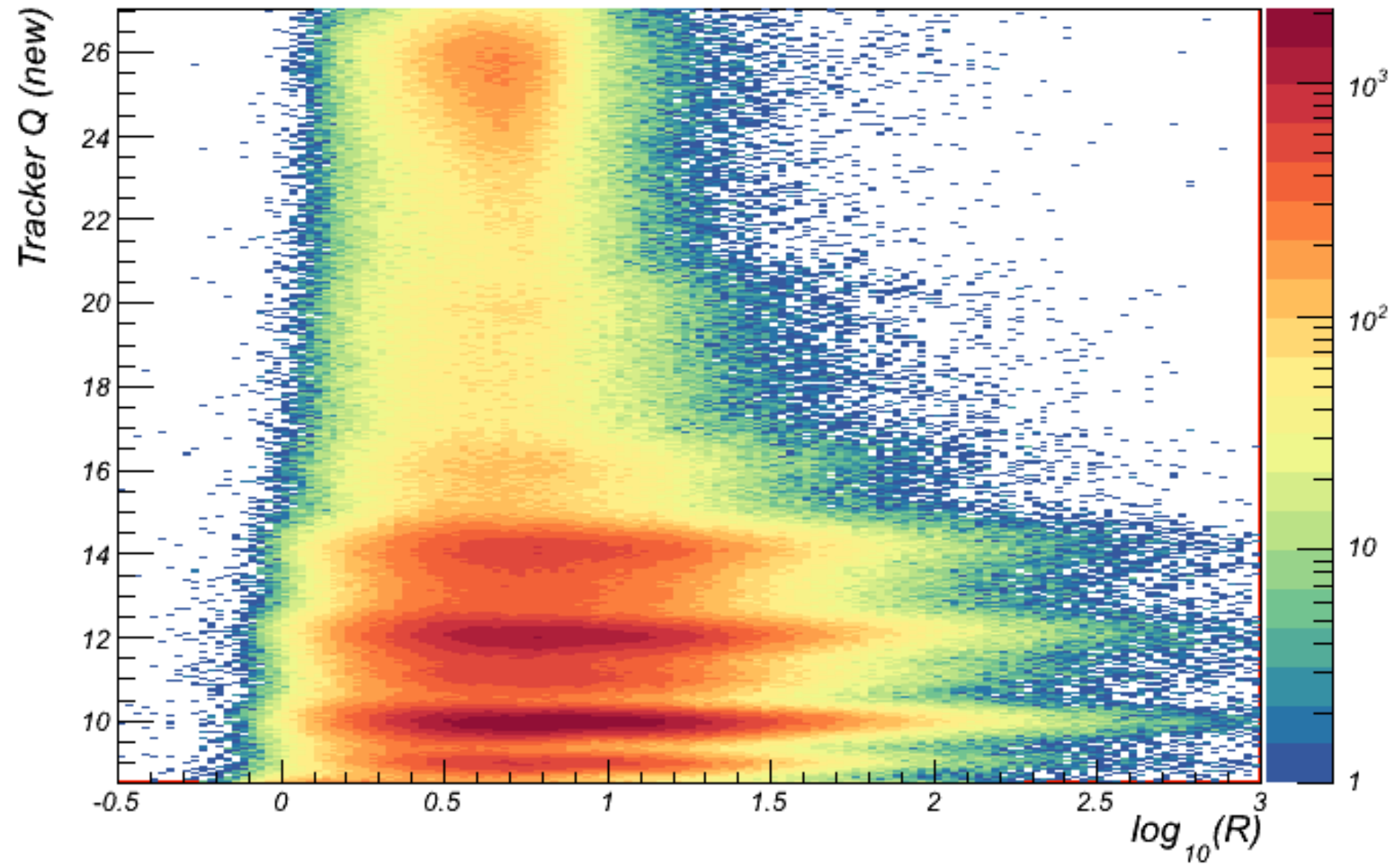
Velocity corrections to the measured dE/dX

To eliminate the velocity dependence of the charge, usually a velocity correction is applied to the measured energy depositions

$$\Delta E_{\text{corr}} = \Delta E_{\text{meas}} * \beta / F(\beta)$$



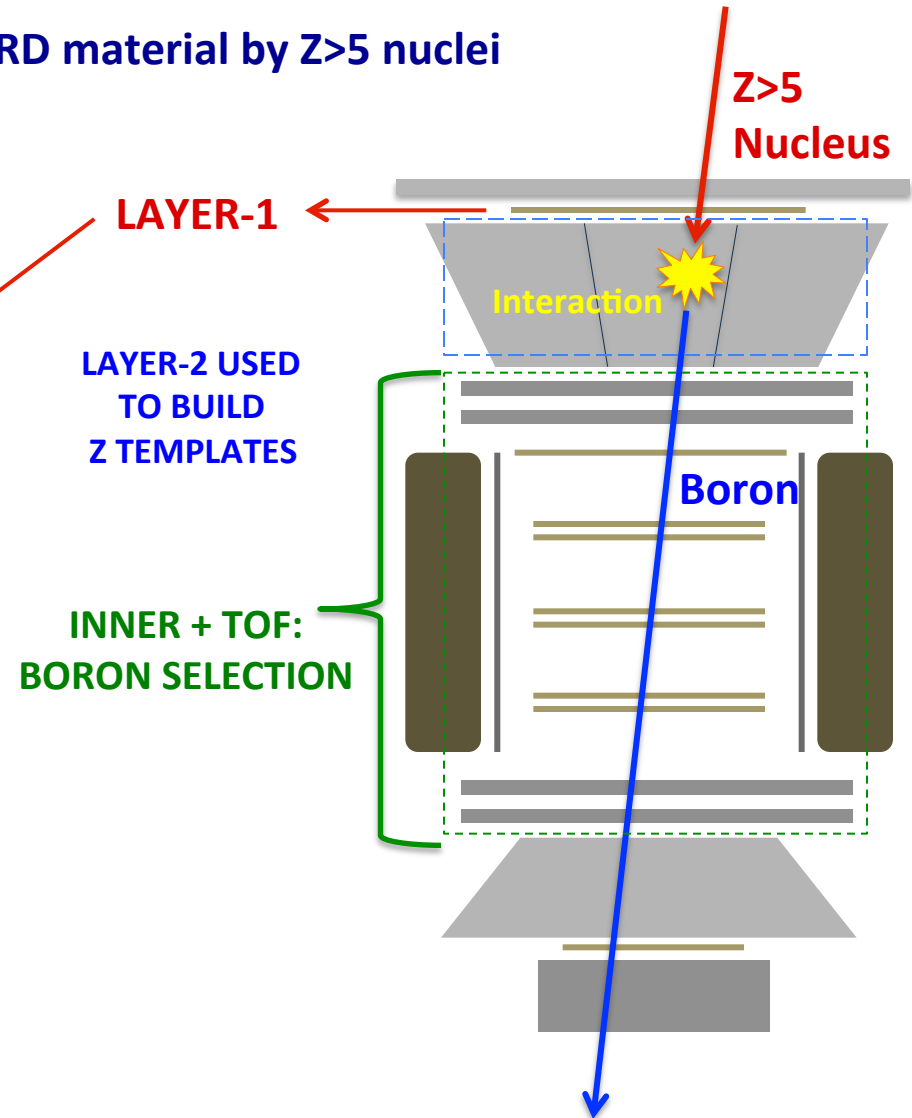
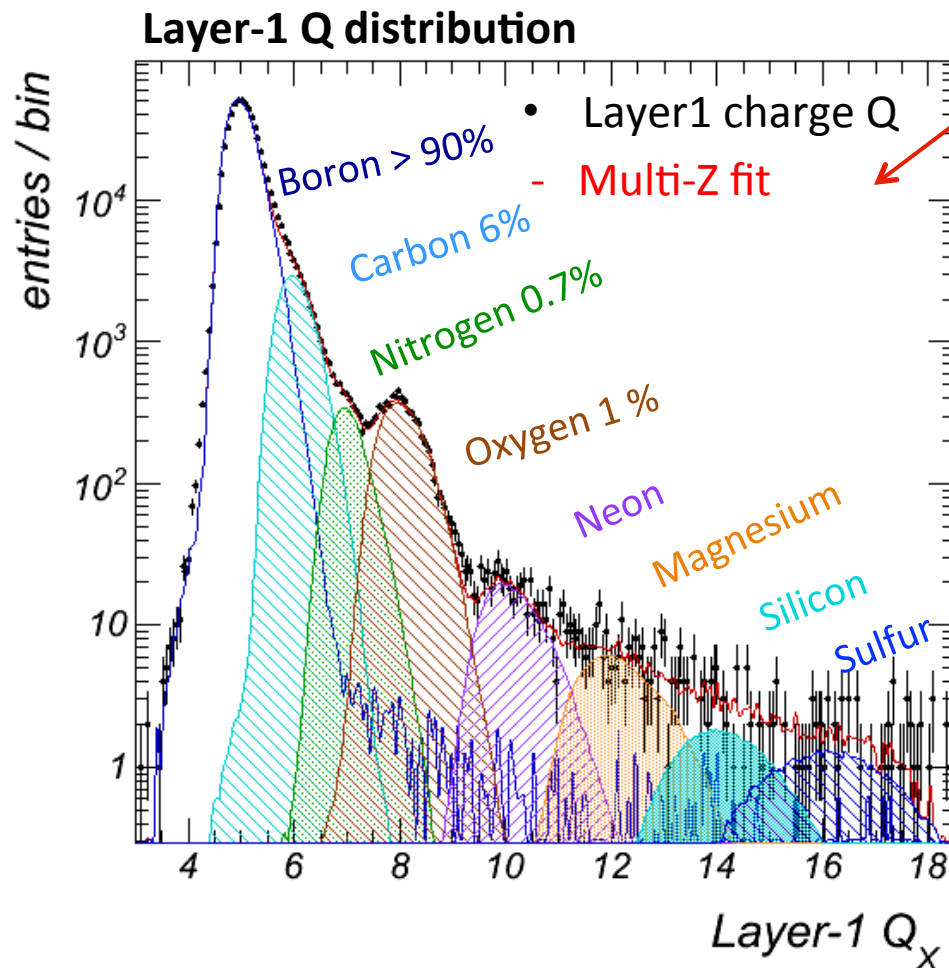
After Rigidity Correction (high-Z)



Fragmentation studies in the detector

Redundancy in Z-measurements allows us to study different fragmentation processes appearing at different levels in the detector.

Example: secondary production of Boron in TRD material by $Z > 5$ nuclei

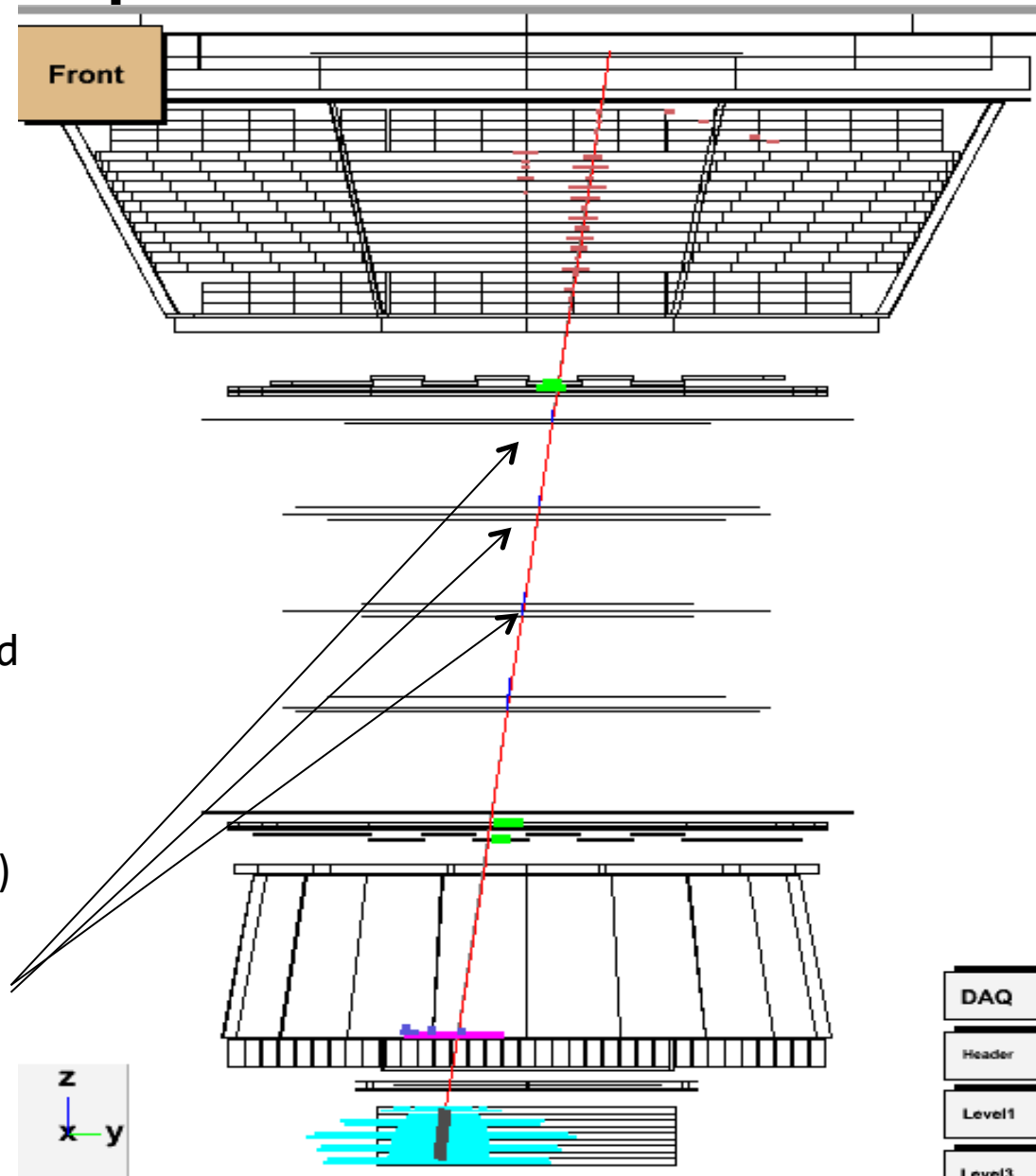


dE/dX and particle ID

dE/dX can (and it IS used) to signal the passage of a charged particle through an active medium, as the active planes (where the energy is deposited and converted in an electrical signal) of a tracking device (eg immersed in a magnetic field)

Each energy deposit is $\propto z^2$ and therefore they can be used to measure the particle $|z|$, provided that an independent measurement of β is available (since energy deposit $\propto z^2/\beta^2$)

AMS Event display for a high energy electron: the passage at a given point is signalled by the energy deposits in the tracking planes

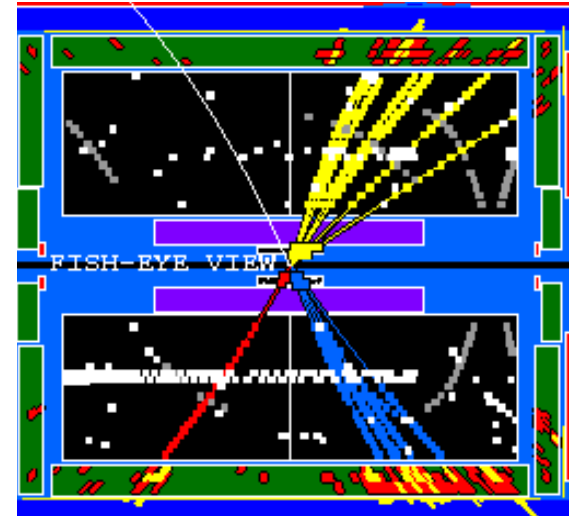
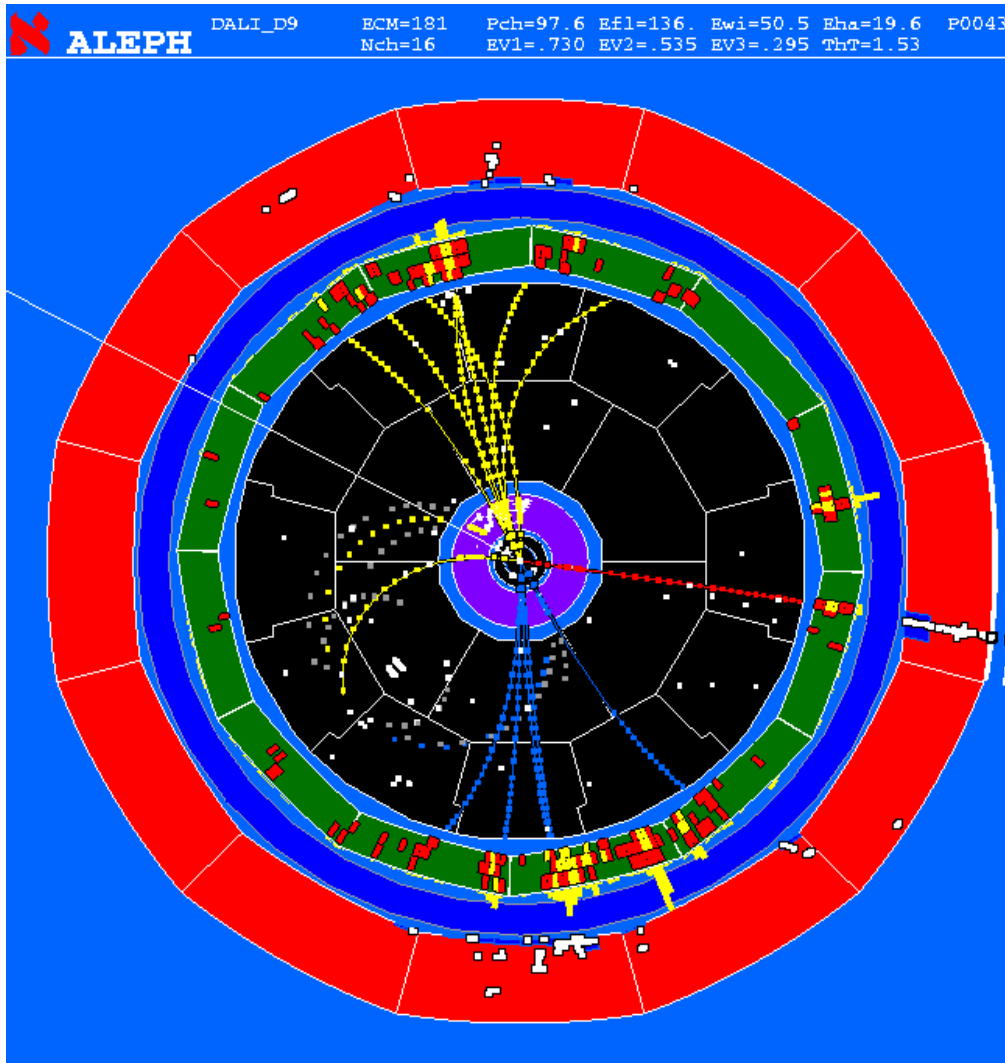


A W^+W^- decay in ALEPH (a LEP experiment)

e^+e^- ($\sqrt{s}=181$ GeV)

$\rightarrow W^+W^- \rightarrow qq\mu\nu_\mu$

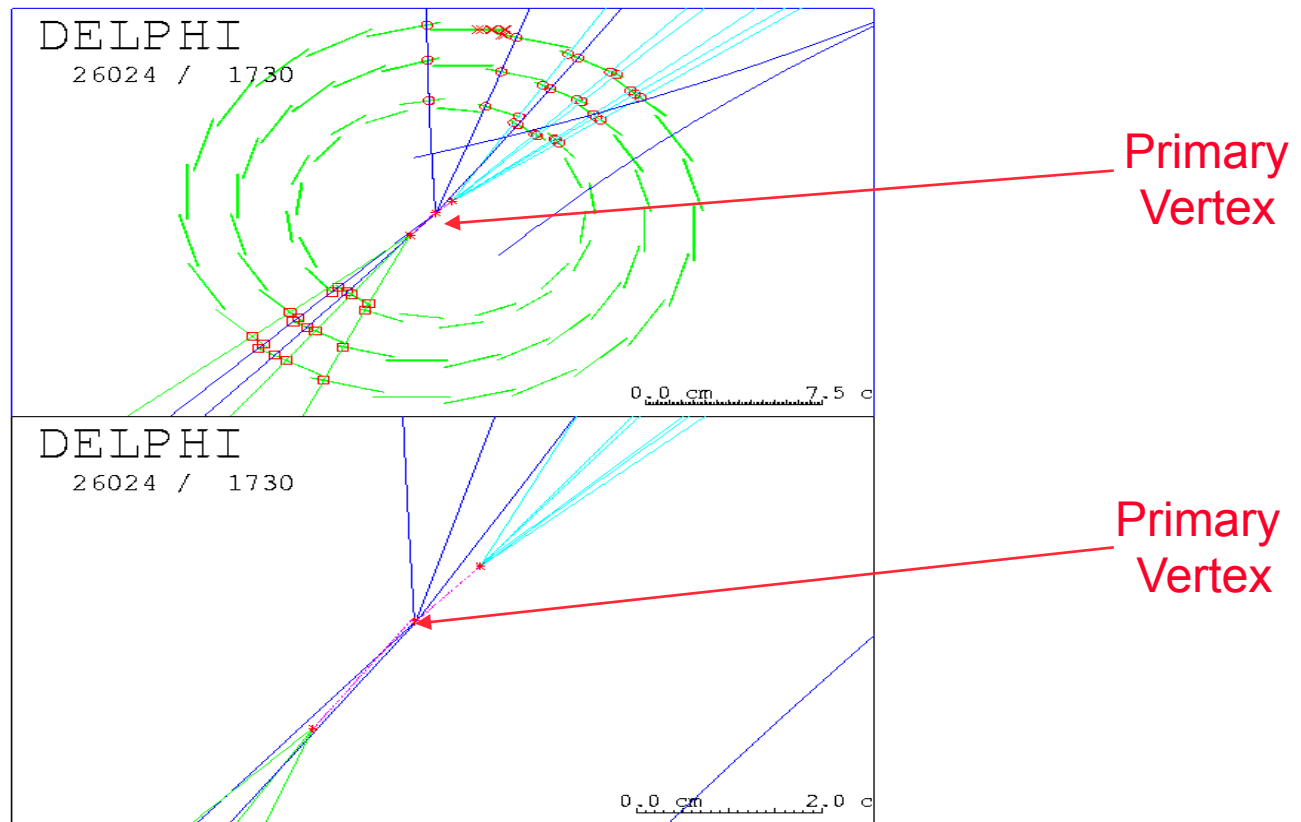
$\rightarrow$ 2 hadronic jets + μ + missing momentum



Most of position sensitive detectors are based on the energy deposit in the active medium by ionization, as tracking detectors used to reconstruct trajectories the particles

Reconstructed B-mesons in the DELPHI micro vertex detector

$$\tau_B \approx 1.6 \text{ ps} \quad l = c\tau\gamma \approx 500 \mu\text{m} \cdot \gamma$$



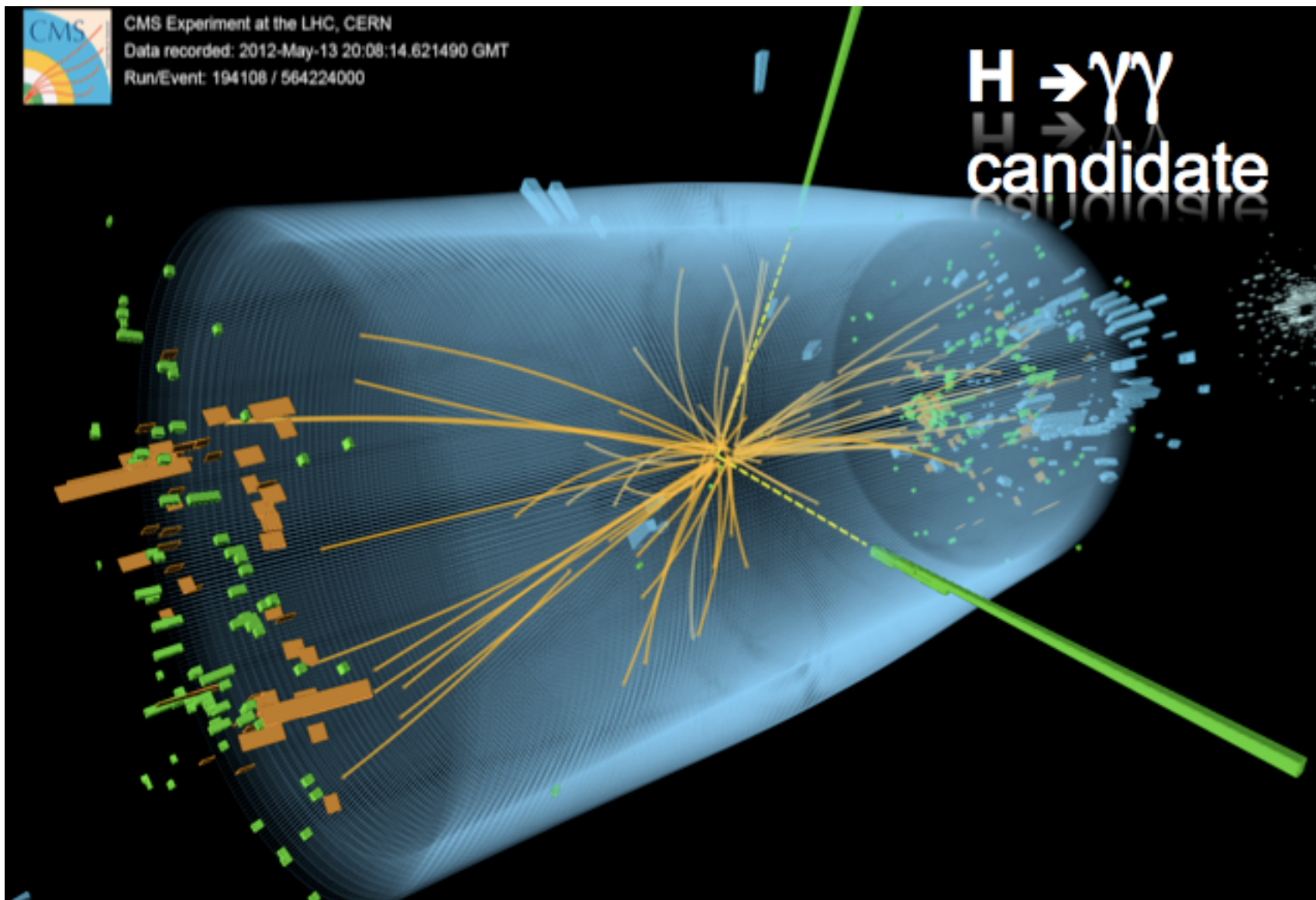


CMS Experiment at the LHC, CERN

Data recorded: 2012-May-13 20:08:14.621490 GMT

Run/Event: 194108 / 564224000

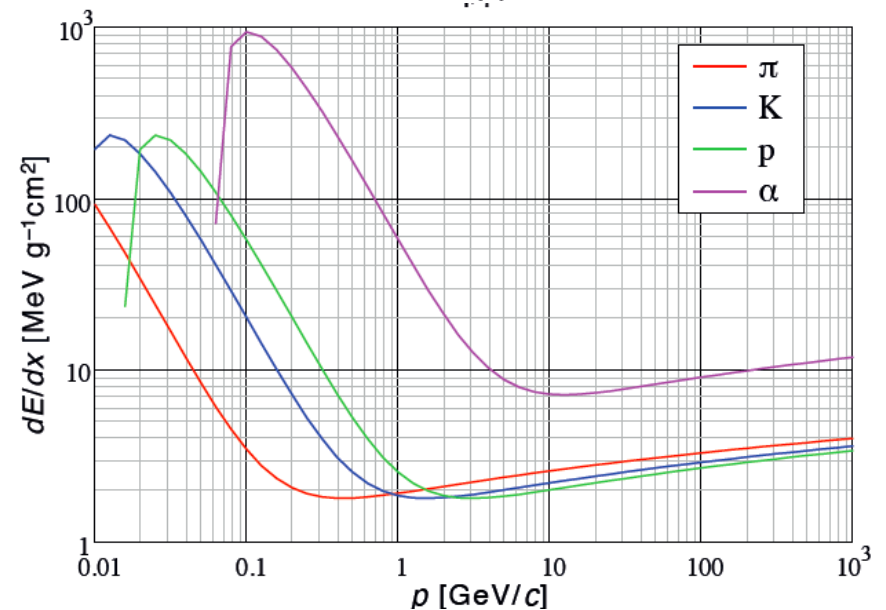
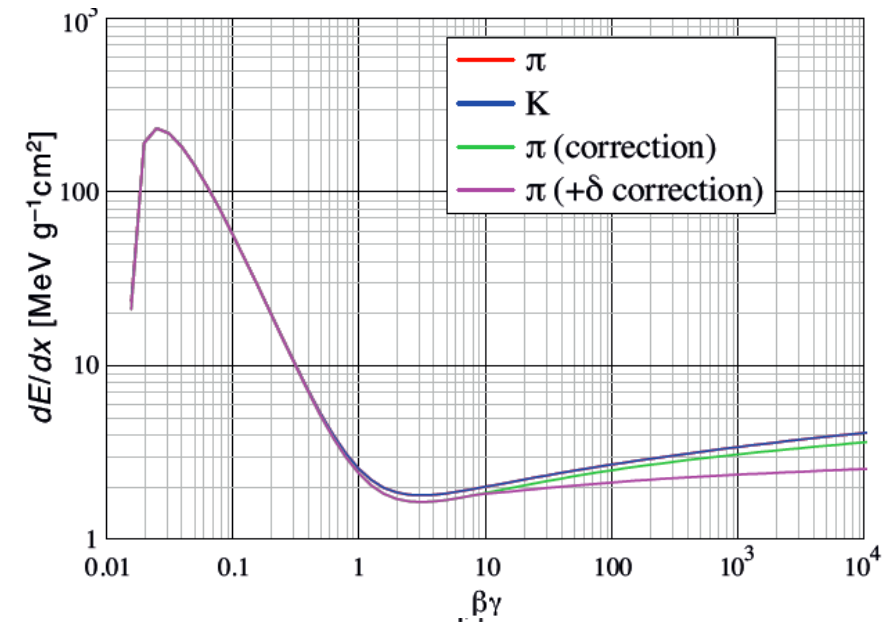
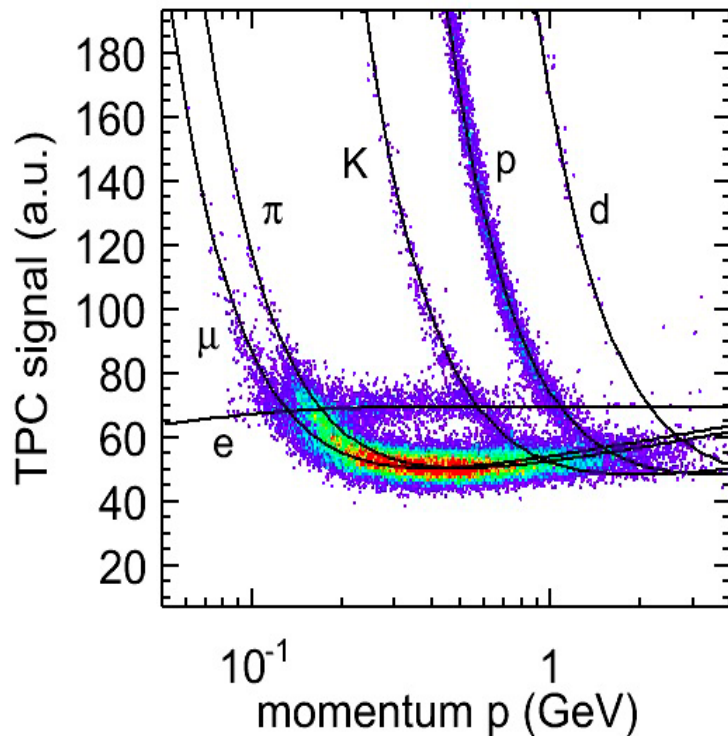
$H \rightarrow \gamma\gamma$
candidate



dE/dx and particle ID

For a given medium, dE/dX depends only on β

But for a given momentum, $\beta\gamma$ and hence dE/dX are different for particles with different masses:

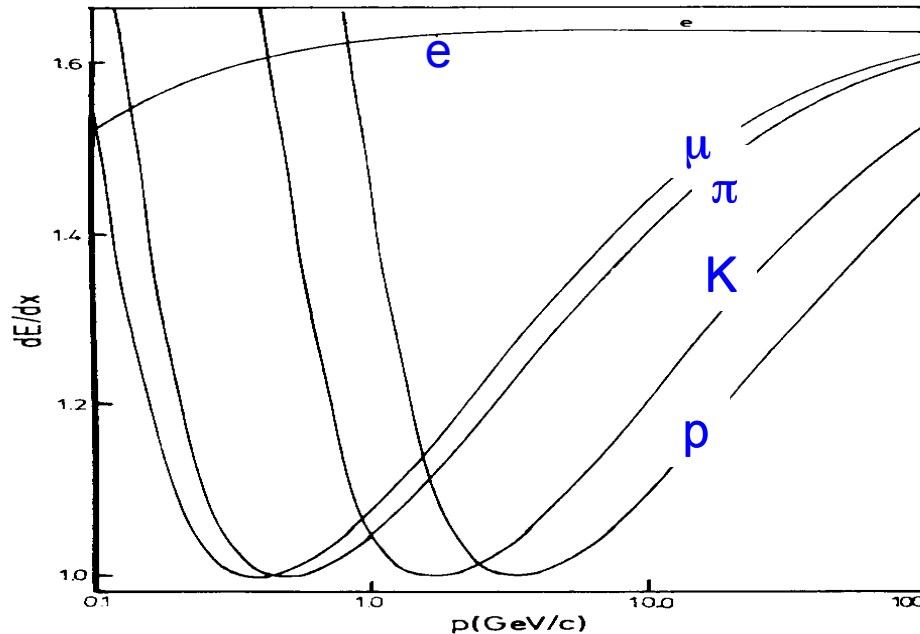


Particle ID using the specific energy loss dE/dx

$$p = m_0 \beta \gamma c$$

$$\frac{dE}{dx} \propto \frac{1}{\beta^2} \ln(\beta^2 \gamma^2)$$

} Simultaneous measurement of p and dE/dx defines mass m , hence the particle identity



} π/K separation (2σ) requires a dE/dx resolution of $< 5\%$

Average energy loss for e, μ, π, K, p in 80/20 Ar/CH₄ (NTP)

(J.N. Marx, Physics today, Oct.78)

But: Large fluctuations + Landau tails !

When $\beta \rightarrow 1$, there only a $\ln(\beta\gamma)$ dependence, all the particles tend to have similar loss: difficult to separate particles from single dE/dx
 $\rightarrow$ very good resolution both in p and dE/dx measurements needed

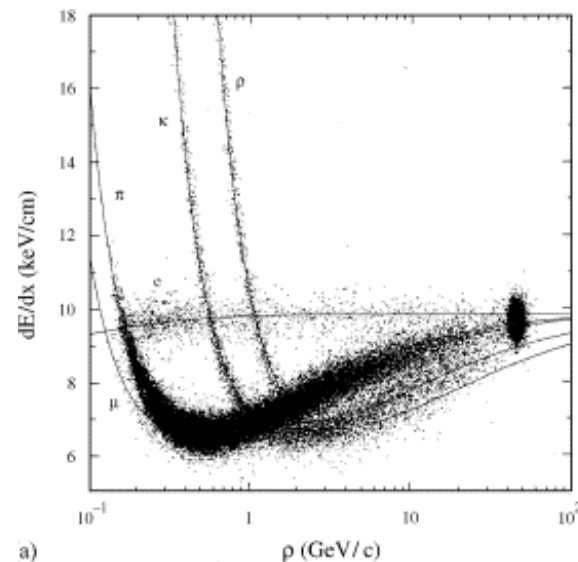
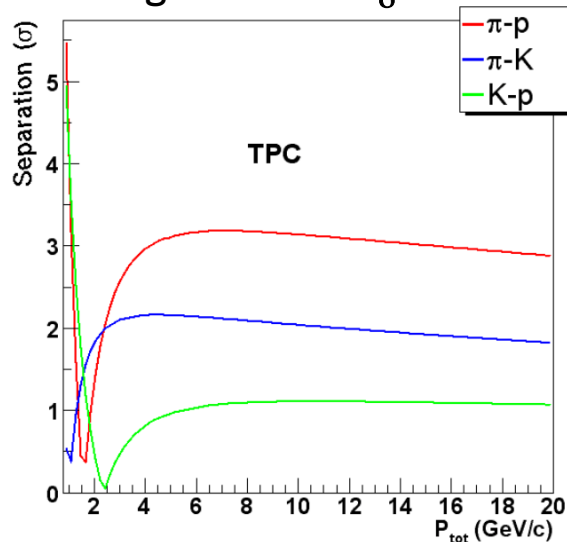
dE/dx and particle ID

Any PID technique is based on the fact that the detector response is different “enough” to different types of particles. Therefore it is worthwhile to introduce the concept of “separation power” of a PID system, defined by the significance of the measurement of the detector response, herein generically indicated with S . The detector response could actually be, according to the method employed, either time (in the case of TOF detectors), the ionization energy loss or the Cherenkov angle.

If S_A and S_B are the mean values of such a quantity measured for particles of type A and B, respectively, and σ_{AB} is the average value between the standard deviations of the measured distributions, the separation power n_σ is given by:

$$n_\sigma = (S_A - S_B) / \sigma_{AB}$$

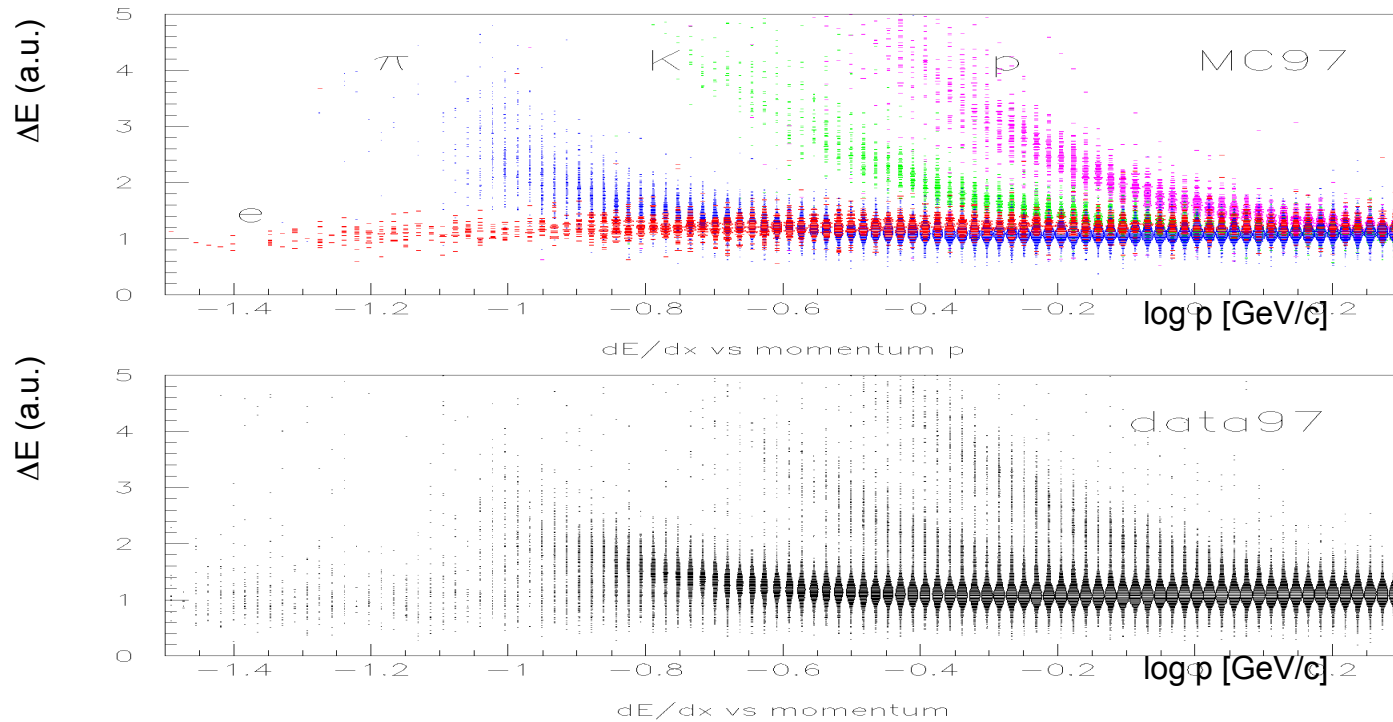
By assuming that the measured quantities S_A and S_B are distributed according to a Gaussian, a particle contamination smaller than 1% requires a separation power larger than $n_\sigma = 4$.



dE/dx and particle ID

dE/dx can also be used in
Silicon detectors

Example DELPHI microvertex detector (3 x 300 μm Silicon)



Time of flight

Particle ID using Time Of Flight (TOF)

$$t = \frac{L}{\beta c}$$


Combine TOF with momentum measurement

$$m = p \sqrt{\frac{c^2 t^2}{L^2} - 1}$$

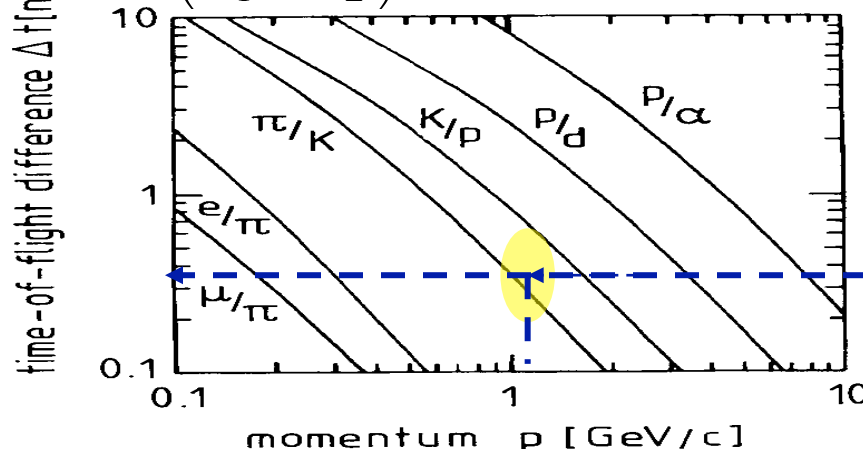
Mass resolution

$$(p = m_0 \beta \gamma)$$

$$\frac{dm}{m} = \frac{dp}{p} \oplus \gamma^2 \left(\frac{dt}{t} + \frac{dL}{L} \right) \quad \text{Quadratic sum}$$

TOF difference of 2 particles at a given momentum

$$\Delta t = \frac{L}{c} \left(\frac{1}{\beta_1} - \frac{1}{\beta_2} \right) = \frac{L}{c} \left(\sqrt{1 + m_1^2 c^2 / p^2} - \sqrt{1 + m_2^2 c^2 / p^2} \right) \approx \frac{Lc}{2p^2} (m_1^2 - m_2^2)$$



Δt for $L = 1$ m path length

$\sigma_t = 300$ ps

π/K separation up to
1 GeV/c

Collisioni fra particelle *cariche:* Multiple Scattering

Riprendiamo una slide già vista

Una particella di massa $\gg$ dell'elettrone
in moto (veloce) in un materiale
collide con:

- **Nuclei** $\rightarrow$ poca energia rilasciata al nucleo, ma angolo di scattering della particella incidente significativo.
- **Elettroni atomici** $\rightarrow$ gli elettroni (leggeri) si prendono energia dalla particella incidente, ma questa fa uno scattering trascurabile $\rightarrow$ Bethe-Bloch, Landau,...

Collisioni fra particelle cariche: Multiple Scattering

- Quindi quando una particella attraversa un mezzo continuo siamo interessati a conoscere:
 - quanta E deposita $\rightarrow$ Bethe-Bloch
 - che deviazione subisce
- Tratteremo i due processi in maniera indipendente (anche se perdite di energia e deviazione NON sono in realta' indipendenti)

Multiple Scattering

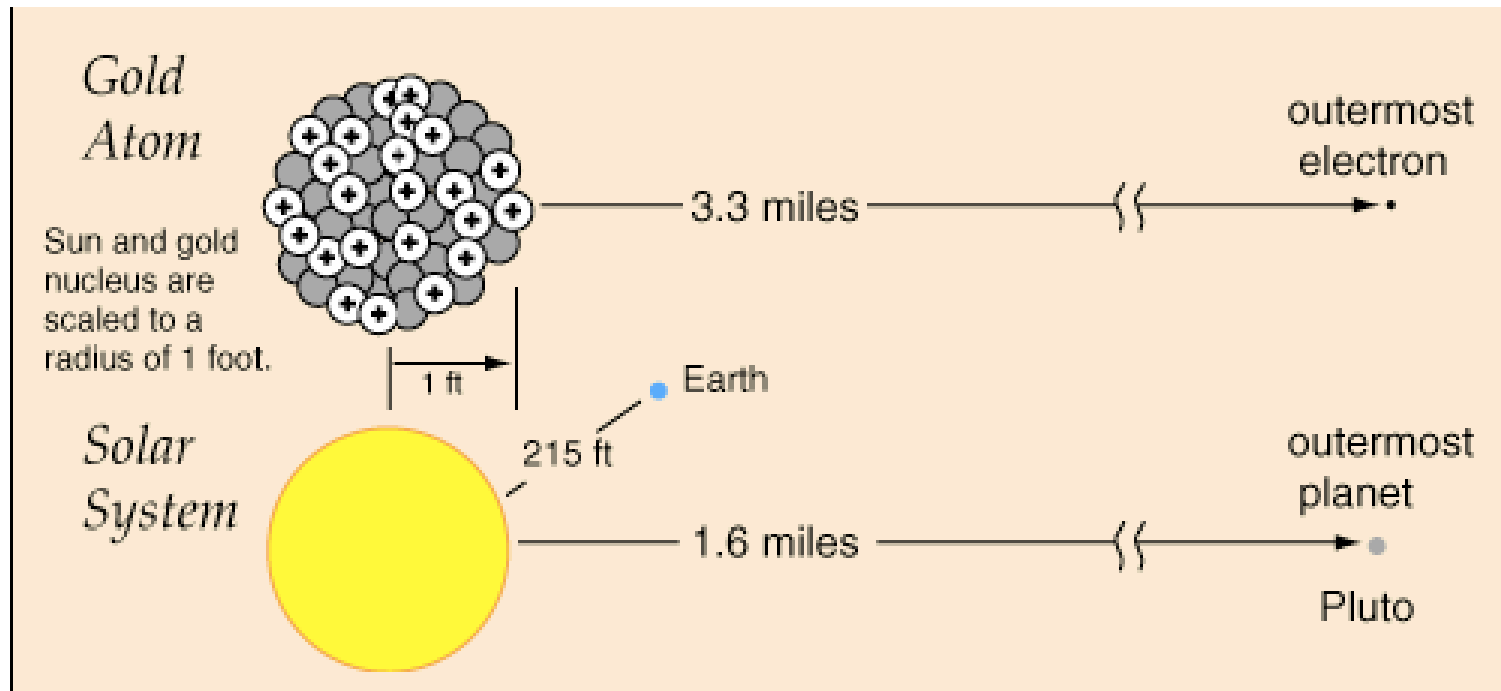
- Looking at dE/dx from ionization, ignore nuclei.
 - Energy transfer small compared to scattering from (lighter) electrons.
- However, scattering from nuclei does change the **direction** of the particles momentum, if not its **magnitude**.
 - Deflection of particle's path limits the accuracy with which the curvature in a magnetic field can be determined, and hence the momentum measured.

Size of Nuclei

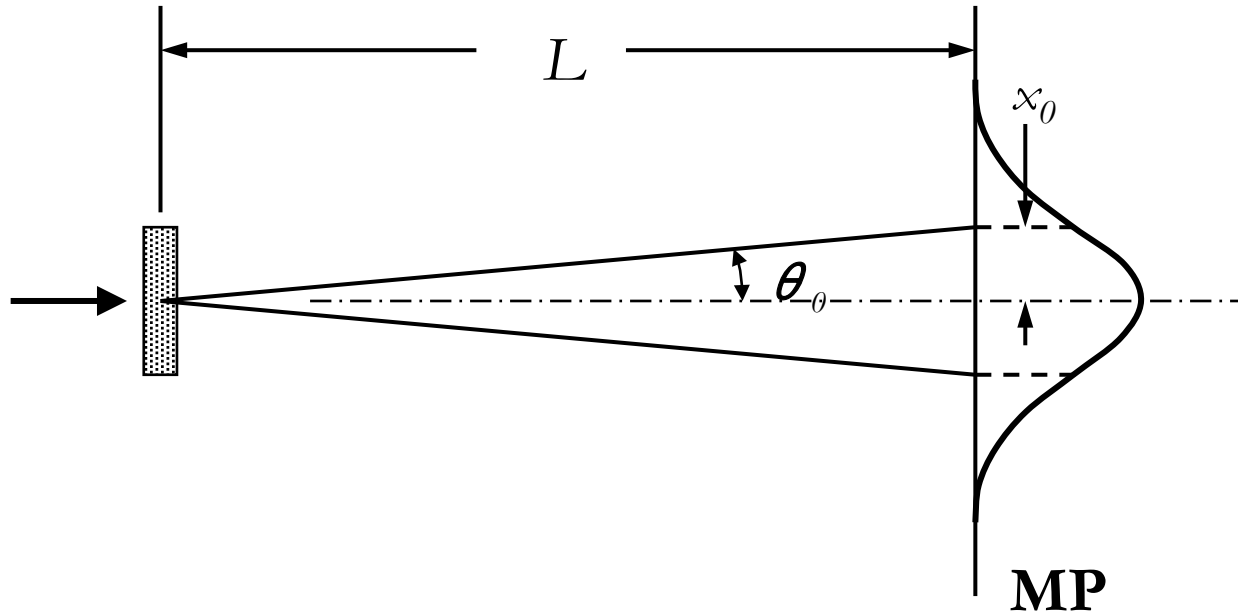
Atomic radius of aluminum = 1.3×10^{-10} m

Nuclear radius aluminum = 3.6×10^{-15} m

The nucleus occupies a tiny fraction of atom volume.



Multiple Scattering



An initially monodirectional beam of particles shows a spread in directions after it crosses a slab of material. This is known as direction straggling or multiple scattering.

When particles, as protons, pass through a slab of material they suffer many collisions with atomic nuclei. The statistical outcome is a *multiple scattering angle* whose distribution is approximately Gaussian. The width parameter of the angular distribution is θ_0 .

The task of multiple scattering theory is to predict θ_0 given the scattering material and thickness, and the incident particle energy.

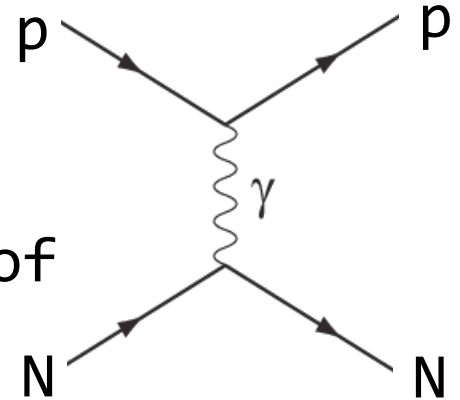
Multiple Scattering

Basic process is the scattering $p + N \rightarrow p + N$ with a probability to scatter in $d\Omega$ around θ, ϕ as $dP = (d\sigma/d\Omega)d\Omega$

Each collision deviates the incident particle direction

We want to know the cumulative effect of many collisions, ie the amount of deviation after crossing a given thickness of material

As usual, two approaches are possible: classical and quantistic



Classical Coulomb cross section

- Il primo passo e' conoscere la probabilita' di diffusione ad un dato angolo in una collisione singola su un nucleo
- Quindi derivare l'effetto statistico delle collisioni dopo un dato spessore x di materiale attraversato

- For **repulsive scattering** (what we mainly look at here) the situation is as shown in the figure:

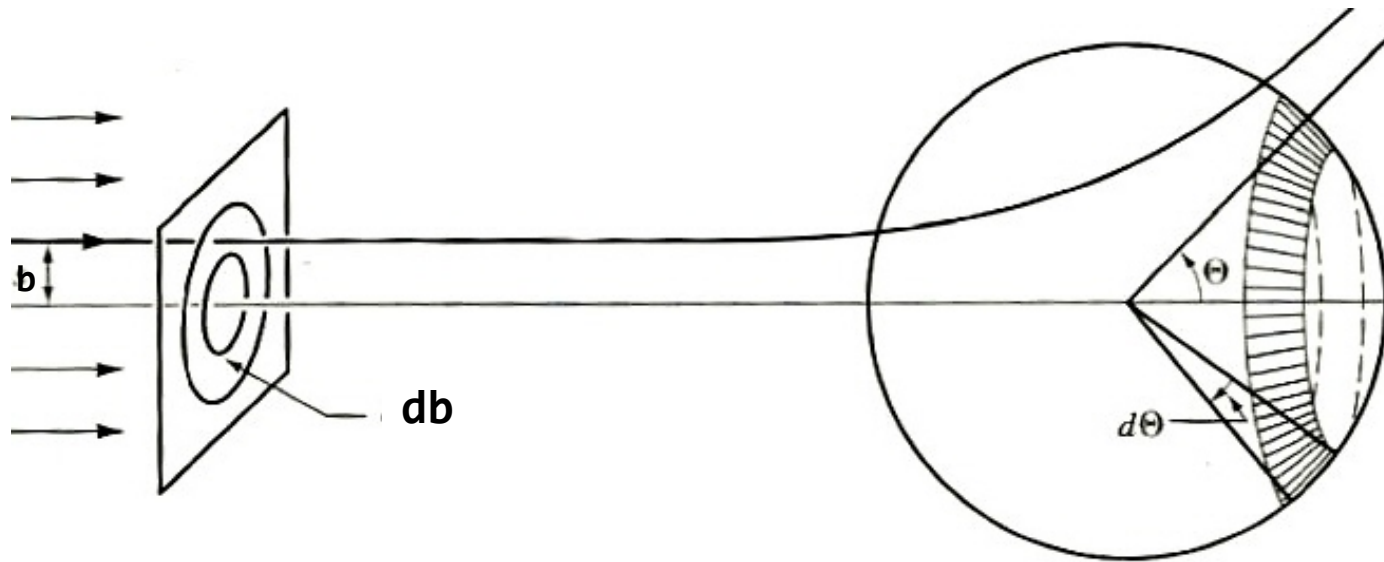


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- Define: Differential Cross Section for Scattering** in a given direction (into a given solid angle $d\Omega$):
 $\sigma(\Omega)d\Omega \equiv (N_s/I)$. With I = incident intensity
 N_s = # particles/time scattered into solid angle $d\Omega$

- **Scattering Cross Section:**

$$\sigma(\Omega)d\Omega \equiv (N_s/I)$$

I = incident intensity

N_s = # particles/time

scattered into angle $d\Omega$

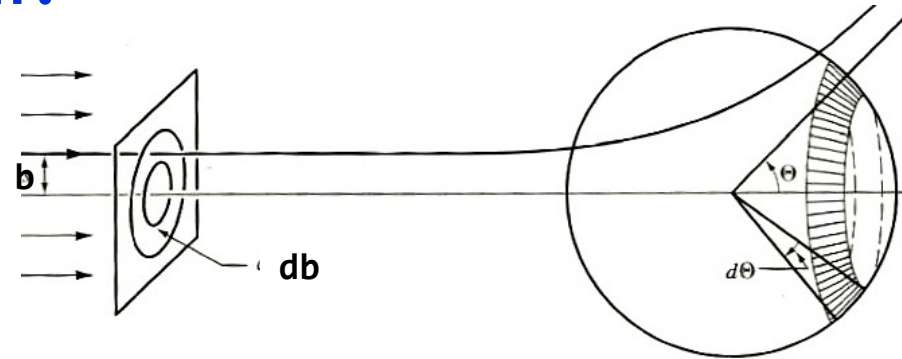


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- In general, the solid angle Ω depends on the spherical angles Θ, Φ . However, for central forces, there must be **symmetry** about the axis of the incident beam

$\Rightarrow \sigma(\Omega) (\equiv \sigma(\Theta))$ is independent of azimuth angle Φ

$\Rightarrow d\Omega \equiv 2\pi \sin\Theta d\Theta$, $\sigma(\Omega)d\Omega \equiv 2\pi \sigma(\Theta) \sin\Theta d\Theta$,

$\Theta \equiv$ Angle between incident & scattered beams, as in the figure.

$\sigma \equiv$ “**cross section**”. It has units of area

Also called the differential cross section.⁴⁴

- As in all Central Force problems, for a given particle, the orbit, & thus the amount of scattering, is determined by the energy E & the angular momentum ℓ

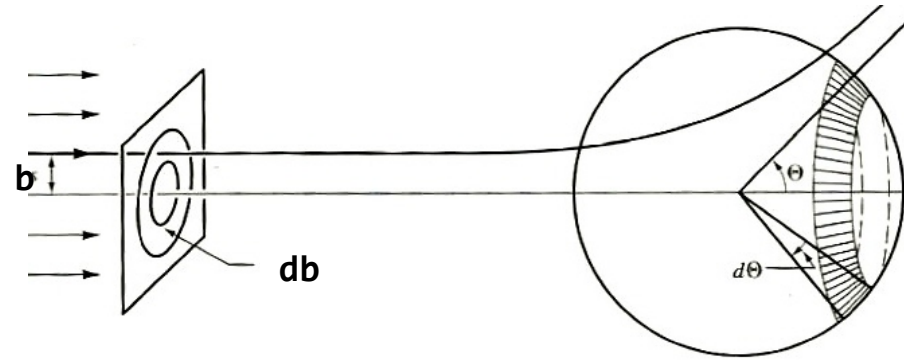


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- **Define: Impact parameter**, b , & express the angular momentum ℓ in terms of E & b .
- Impact parameter $b \equiv$ the $\perp$ distance between the center of force & the incident beam velocity (fig).
- **GOAL: Given** the energy E , the impact parameter b , & the force $f(r)$, what is the cross section $\sigma(\Theta)$?

- **Beam**, intensity I .
 Particles, mass m ,
 incident speed
 (at $r \rightarrow \infty$) = v_θ .

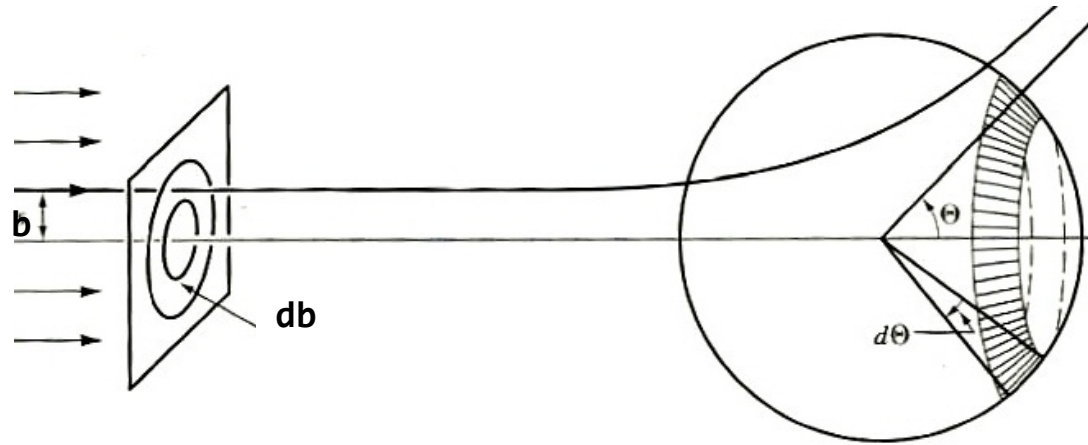


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- **Energy conservation:**

$$E = T + V = \frac{1}{2} m v^2 + V(r) = (\frac{1}{2}) m v_\theta^2 + V(r \rightarrow \infty)$$

- Assume $V(r \rightarrow \infty) = 0 \Rightarrow E = \frac{1}{2} m (v_\theta)^2$
 $\Rightarrow v_\theta = (2E/m)^{\frac{1}{2}}$

- **Angular momentum:** $\ell \equiv m v_\theta b \equiv b (2mE)^{\frac{1}{2}}$

ℓ (vector) is conserved for central forces

- **Angular momentum** $\ell \equiv mv_{\theta}b \equiv b(2mE)^{\frac{1}{2}}$

Incident speed v_{θ} .

- $N_s \equiv$ # particles scattered into solid angle $d\Omega$ between

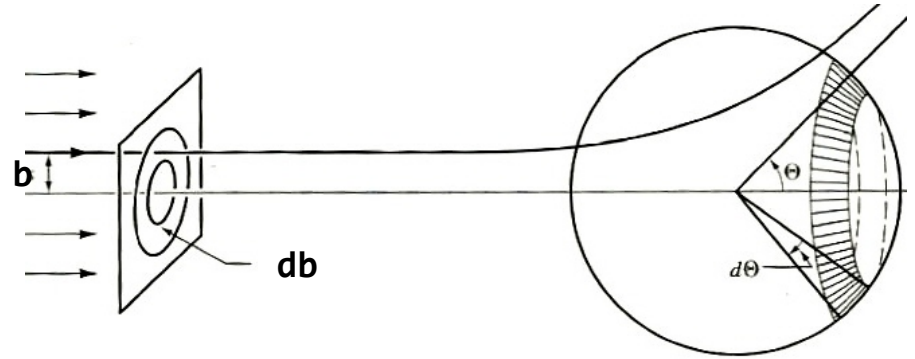


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

Θ & $\Theta + d\Theta$. **Cross section definition**

$$\Rightarrow N_s \equiv I\sigma(\Theta)d\Omega = 2\pi I\sigma(\Theta)\sin\Theta d\Theta$$

- $N_i \equiv$ # incident particles with impact parameter between b & $b + db$. $N_i = 2\pi I b db$

- **Conservation of particle number**

$$\Rightarrow N_s = N_i \text{ or: } 2\pi I\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi I b|db|$$

$2\pi I$ cancels out! (Use absolute values because N 's are always > 0 , but db & $d\Theta$ can have any sign.)

$$\sigma(\Theta)\sin\Theta|d\Theta| = b|db| \quad (1)$$

- b = a function of energy E
- & scattering angle Θ :

$$b = b(\Theta, E)$$

$$(1) \Rightarrow \sigma(\Theta) = (b/\sin\Theta) (|db|/|d\Theta|) \quad (2)$$

- To compute $\sigma(\Theta)$ we clearly need $b = b(\Theta, E)$
- Get $\Theta = \Theta(b, E)$ from the orbit eqtn. For general central force (θ is the angle which describes the orbit $\mathbf{r} = \mathbf{r}(\theta)$; $\theta \neq \Theta$)

$$\theta(r) = \int (\ell/r^2)(2m)^{-1/2} [E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2} dr$$

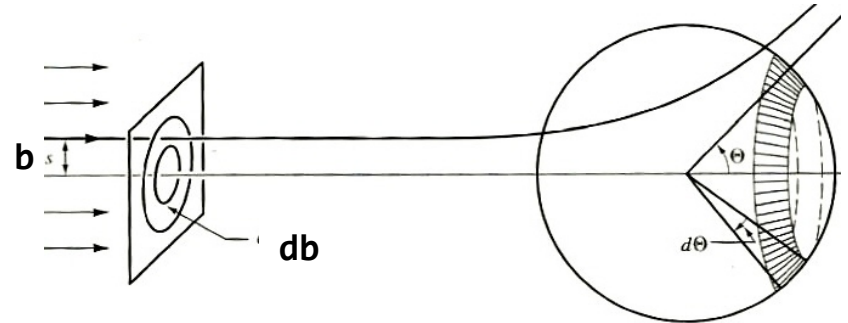


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- Orbit eqtn. **General central force:**

$$\theta(r) = \int (\ell/r^2)(2m)^{-1/2} [E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2} dr \quad (3)$$

- Consider purely **repulsive scattering**. See figure:

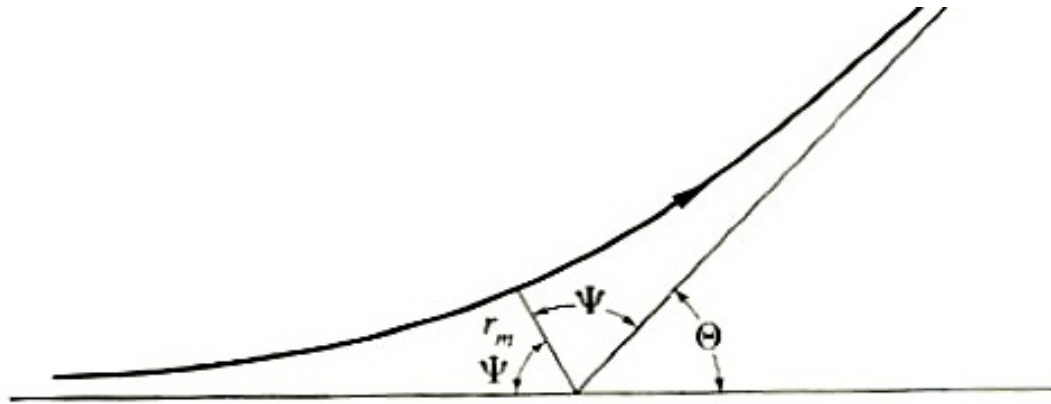


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- Closest approach distance** $\equiv \mathbf{r}_m$. Orbit must be symmetric about $\mathbf{r}_m \Rightarrow$ Sufficient to look at angle Ψ (see figure): Scattering angle $\Theta \equiv \pi - 2\Psi$. Also, orbit angle

$$\theta = \pi - \Psi \text{ in the special case } r = r_m$$

⇒ After some **manipulation** can write (3) as:

$$\Psi = \int (dr/r^2) [(2mE)/(\ell^2) - (2mV(r))/(\ell^2) - 1/(r^2)]^{-1/2} \quad (4)$$

- Integrate from r_m to $r \rightarrow \infty$
- Angular momentum in terms of impact parameter

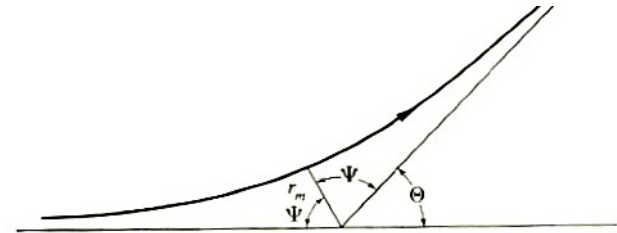


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

b & energy **E**: $\ell \equiv mv_0 b \equiv b(2mE)^{1/2}$. Put this into (4) & get for scattering angle Θ (after **manipulation**):

$$\Theta(b) = \pi - 2 \int dr (b/r) [r^2 \{1 - V(r)/E\} - b^2]^{-1/2} \quad (4')$$

Changing integration variables to $u = 1/r$:

$$\Theta(b) = \pi - 2 \int b du [1 - V(r)/E - b^2 u^2]^{-1/2} \quad (4'')$$

- Integrate from $u = 0$ to $u = u_m = 1/r_m$

- **Summary:** Scattering by a *general central force*:
- **Scattering angle** $\Theta = \Theta(b, E)$ (b = impact parameter, E = energy):

$$\Theta(b) = \pi - 2 \int b du [1 - V(r)/E - b^2 u^2]^{-1/2} \quad (4'')$$

Integrate from $u = 0$ to $u = u_m = 1/r_m$

- **Scattering cross section:**

$$\sigma(\Theta) = (b/\sin\Theta) (|db|/|d\Theta|) \quad (2)$$

- **Recipe**: To solve a scattering problem:
 1. Given force $\mathbf{f}(\mathbf{r})$, compute potential $V(\mathbf{r})$.
 2. Compute $\Theta(b)$ using (4'').
 3. Compute $\sigma(\Theta)$ using (2).

Coulomb Scattering

- *Special case: Scattering by r^{-2} repulsive forces:*
 - **For this case**, as well as for others where the orbit eqn $\mathbf{r} = \mathbf{r}(\theta)$ is known analytically, instead of applying (4'') directly to get $\Theta(\mathbf{b})$ & then computing $\sigma(\Theta)$ using (2), we make use of the known expression for $\mathbf{r} = \mathbf{r}(\theta)$ to get $\Theta(\mathbf{b})$ & then use (2) to get $\sigma(\Theta)$.
- Repulsive scattering by r^{-2} repulsive forces =
Coulomb scattering by like charges
 - Charge $+Ze$ scatters from the center of force, charge $Z'e$
$$\Rightarrow \mathbf{f}(\mathbf{r}) \equiv (ZZ'e^2)/(\mathbf{r}^2) \equiv -\mathbf{k}/\mathbf{r}^2$$
 - Gaussian E&M units! Not SI! For SI, multiply by $(1/4\pi\epsilon_0)!$
$$\Rightarrow \text{For } \mathbf{r}(\theta), \text{ in the previous formalism for } \mathbf{r}^{-2} \text{ attractive forces, make the replacement } \mathbf{k} \rightarrow -ZZ'e^2$$

- **Like charges:** $f(r) \equiv (ZZ'e^2)/(r^2)$
 $\Rightarrow k \rightarrow -ZZ'e^2$ in orbit eqtn $r = r(\theta)$
- Orbit eqtn for r^{-2} force is a conic section:

$$[\alpha/r(\theta)] = 1 + \varepsilon \cos(\theta - \theta') \quad (1)$$

With: **Eccentricity** $\equiv \varepsilon = [1 + \{2E\ell^2/(mk^2)\}]^{1/2}$ &

$2\alpha = [2\ell^2/(mk)]$. Eccentricity = ε to avoid confusion with electric charge e . $E > 0 \Rightarrow \varepsilon > 1 \Rightarrow$ **Orbit is a hyperbola**, by previous discussion.

- Choose the integration const $\theta' = \pi$ so that $r_{\min}$ is at $\theta = 0$
- Make the changes in notation noted:

$$\Rightarrow [1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon \cos\theta - 1) \quad (2)$$

$$f(r) = (ZZ'e^2)/(r^2)$$

$$[1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon \cos\theta - 1) \quad (2)$$

- **Hyperbolic orbit.**

- With change of notation, eccentricity is

$$\varepsilon = [1 + \{2E\ell^2/(mZ^2Z'^2e^4)\}]^{1/2}$$

- Using the relation between angular momentum, energy, & impact parameter, $\ell^2 = 2mEb^2$ this is:

$$\varepsilon = [1 + (2Eb)^2/(ZZ'e^2)^2]^{1/2}$$

$$[1/r(\theta, b)] = [(mZZ'e^2)/(\ell^2)](\epsilon \cos \theta - 1) \quad (2)$$

$$\epsilon = [1 + (2Eb)^2/(ZZ'e^2)^2]^{1/2} \quad (3)$$

- From (2) get $\theta(r, b)$. Then, use relations between orbit angle θ scattering angle Θ , & auxillary angle Ψ in the scattering problem, to get $\Theta = \Theta(b)$ & thus **the scattering cross section**.

$$\Theta = \pi - 2\Psi$$

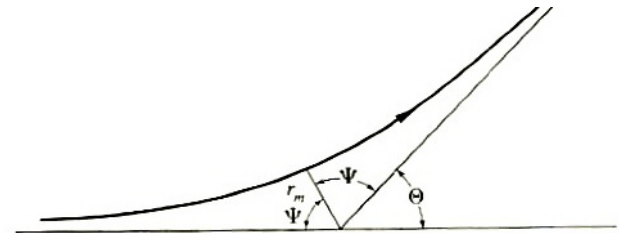


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- Ψ = direction of incoming asymptote. Determined by $r \rightarrow \infty$ in (2) $\Rightarrow \cos \Psi = (1/\epsilon)$.
- In terms of Θ this is: $\sin(1/2\Theta) = (1/\epsilon)$. (4)

$$\varepsilon = [1 + (2Eb)^2/(ZZ'e^2)^2]^{1/2} \quad (3)$$

$$\sin(1/2\Theta) = (1/\varepsilon) \quad (4)$$

- Put (4) into left side of (3) and solve for b:

$$\Rightarrow \quad \mathbf{b} = \mathbf{b}(\Theta, \mathbf{E}) = (ZZ'e^2)/(2E) \cot(1/2\Theta) \quad (7)$$

(7), the impact parameter as function of Θ & $\mathbf{E}$ for Coulomb scattering is an **important result!**

$$b = b(\Theta, E) = (ZZ'e^2)/(2E) \cot(1/2\Theta) \quad (7)$$

- Now, use (7) to compute the **Differential Scattering Cross Section for Coulomb Scattering.**

- We had: $\sigma(\Theta) = (b/\sin\Theta) (|db|/|d\Theta|)$ (8)

(7) & (8) (after using trig identities):

$$\Rightarrow \sigma(\Theta) = (1/4)[(ZZ'e^2)/(2E)]^2(1/\sin^4(1/2\Theta)) \quad (9)$$

(9) $\equiv$ *The Rutherford Scattering Cross Section*

- Get the same results in a (non-relativistic) QM derivation!
- Valid for spin 0 particles

Multiple Scattering

- Il fatto che il nucleo domini la diffusione angolare si vede anche dal fatto che la sez d'urto sul nucleo va come Z^2e^4
- e come e^4 per i singoli e- atomici; se ci sono Z elettroni, la sez d'urto sugli e- va come Ze^4
- quindi la diffusione sul nucleo e' Z volte quella degli elettroni

Single Scattering

- Individual collisions governed by Rutherford scattering formula:

$$\frac{d\sigma}{d\Omega} = z^2 Z^2 r_e^2 \left(\frac{m_e c}{\beta p} \right)^2 \frac{1}{4 \sin^4(\theta/2)} \quad **$$

- Does not take into account spin effects or screening
- Although single large angle scattering can occur for very small impact parameter, probability that a single interaction will scatter through a significant angle is very small due to $1/\sin^4(\theta/2)$ dependence.
- For large impact parameter (much more probable), scattering angle is further reduced w.r.t. Rutherford formula due to partial screening of nuclear charge by atomic electrons.
- **NB: the form is equivalent to the previous one, provided that $Mv^2 = cp\beta$, M is the mass of incident particle and $r_e = e^2/m_e c$ is the classical electron radius

Scattering elastico

Nel limite di piccoli angoli $\theta \ll 1$, $\sin\theta \sim \theta \rightarrow$ la sezione d'urto si riduce a

$$\frac{d\sigma}{d\Omega} \approx 4z^2 Z^2 r_e^2 \left(\frac{m_e c}{\beta p} \right)^2 \frac{1}{\theta^4}$$

Questa formula ci dice che la sezione d'urto diverge a piccolo angolo.

Ma esiste un θ minimo ed un θ massimo

Scattering elastico: quick calc per $\theta \ll 1$

La sezione d'urto e' = all'elemento di area trasversa $d\sigma = 2\pi b db$

Il momento scambiato durante l'interazione e'

$P_t = 2zZe^2/bv$, perpendicolare all'impulso iniziale

La deflessione che la particella subisce e'

$\tan \theta \approx \theta = p_t/p$ per piccoli angoli

$\theta \approx 2zZe^2/bvp$ ovvero $b = (2zZe^2/vp) (1/\theta)$

La sezione d'urto infinitesima e'

$d\sigma = 2(2zZe^2/vp)^2 (d\theta/\theta^3)$

Perciò $d\sigma/d\Omega = 2\pi (2zZe^2/vp)^2 (1/\theta^3) (d\theta/d\Omega)$

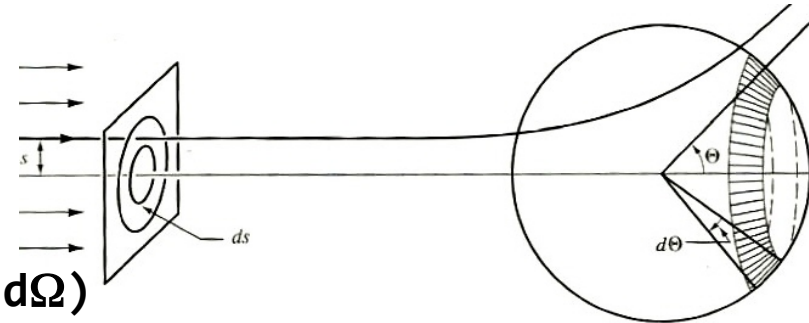


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$d\Omega = 2\pi \sin(\theta) d\theta \sim 2\pi \theta d\theta$ (per piccoli angoli) $\rightarrow d\theta/d\Omega = 1/2\pi\theta$

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{vp} \right)^2 \frac{1}{\theta^4}$$

Calcolo classico, ma viene esattamente lo stesso risultato in meccanica quantistica.

NB: diverge a piccoli angoli, ma l'approx si applica fra un angolo minimo e uno massimo di deflessione

Scattering elastico: angolo min e max

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{vp} \right)^2 \frac{1}{\theta^4}$$

X-section diverges for $\theta \rightarrow 0$ because Coulomb interaction is long range for an single target-projectile pair.

But in a medium, also atomic electrons are usually present that screen nuclear charge for distances r (that is impact parameter b) $>$ atomic radius a and the x-section remains finite for $\theta = 0$

The simplest classical approximation is to truncate the coulomb potential at $r = a \rightarrow$ The scattering angle reaches a minimum, $\theta_{\min}$, and the x-section tends to a constant value for diffusion angles below a minimum deflection $\theta_{\min}$

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{\beta p} \right)^2 \frac{1}{(\theta^2 + \theta_{\min}^2)^2}$$

$\theta_{\min}$ is obtained from $\theta \approx 2zZe^2/bvp$ with $b = a$
 $\theta_{\min} = 2zZe^2/vpa$

In terms of impact parameter, b is max (cioe' θ e' min) when $b_{\max} = R_{\text{atomo}}$

- $b_{\max} = a_0 = r_e/\alpha^2$ for hydrogen
- $b_{\max} = r_a \sim [(h/2\pi\alpha mc)] Z^{-1/3}$ for materials heavier than H (Fermi-Thomas model)

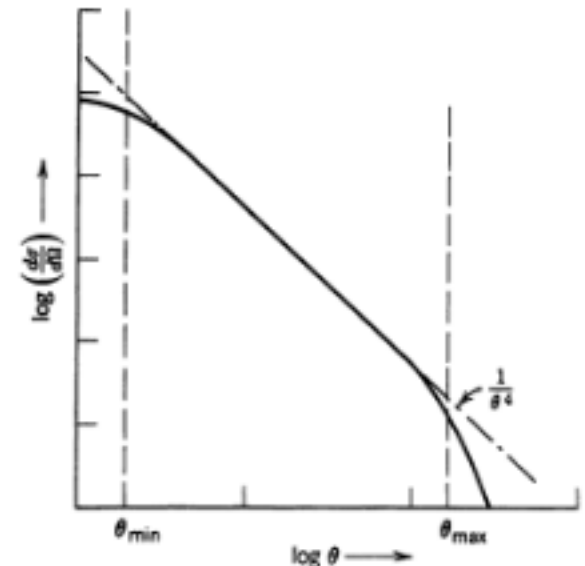
Scattering elastico: angolo min e max

Quantisticamente, affinche' lo scattering possa avvenire, la posizione della particella incidente deve essere localizzata entro $\delta x \sim R_{\text{atomo}}$ per avere una probabilita' apprezzabile di essere deviata.

Allora $\Delta p \approx h/(2\pi r_{\text{atomo}})$ e $\Delta \theta \approx \Delta p/p$ e quindi

$$\theta_{\min} \approx h/(2\pi r_{\text{atomo}} p) = (Z^{1/3}/192) (mc/p)$$

Dei due minimi, classico e quantistico, si sceglie il piu' grande (analogamente al caso della BB per la scelta del $b_{\min}$). Usualmente e' quello quantistico.



Scattering elastico: angolo min e max

Abbiamo anche un $\theta_{\max}$ (che corrisponde a $b_{\min}$).

Ad angoli grandi rispetto a $\theta_{\min}$ (ma piccoli comunque in assoluto), la sezione 'urto devia dal valore previsto da Rutherford a causa della dimensione finita del nucleo.

Le cause sono diverse:

- per i leptoni e, μ, τ , l'influenza delle dimensioni nucleari e' di natura puramente elettromagnetica,

- per gli adroni ($\pi, K, p, \alpha, \dots$) sono in genere presenti reazioni anelastiche dovute alle interazioni nucleari forti.

L'effetto netto complessivo e' quello di ridurre la la sezione d'urto rispetto a quella elastica di Rutherford.

Ci limitiamo a considerare solo l'aspetto elettromagnetico, non quello delle interazioni nucleari

Scattering elastico: angolo min e max

Abbiamo anche un $\theta_{\max}$ (che corrisponde a $b_{\min}$) ($\theta \approx 2zZe^2/bvp$).

Ad angoli grandi rispetto a $\theta_{\min}$ (ma piccoli comunque in assoluto), la sezione 'urto devia dal valore previsto da Rutherford a causa della dimensione finita del nucleo.

Quando $\lambda = h/p \sim R_{\text{nucleo}}$, la diffusione diventa diffrattiva e confinata entro un angolo $\theta = \lambda/R = h/pR$

Alternativamente, con il principio di indetermin., $\Delta p = h/R$ e quindi $\theta \approx \Delta p/p = h/pR$

Una stima di R e' $R_n \sim 1.4 \times 10^{-15} A^{1/3}$

$$\Rightarrow \theta_{\max} \cong \frac{274}{A^{1/3}} \frac{mc}{p}$$

Si noti che $1 \gg \theta_{\max} \gg \theta_{\min}$ per tutti valori fisici di A e Z

$\theta_{\max}/\theta_{\min} = 274 \times 192/(AZ)^{1/3} \approx 2000$
per l'Uranio

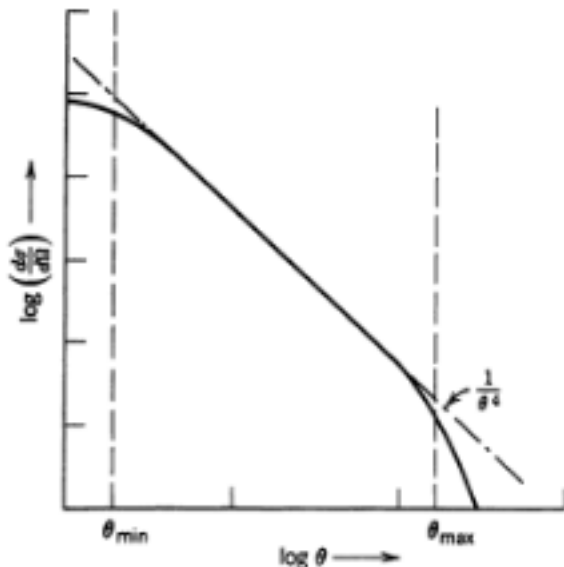


Figure 13.6 Atomic scattering, including effects of electronic screening at small angles and finite nuclear size at large angles.

Scattering elastico: sezione d'urto totale

La sezione d'urto totale e' finita e puo' essere ottenuta integrando sull'angolo solido quella differenziale

$$\Rightarrow \sigma = \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\sigma}{d\Omega} 2\pi \vartheta d\theta \cong \left(\frac{2zZe^2}{\beta p} \right)^2 \frac{1}{\theta_{\min}^2}$$

$$\theta_{\min} \approx (Z^{1/3}/192) (mc/p)$$

$$\Rightarrow \sigma = \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\sigma}{d\Omega} 2\pi \vartheta d\theta \cong \pi \left(\frac{2zZe^2}{\beta} \right)^2 \left(\frac{192}{Z^{1/3}} \right)^2 \frac{1}{(mc)^2}$$

La sezione d'urto totale di deflessione per singolo urto decresce aumentando β .

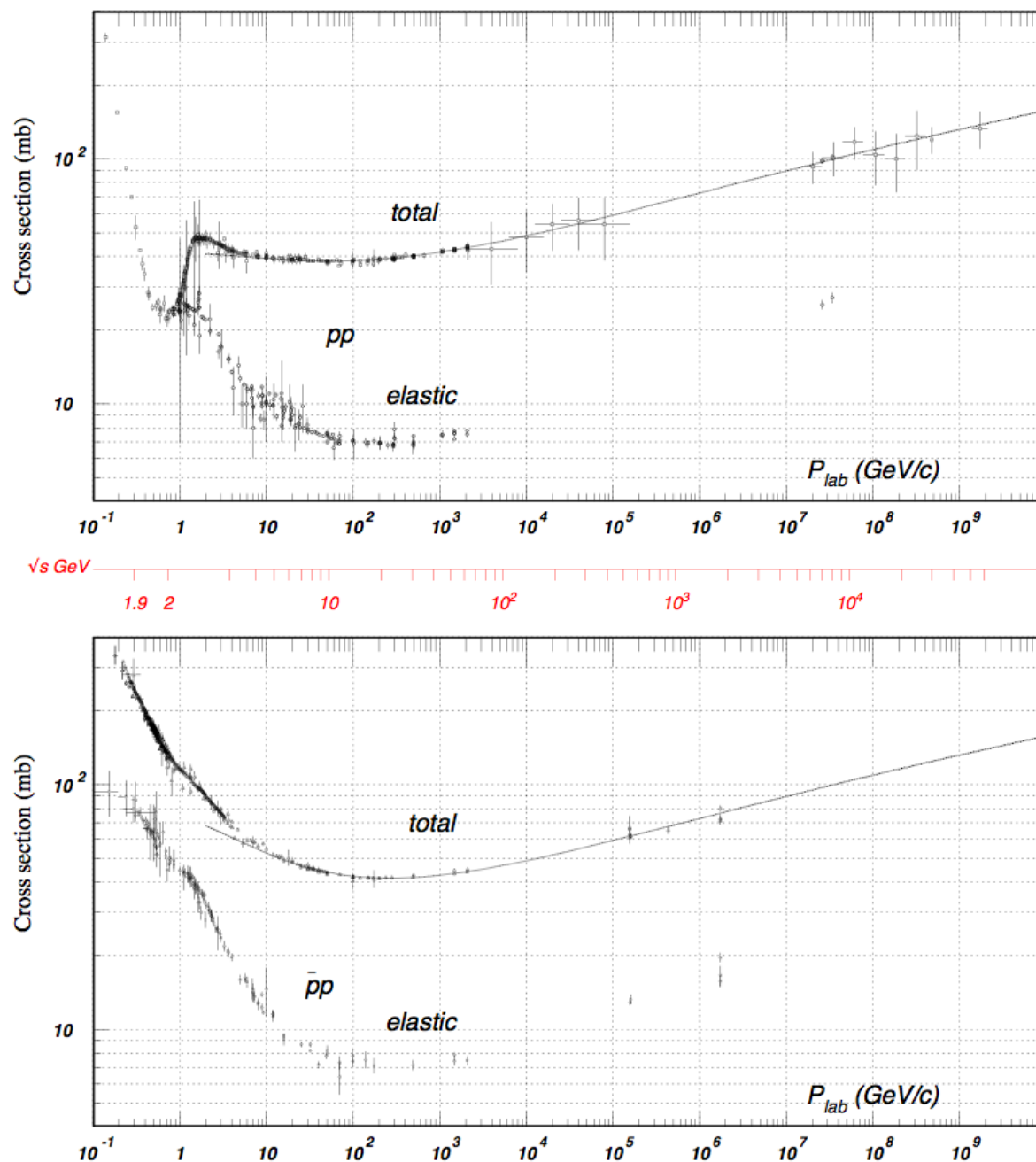


Figure 49.9D11: Total and elastic cross sections for pp and $\bar{p}p$ collisions as a function of laboratory beam momentum and total center-of-mass energy. Corresponding data files may be found at