Particle Detectors

Lecture 4
15/03/17

a.a. 2016-2018

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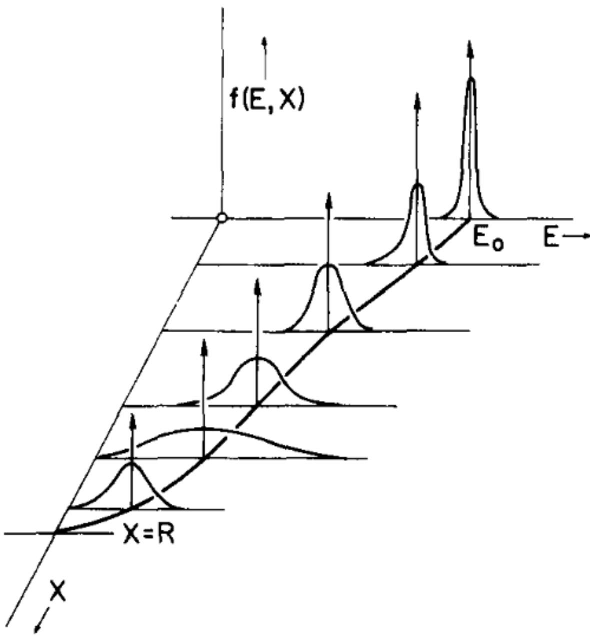
Energy loss fluctuations

We learned that as a beam of particles travels in a slab of material loses energy.
Because of statistical nature of the process, not all the particles have the same loss (eg different nbr of collisions and different energy loss in each collision) at the same depth: BB describes the **average energy loss**, not the actual loss in a single collision, that is $\langle \Delta E \rangle = \langle (dE/dX) \rangle \Delta X$ if medium is homogeneous.

Therefore, a spread in energies always results after a beam of monoenergetic charged particles has passed through a given thickness of absorber. The width of this energy distribution is a measure of energy straggling, which varies with the distance along the particle track.

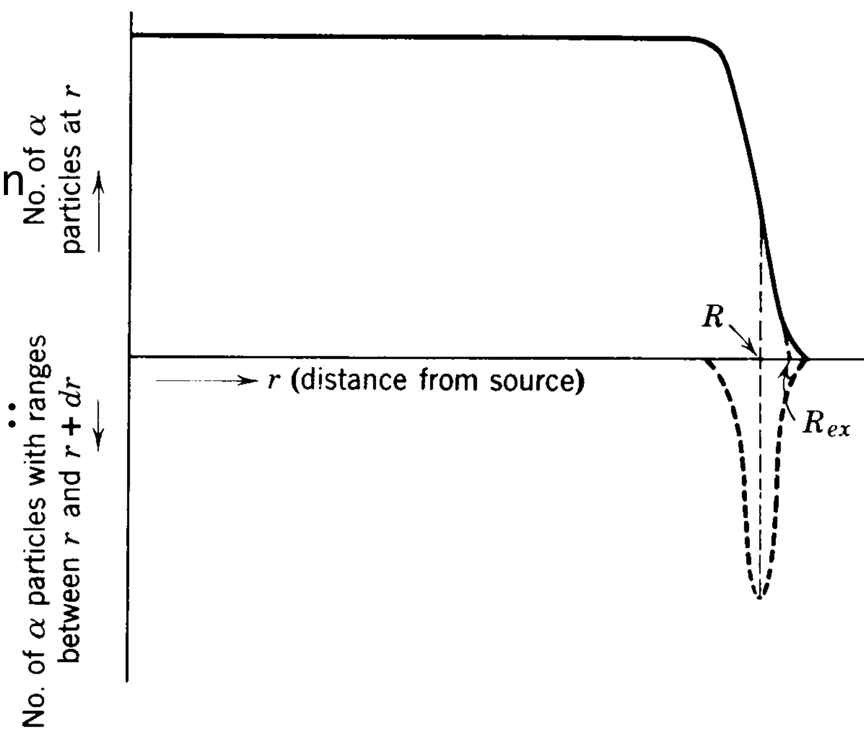
If the slab is thick enough, the particles will lose all their energy and will be stopped in the absorber. For the same reason as above, not all the particles with the same initial energy will travel the same distance in the absorber before stopping



Range

- Since particles loose energy in matter, a particle will travel a finite distance, called "range", before it loses all its energy.
- Experimentally, the range can be determined by passing a beam of particles through different thicknesses L and measuring the ratio **r** of transmitted to incident particles

- For small L, **r**=1: all (or practically all) the particles pass through.
- At the range R, r drops to 0
- The drop is not sharp, but slopes down over a certain spread of L, since the energy loss process is statistical. Particles with the same energy will travel different distances because of:
  - ✓ different nbr of collisions and
  - ✓ different energy loss in each collision.
- This is known as "**range straggling**"

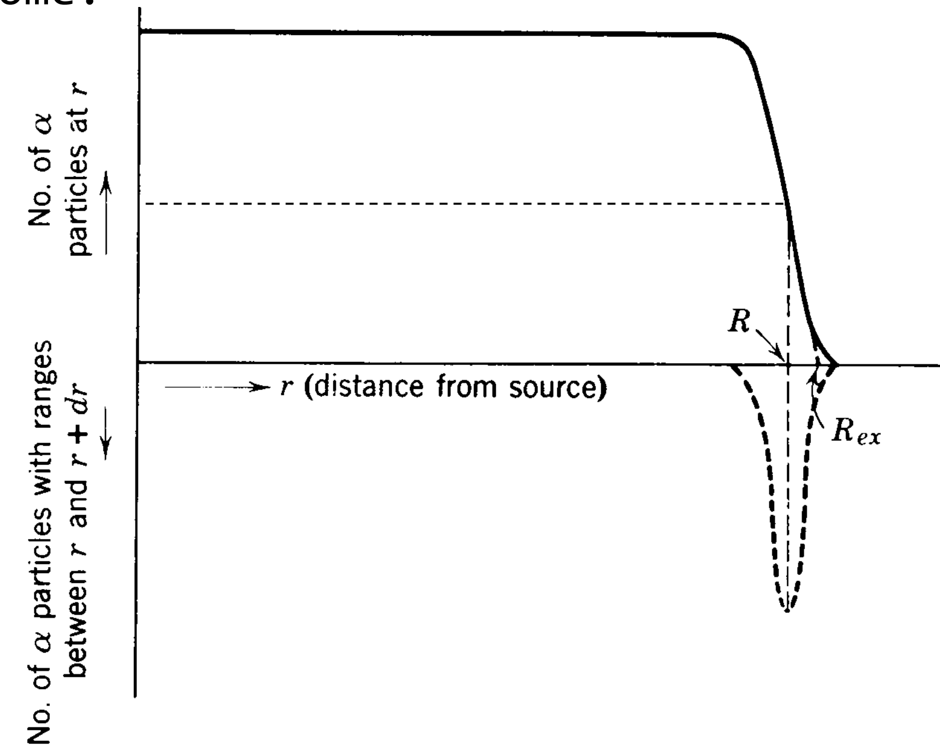


Range

Dato un fascio monoenergetico, **la profondita' alla quale le particelle sono ridotte alla meta' si chiama range medio.**

Formalmente il range e' definito come:

$$R(E) = \int_E^0 \frac{1}{dE/dx} dE$$



Il range rappresenta la distanza percorsa ed e' diversa dallo spessore a causa delle deviazioni che subisce negli urti. Si misura in gr/cm^2.
La dispersione intorno al range medio e' gaussiana. Il calcolo e' complicato. (Evans, Atomic Nucleus, pp. 600-667, available on the web, 1002 pages)

Distance a 10 MeV particle can cross

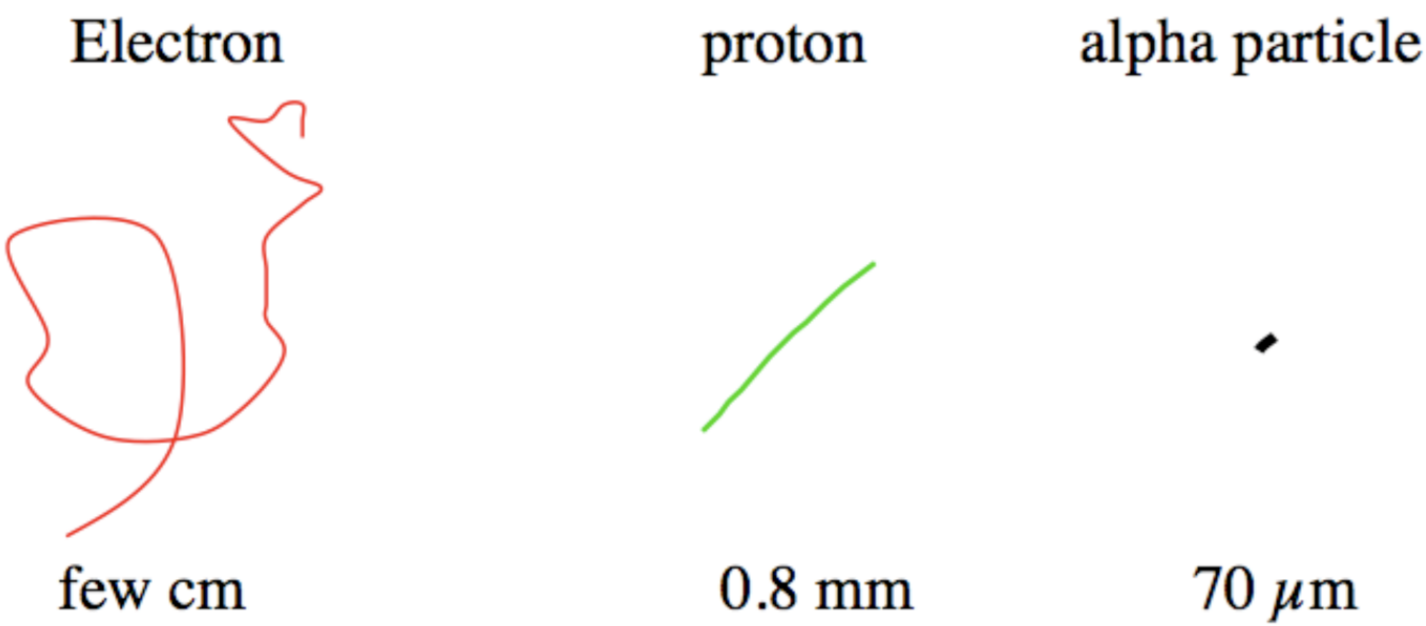


Figure 2.2.2: *A typical trajectory for an electron, a proton and an alpha particle of 10 MeV in silicon. The electron trajectory is drawn on a scale 10 times smaller than the trajectory of the proton and the alpha particle.*

Note that the trajectory is not a straight line because of the collisions against nuclei, i.e. multiple scattering (later).

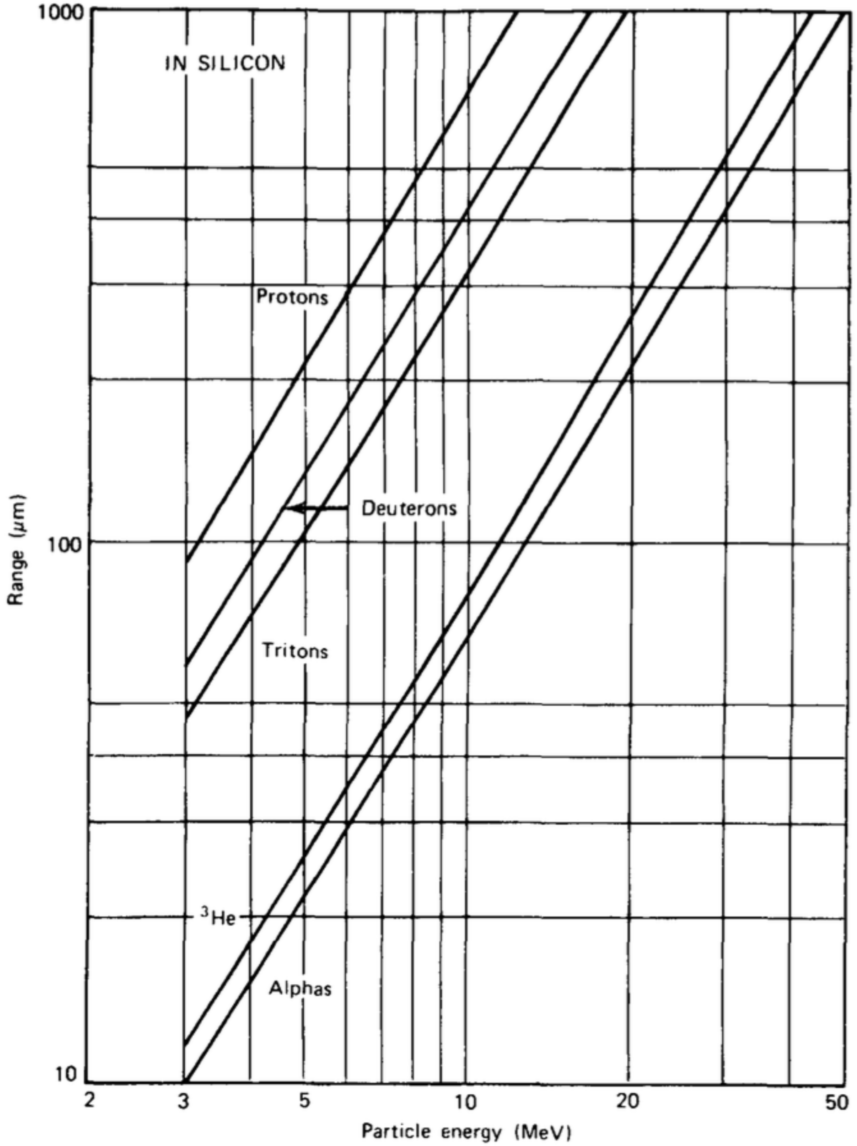


Figure 2.7 Range–energy curves calculated for different charged particles in silicon. The near-linear behavior of the log–log plot over the energy range shown suggests an empirical relation to the form $R = aE^b$, where the slope-related parameter b is not greatly different for the various particles. (From Skyrme.[4])

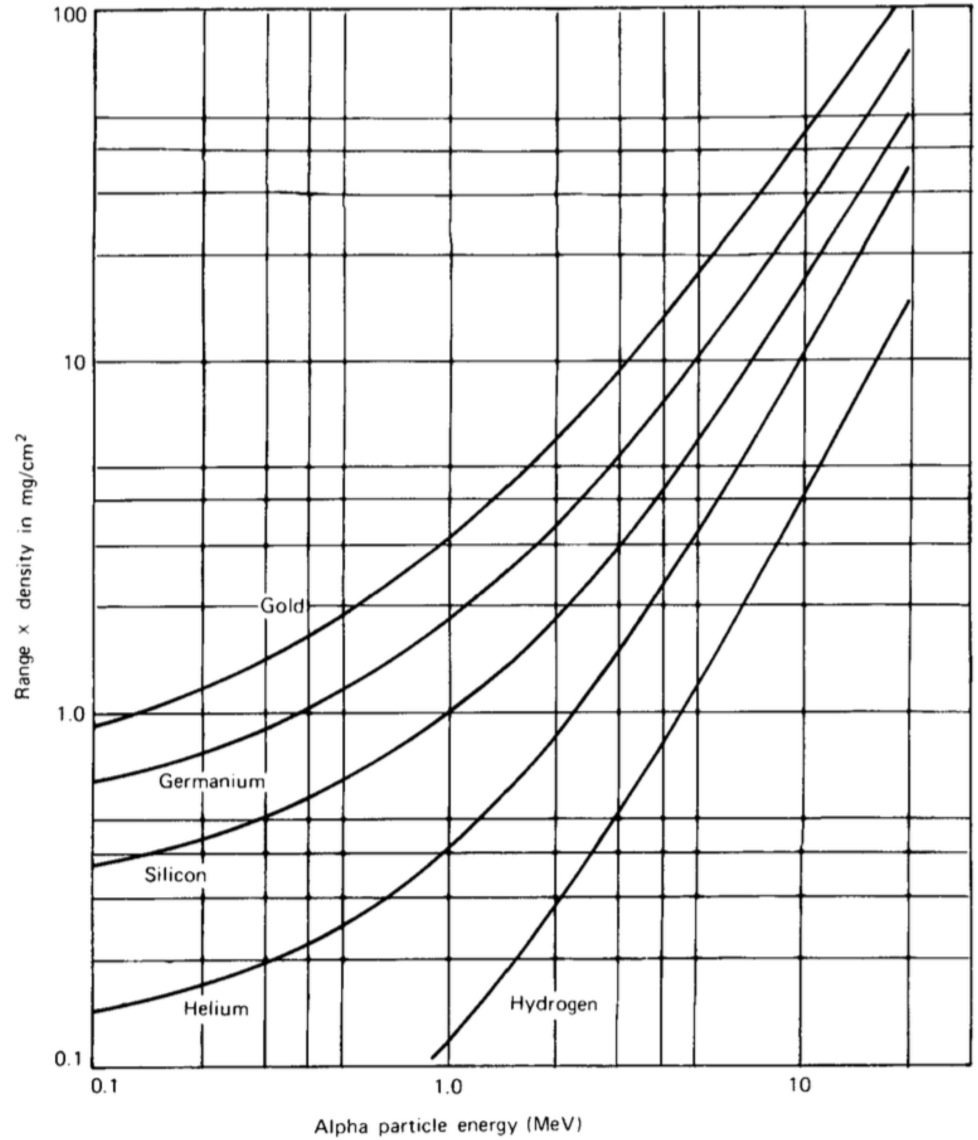


Figure 2.8 Range–energy curves calculated for alpha particles in different materials. Units of the range are given in mass thickness (see p. 54) to minimize the differences in these curves. (Data from Williamson et al.[5])

Range

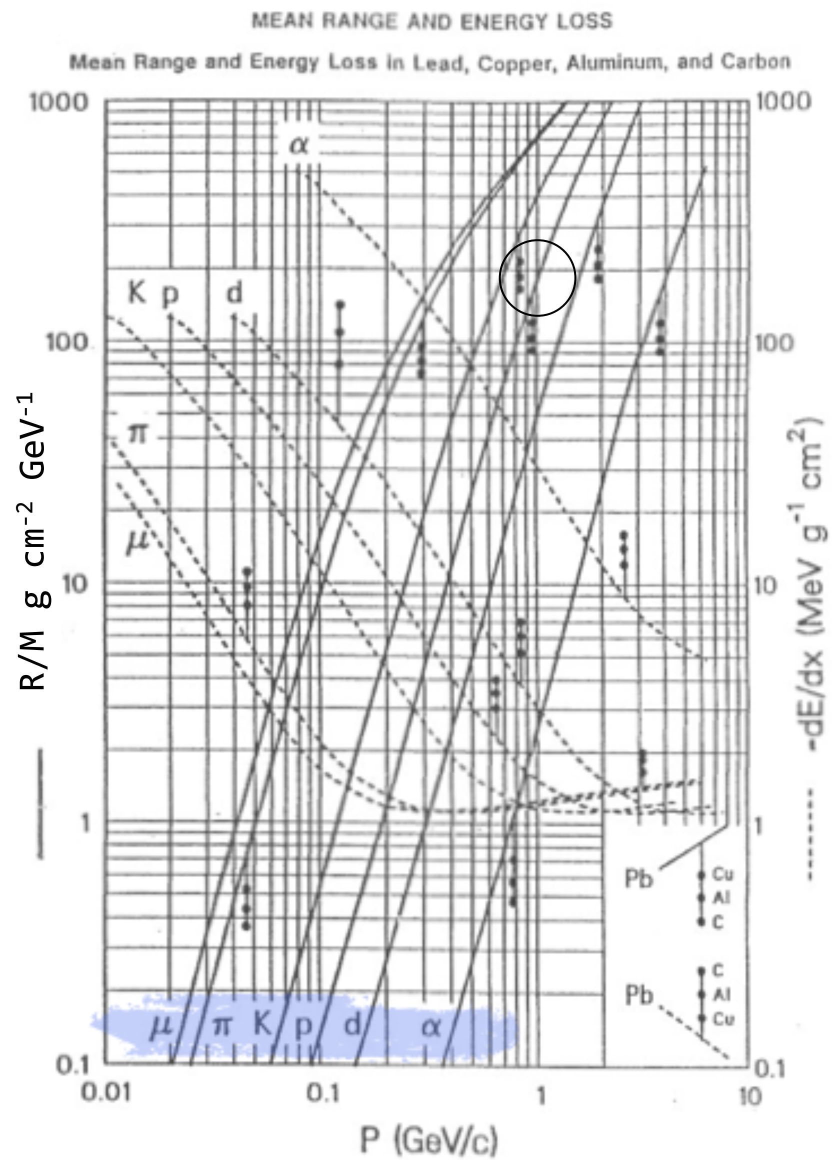
Integrate over energy loss
from E down to 0

$$R = \int_E^0 \frac{dE}{dE/dx}$$

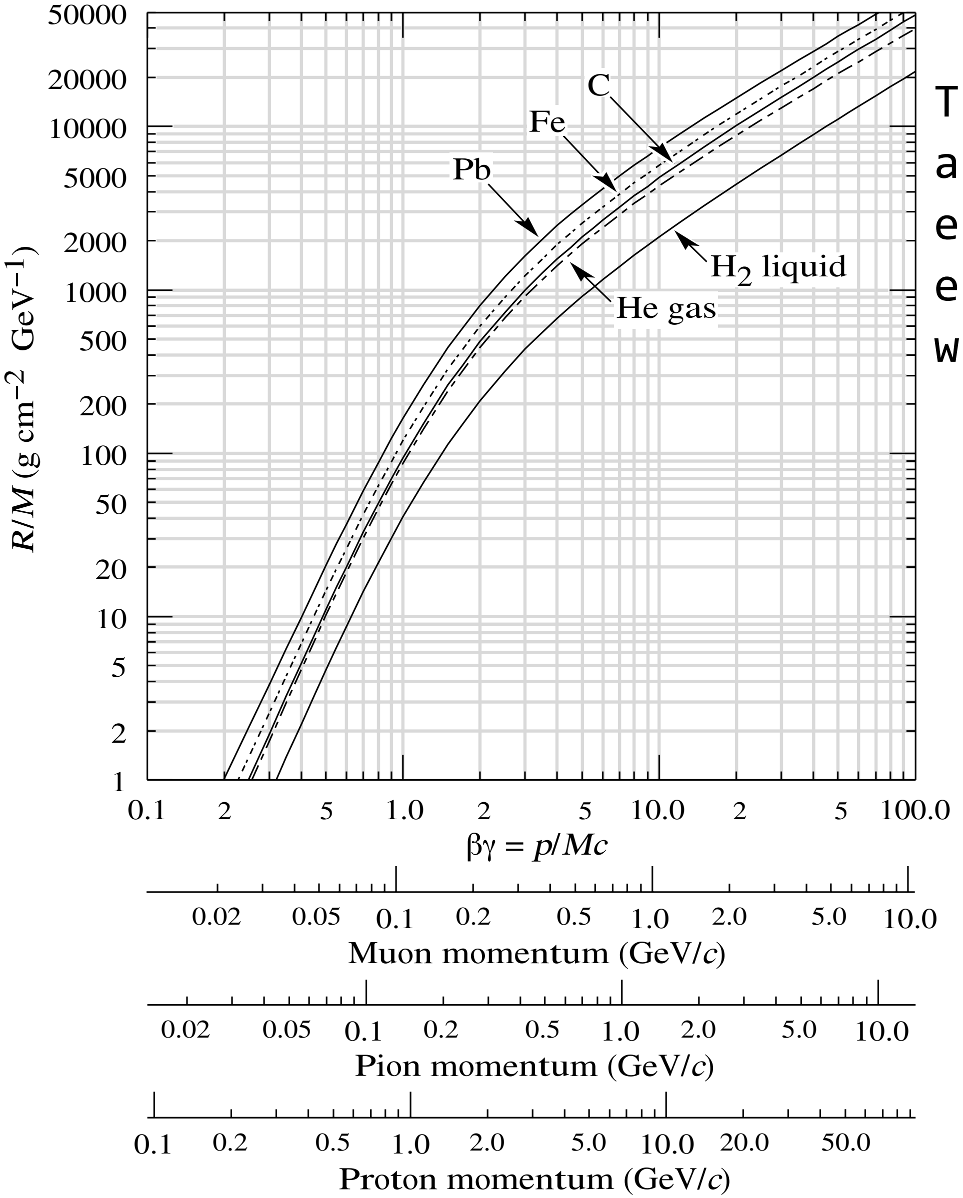
Example:

Proton with p = 1 GeV
Target: lead with ρ = 11.34 g/cm^3

R/M = 200 g cm^{-2} GeV^{-1}
→ R = 200/11.34/1 cm ~ 20 cm



Percorso delle particelle (Range)



The range-energy relationships are often expressed as $R(E)=(E/E_o)^n$ e.g. the range in meters of low energy protons can be approximated with $n=1.8$ and $E_o=9.3$ MeV.

$$R(E) \cong \left(\frac{E}{9.3}\right)^{1.8}$$

E in MeV, R in meters of air

Scaling laws

- Sometimes data are not available on the range or energy loss characteristics of precisely the same particle-absorber combination needed in a given experiment. Recourse must then be made to various approximations, most of which are derived based on the Bethe formula and on the assumption that the dE/dX per atom of compounds or mixtures is additive. This latter assumption, known as the Bragg-Kleeman rule, may be written

$$\frac{1}{N_c}\left(\frac{dE}{dx}\right)_c = \sum_i W_i \frac{1}{N_i}\left(\frac{dE}{dx}\right)_i$$

- In this expression, N is the atomic density, and *Wi* represents the atom fraction of the ith component in the compound C.
- As an example of the application of BB, the linear stopping power of alpha particles in a metallic oxide could be obtained from separate data on dE/dX in both the pure metal and in oxygen.
- Some caution should be used in applying such results, however, since some measurements for compounds have indicated a stopping power differing by as much as 10-20% from that calculated from BB.

dE/dx per composti e miscugli.

Una buona approssimazione della perdita di energia per composti e miscugli è data dalla regola di Bragg: una media pesata delle perdite di energia degli elementi i del composto M, pesate con la frazione di elettroni dell'elemento

$$\frac{1}{\rho}\frac{dE}{dx} = \frac{w_1}{\rho_1}\left(\frac{dE}{dx}\right)_1 + \frac{w_2}{\rho_2}\left(\frac{dE}{dx}\right)_2 + \cdots \qquad w_i = a_i \frac{A_i}{A_M} \qquad A_M = \sum_i a_i A_i$$

Dove a_1, a_2, … e' il nr. di elettroni nello i-esimo elemento del composto M e A_i e' il nr atomico dell'elemento

Possiamo definire dei valori efficaci come segue:

$$Z_{eff} = \sum a_i Z_i \qquad A_{eff} = \sum a_i A_i$$

$$\ln I_{eff} = \sum \frac{a_i Z_i \ln I_i}{Z_{eff}} \qquad \delta_{eff} = \sum \frac{a_i Z_i \delta_i}{Z_{eff}}$$

E riscrivere la dE/dx in termini dei valori efficaci.

dE/dX vs. depth

A plot of the specific energy loss along the track of a charged particle is known as a *Bragg* curve.

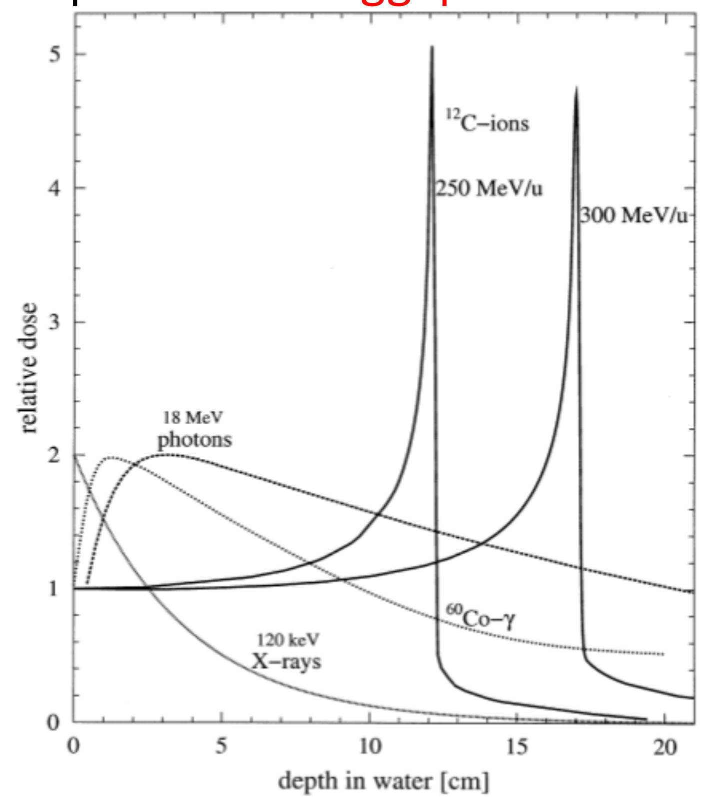
Until the particle is a MIP, its energy loss stays constant (more exactly it varies slowly –logarithmically-with $\beta\gamma$. Remember: $dE/dX \propto 1/\beta^2$ and $\beta \approx 1$ for MIPs)

As $\beta\gamma$ goes down below MIP, the E loss increases very rapidly because of $1/\beta^2$ dependence or as 1/T (kinetic E) as the particle becomes non relativistic and there is a peak in energy deposit : Bragg peak

$\beta\gamma > 3.5$: $\langle dE/dX \rangle \approx (dE/dX)_{min}$

$\beta\gamma < 3.5$: $\langle dE/dX \rangle >> (dE/dX)_{min}$

Near the end of the track, the charge is reduced through electron pickup and the curve falls off: particle becomes very slow and easily captures electrons from medium and becomes neutral before stopping.



dE/dX vs. depth

Particles with high charge begin to pick up electrons early in their slowing-down process.
Note that in an Al absorber, singly charged H ions (protons) show strong effects of charge pickup below about 100 keV, but doubly charged ^{3}He ions show equivalent effects at about 400 keV.

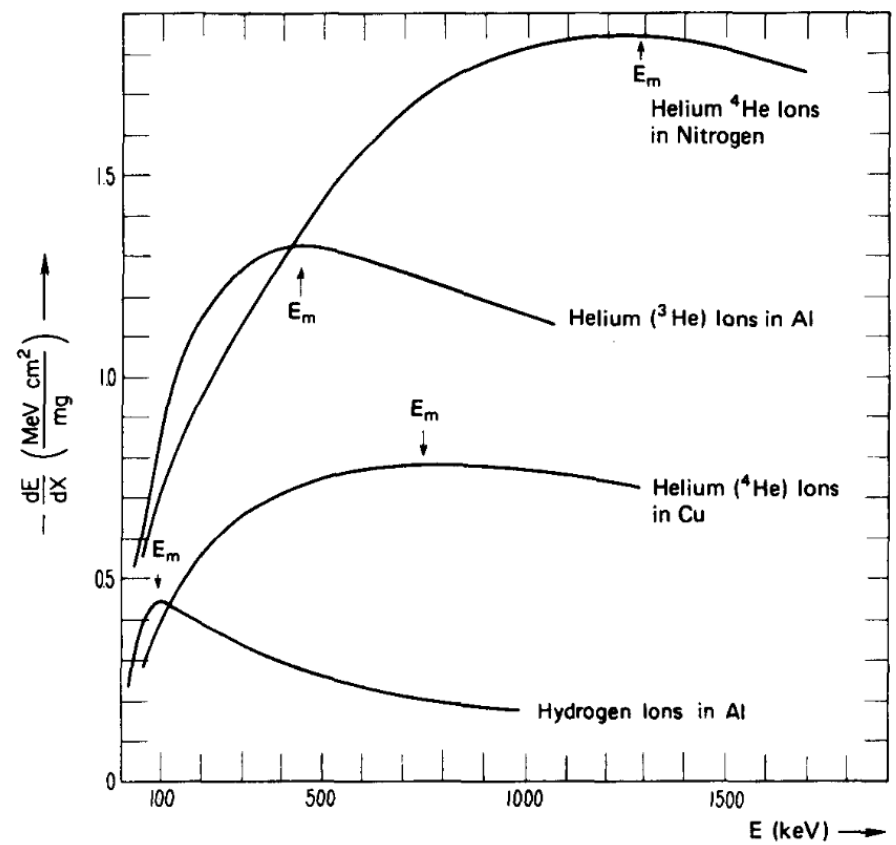


Figure 2.3 Specific energy loss as a function of energy for hydrogen and helium ions. E_m indicates the energy at which dE/dx is maximized. (From Wilken and Fritz.[3])

dE/dx phenomena

$$\frac{dE}{dx} = 4\pi N_A r_e^2 m_e c^2 \frac{z^2}{\beta^2}\frac{Z}{A}\left\{\frac{1}{2}\ln\left[\frac{2m_e c^2\beta^2\gamma^2 T_{\max}}{I^2}\right] - \beta^2 - \frac{\delta(\beta\gamma)}{2}\right\}$$

- Slow particles lose most of their energy in a short distance, since kinetic energy T ~ β^2

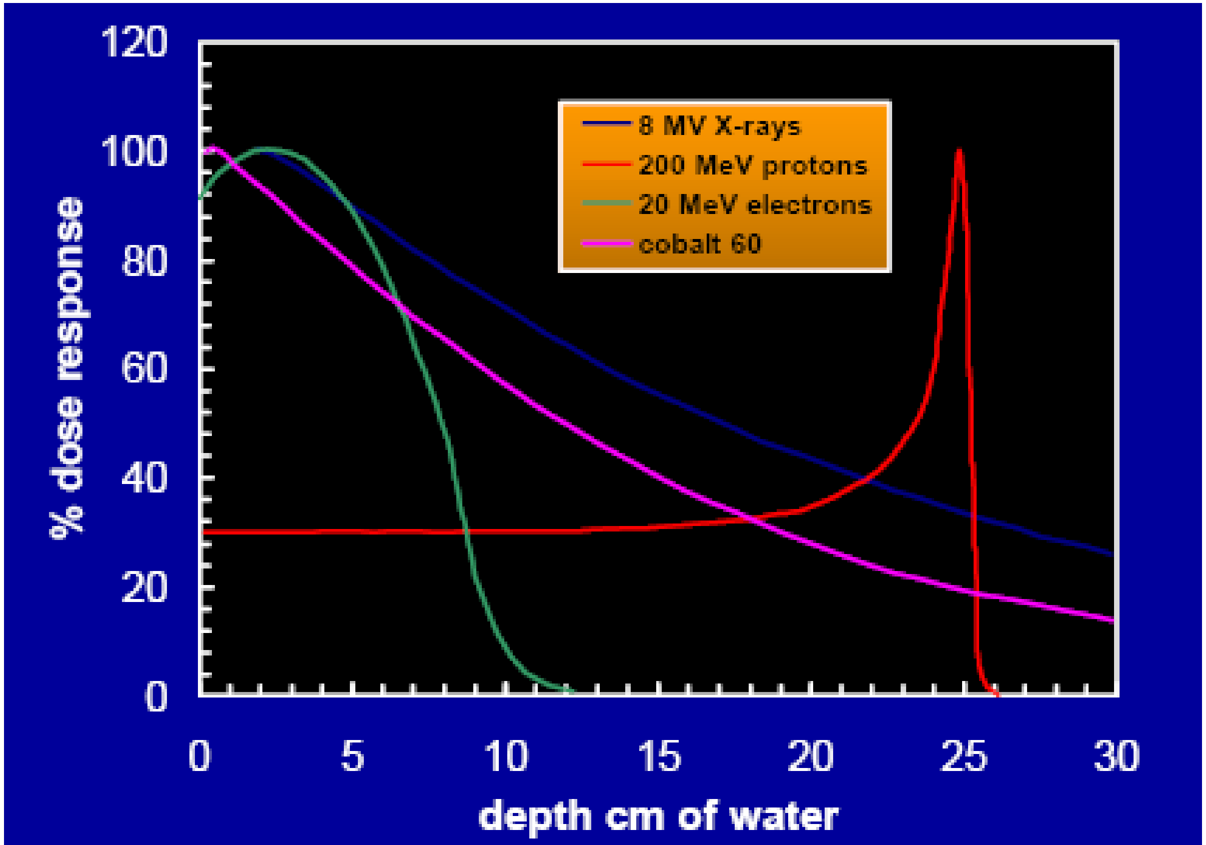
$$-\frac{dE}{dx} = -\frac{dT}{dx} = \left\langle\frac{dE}{dx}\right\rangle_{T_0}\frac{T_0}{T} \qquad \Delta X = -\int_{T_0}^{0}\frac{dT}{dT/dx} \qquad \Delta X = -\int_{T_0}^{0}\frac{TdT}{(dE/dx)_o T_0}$$

$$\Delta X = -\frac{1}{(dE/dx)_o T_0}\int_{T_0}^{0} T\,dT = -\frac{T_0}{2(dE/dx)_o} = \rho\Delta y$$

- For 30 MeV protons in water, <dE/dx> ~ 50 MeV cm^2/gm, so Δy ~ 0.3 cm

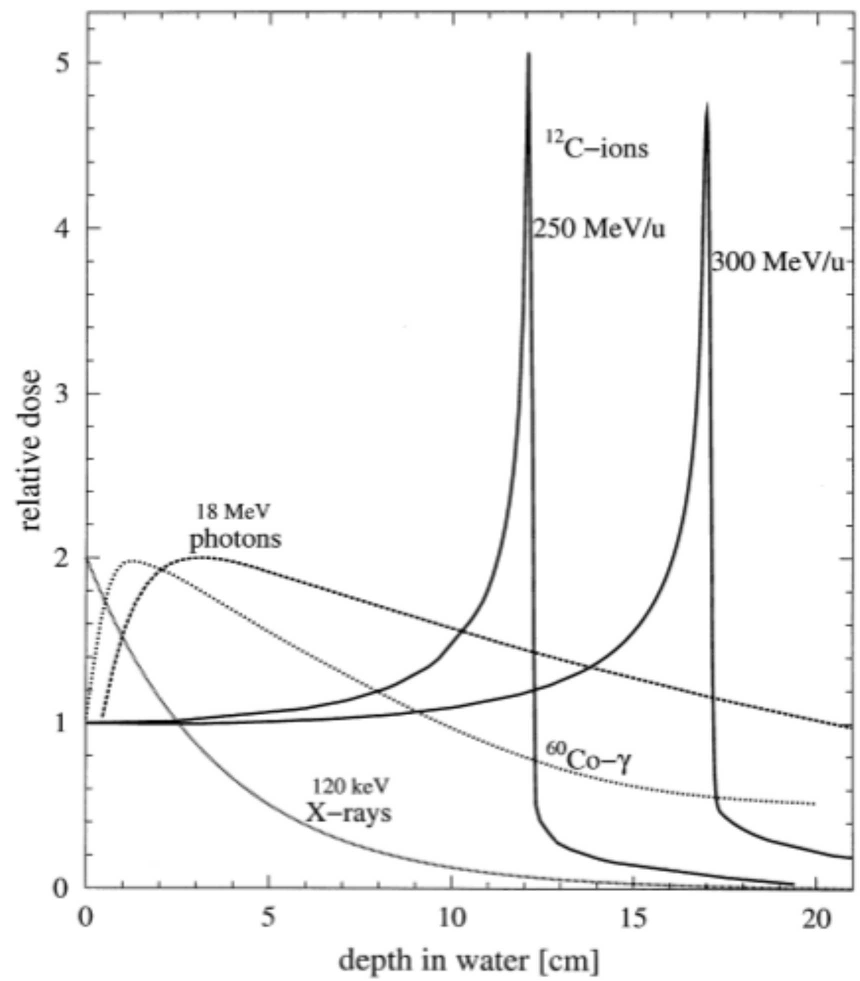
Application of Range

- The localized energy deposition of heavy charged particles can be useful therapeutically = proton radiation therapy



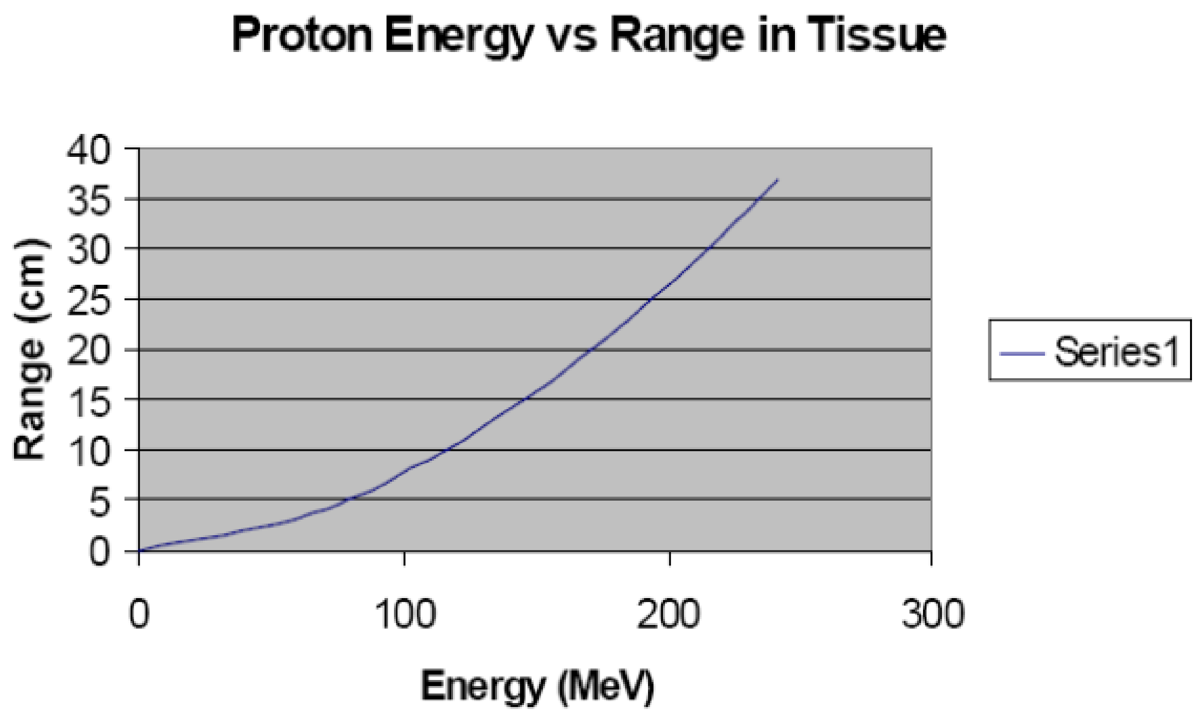
Application of Range

- Monoenergetic proton beam loses energy more rapidly as it slows down; gives sharp Bragg peak in ionization versus depth
- Using a range of proton energies allows a varied profile versus depth
- Photon beam (x-rays) deposits most energy near entrance into tissue
- Tumor therapy with hadrons (adro-therapy)



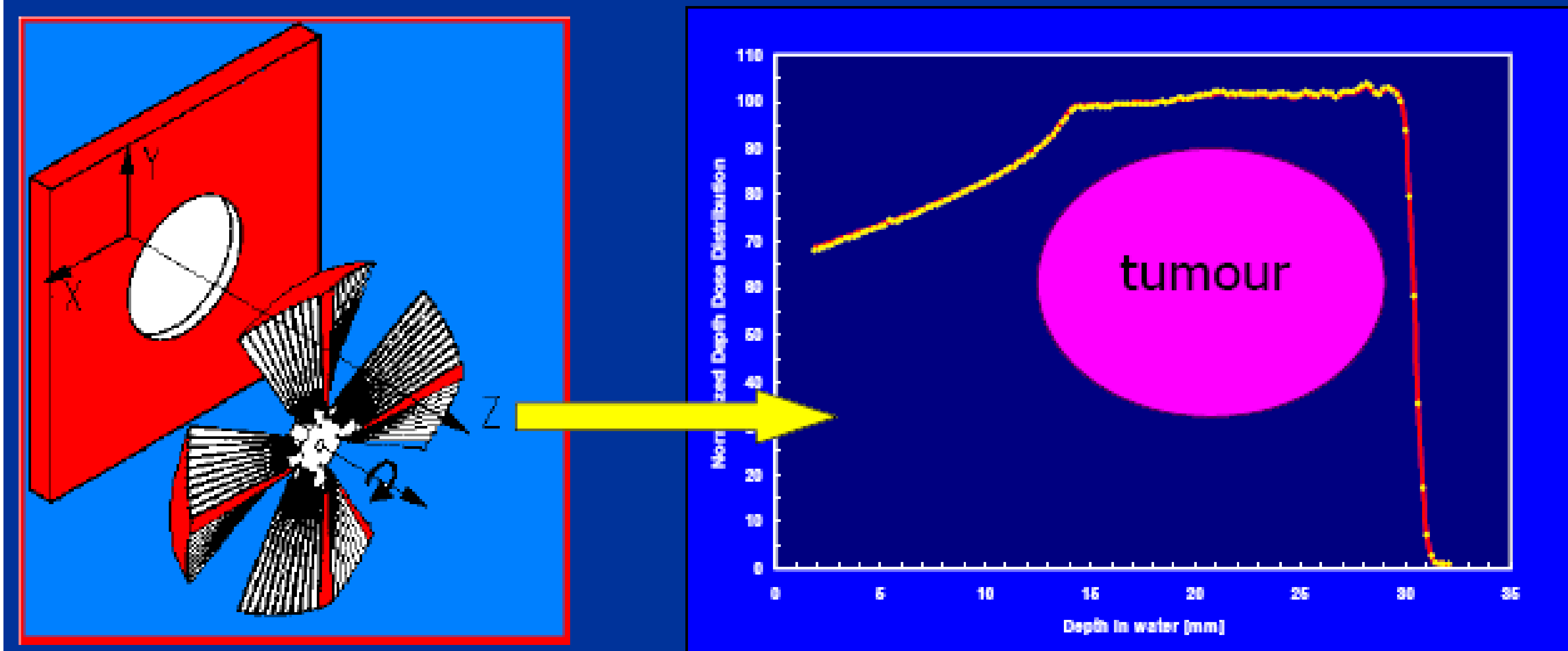
Proton Therapy

- Energy range of interest from 50 (eye) – 250 (prostate) MeV

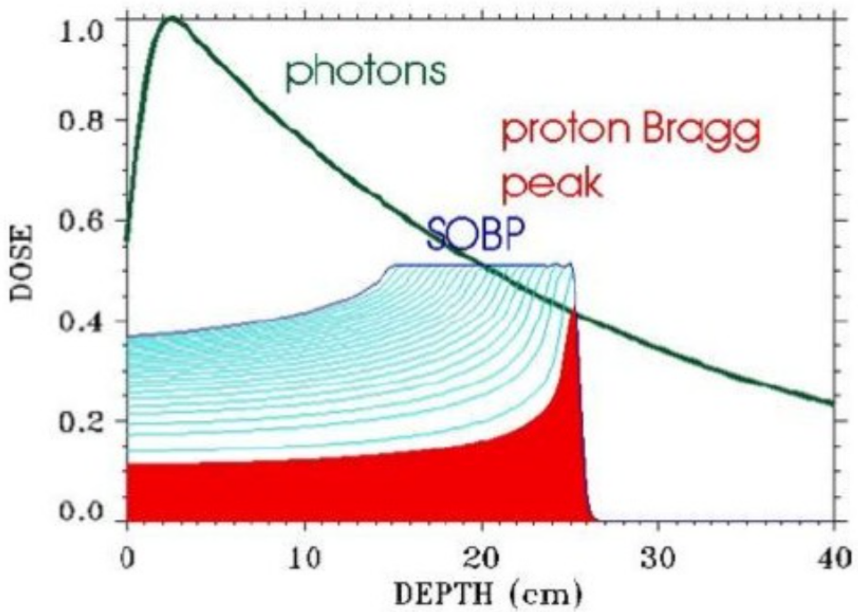


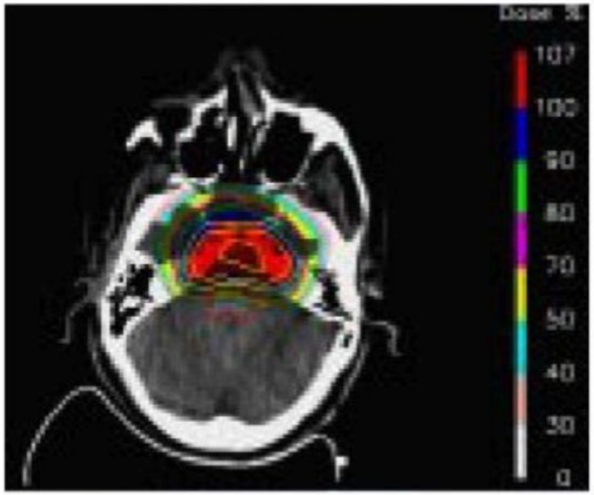
Proton Therapy

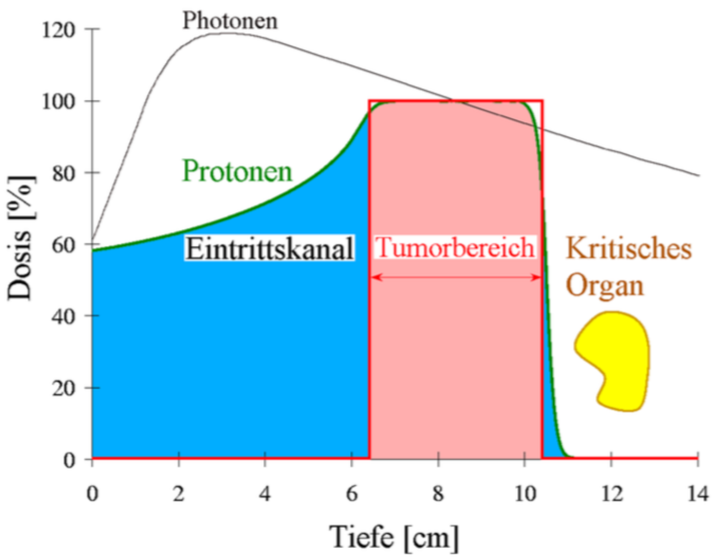
- Simulation of the Modulator Wheel to Obtain the Therapeutical Spread Out Bragg Peak



Proton therapy at PSI



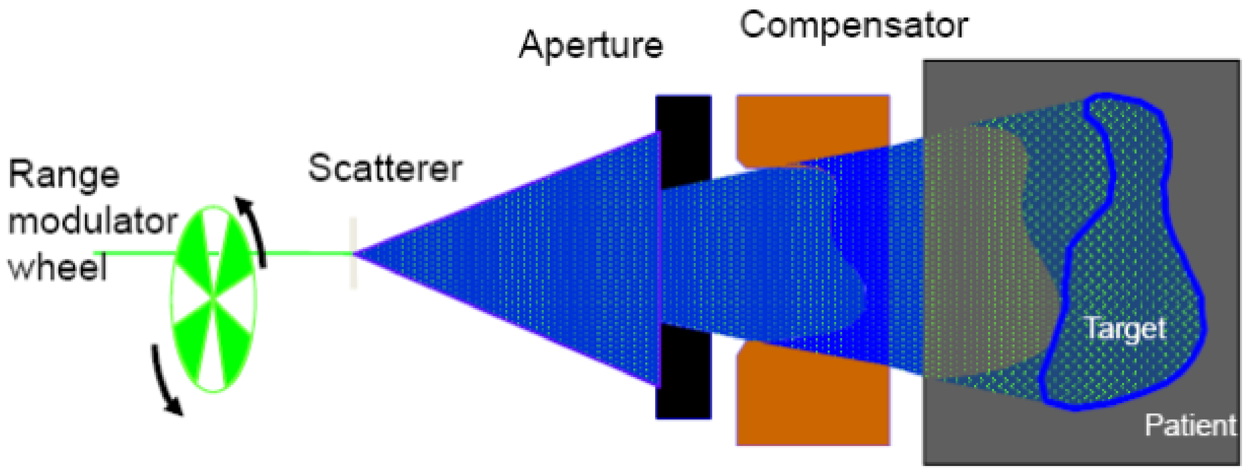




Spread out Bragg peaks (SOBP)
Different absorbers → almost constant dose in region of tumour

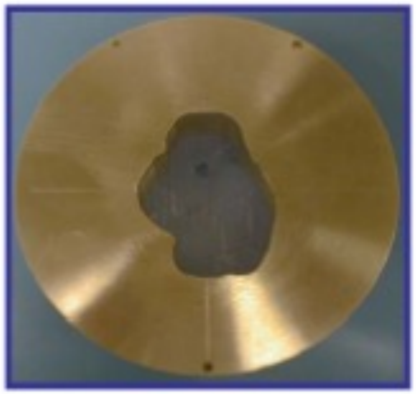
Proton Therapy

- Schematic apparatus for hadron-therapy

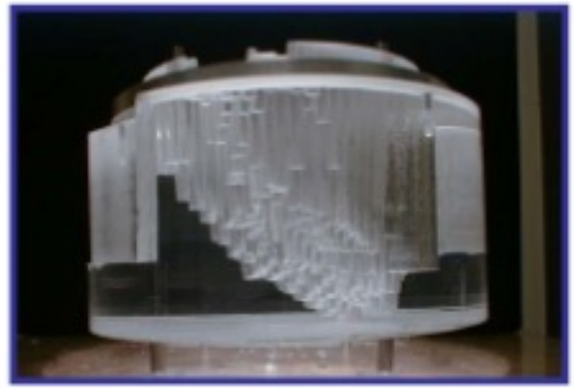


Proton Therapy

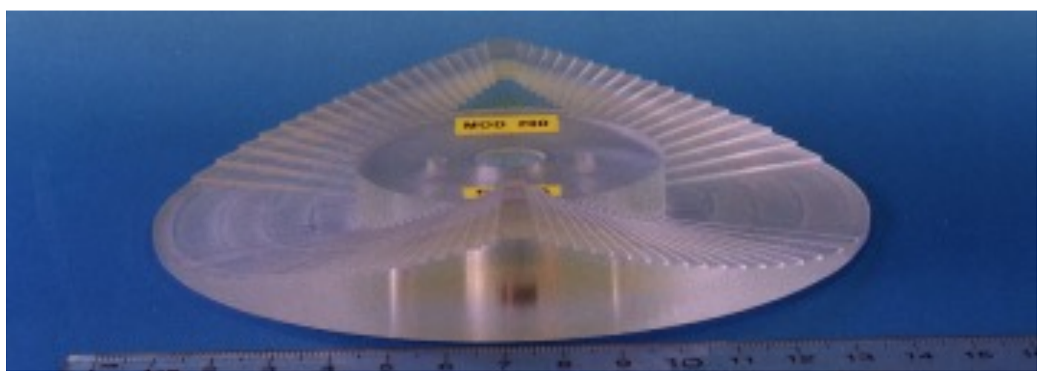
- Modulator, aperture, and compensator



Brass aperture

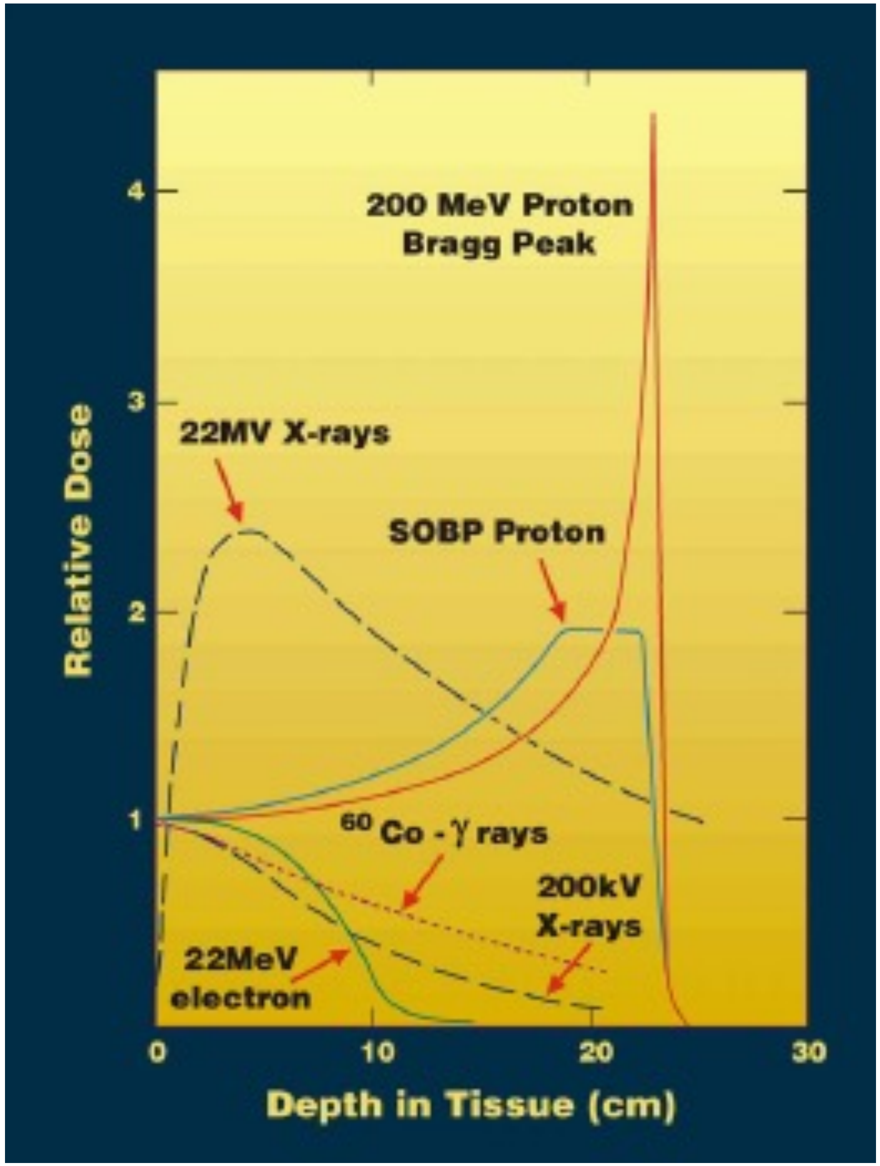


Lucite range compensator



Modulator

Proton Therapy

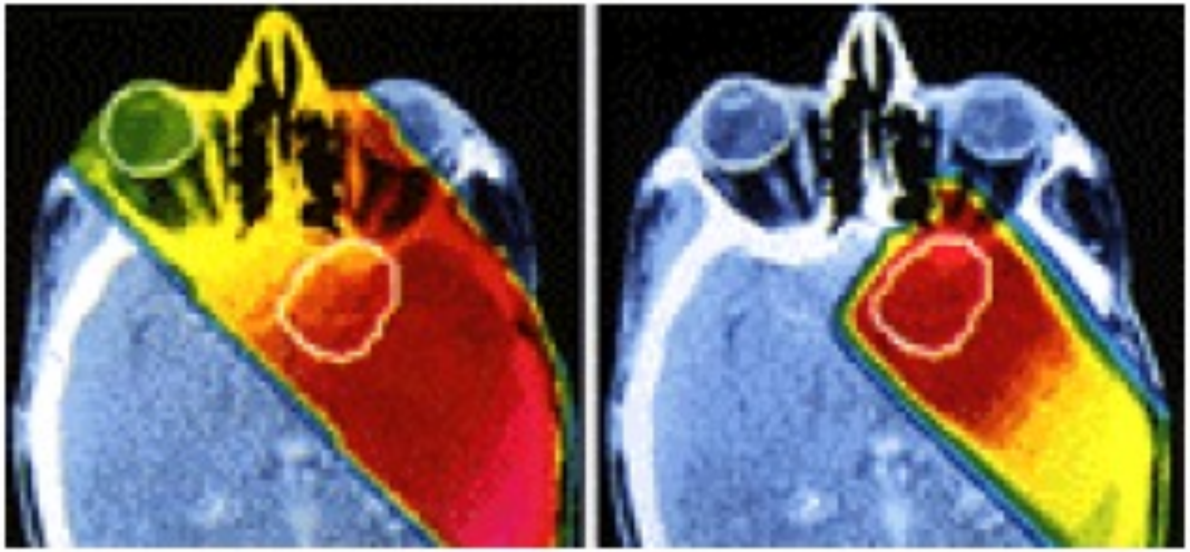


Proton Therapy

Comparison of Treatment Planning using Protons vs. X-rays

Highest dose
- red

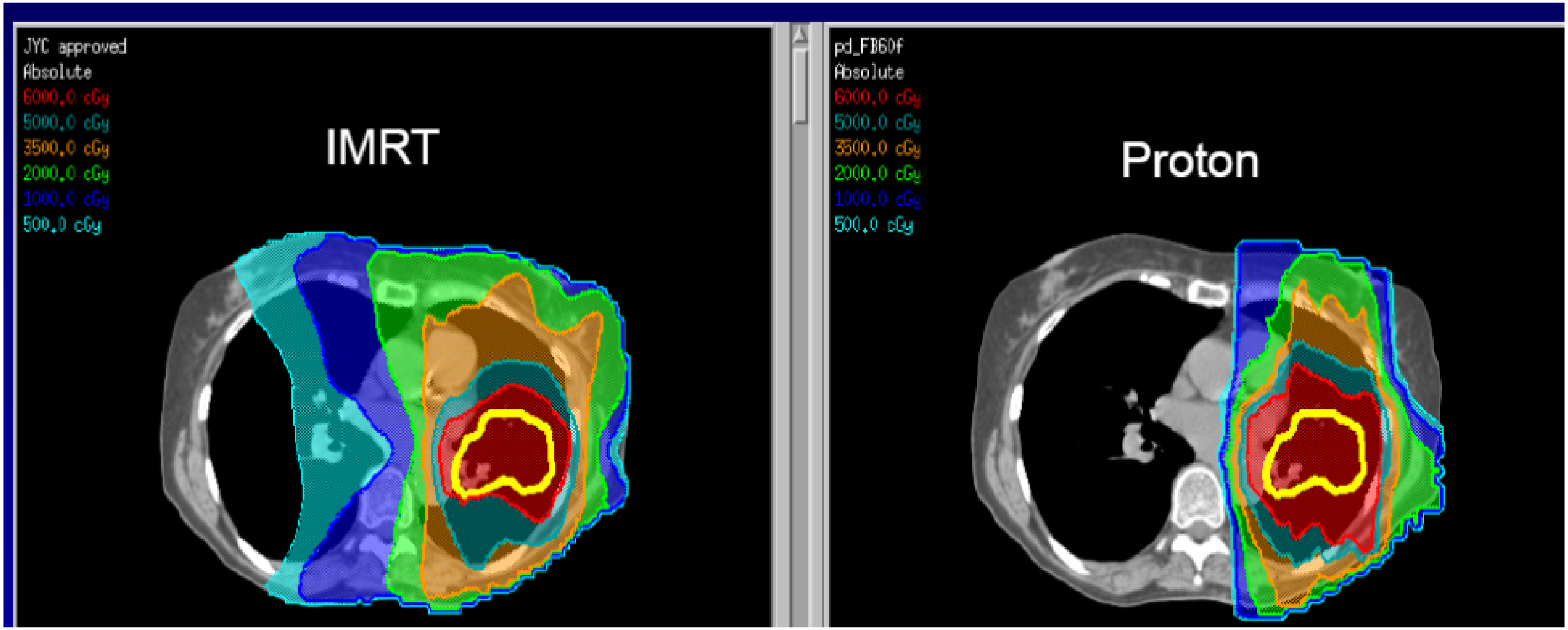
Lowest dose
- yellow



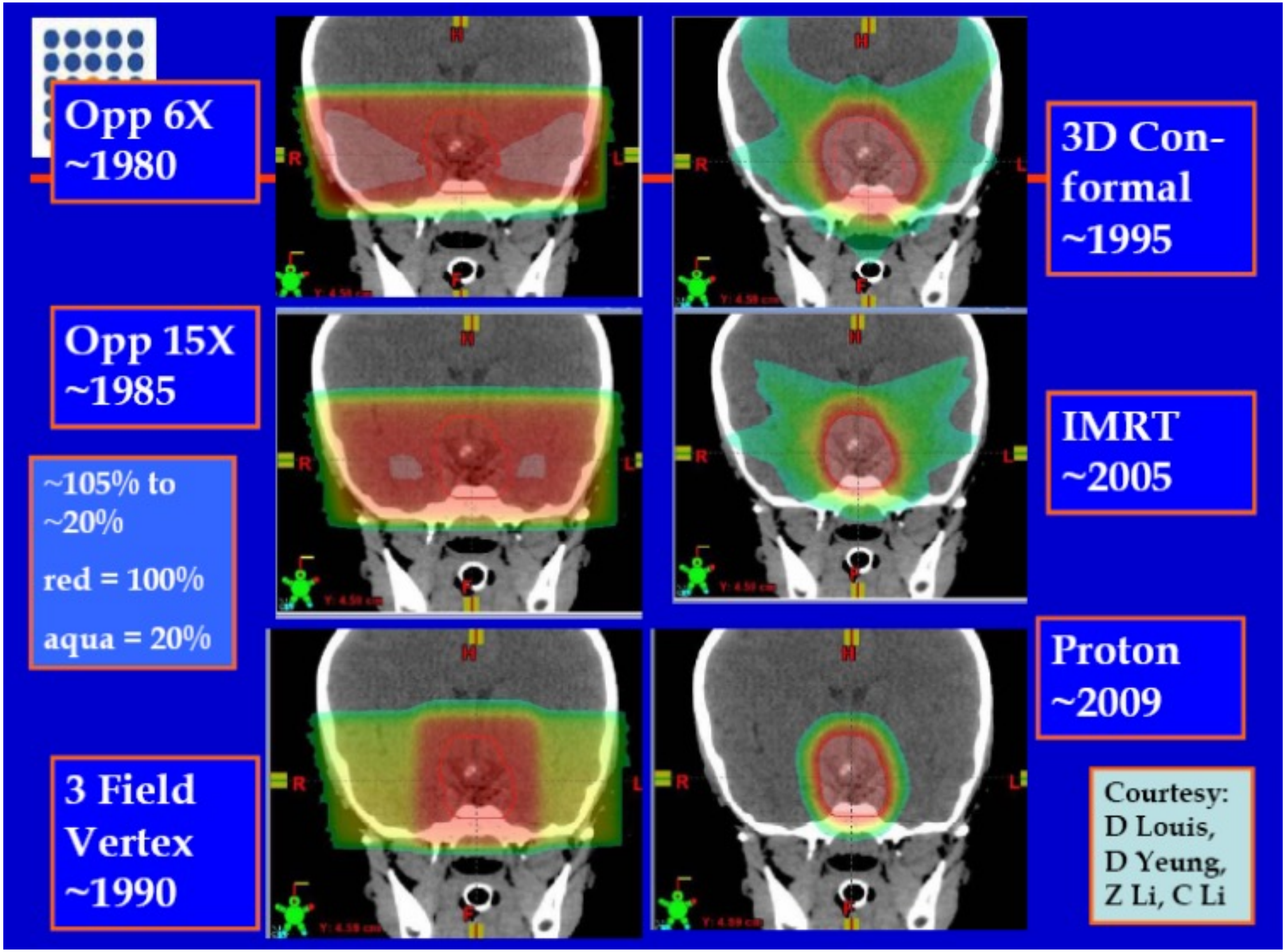
X-rays　　Protons

Proton Therapy

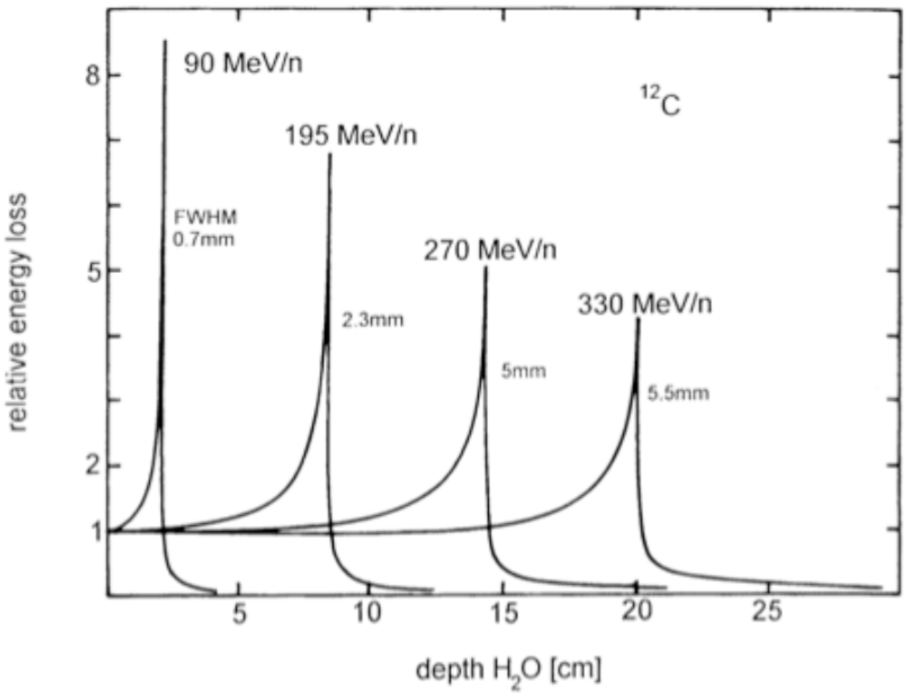
- Lung cancer treatment
  - Intensity modulated radiation therapy vs proton therapy



Proton Therapy



Hadron-therapy for cancer

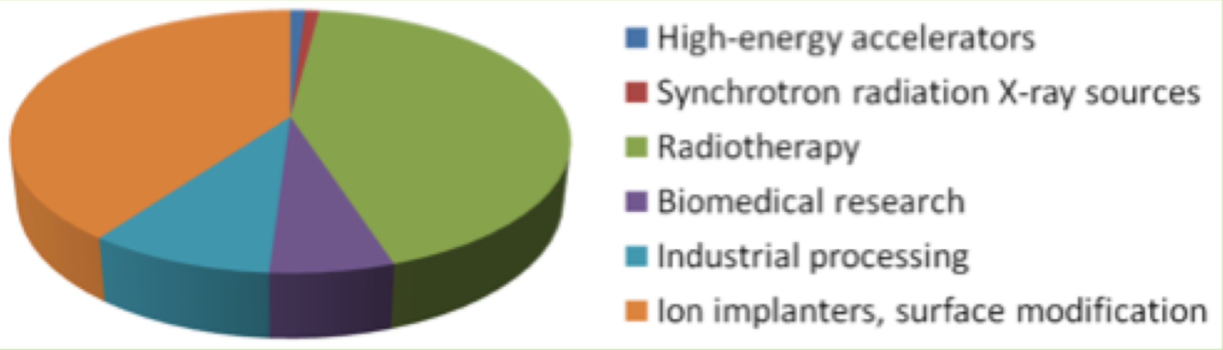


^{12}C ions in water. Treatment of deep-seated tumors.

Proton Cancer Therapy

Accelerators Worldwide

The number of accelerators worldwide exceed 20000



- Market for **medical and industrial** accelerators exceeds **$3.5 billion.** All products that are processed, treated, or inspected by particle beams have a collective annual value of more than $500 billion [1]

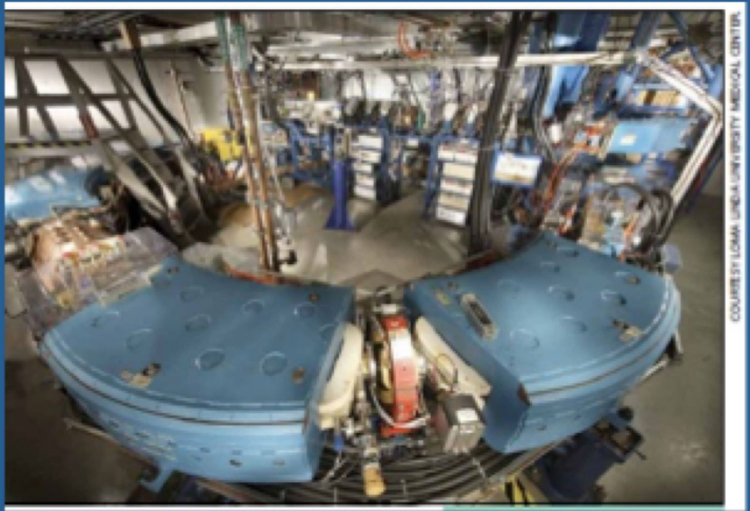
[1] http://www.acceleratorsamerica.org/

Accelerators are not only for high-energy physics

Accelerators for Medicine

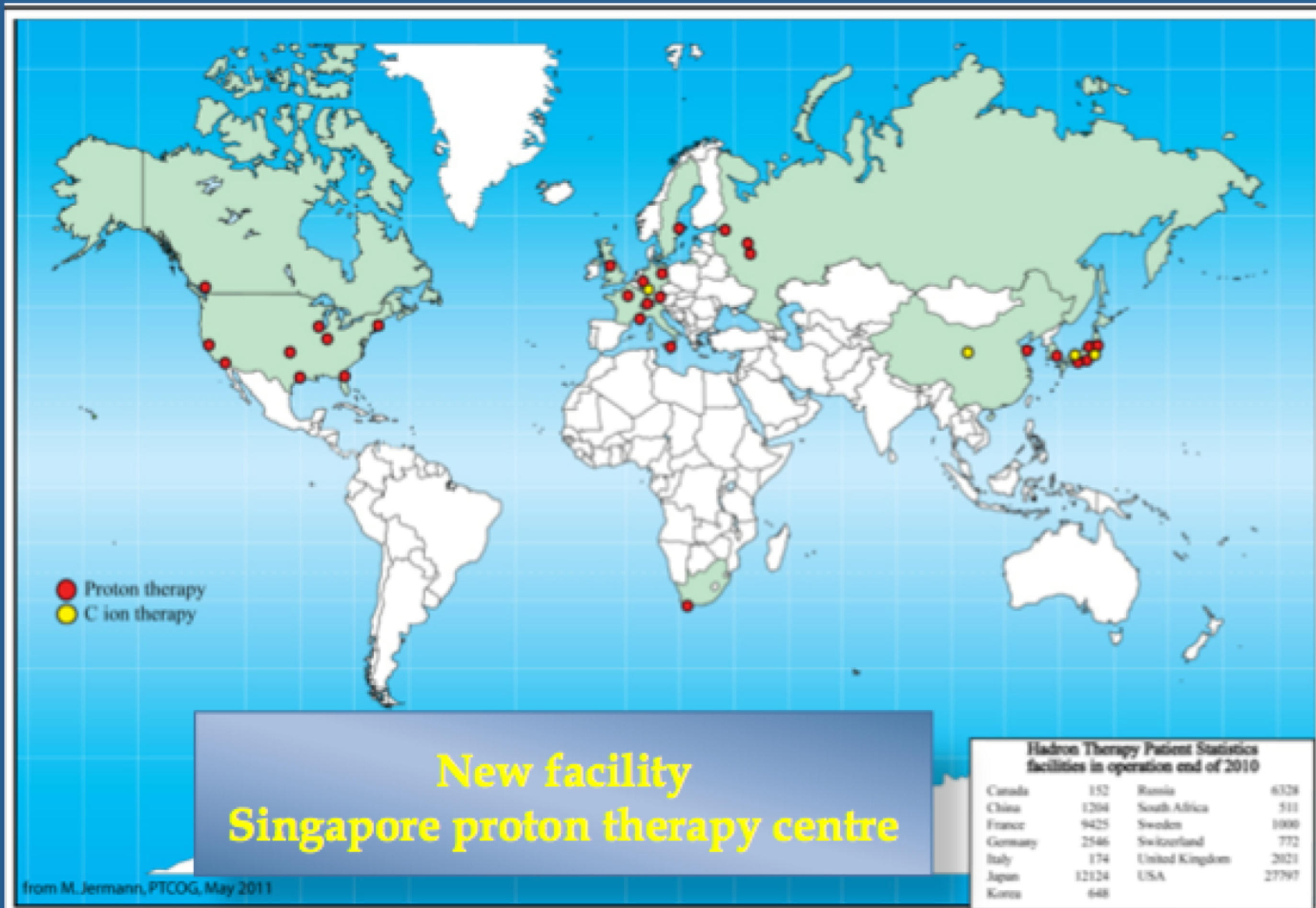
- Medical Therapy
  - X-rays have been used for decades to destroy tumours.
  - For deep-seated tumours and/or minimizing dose in surrounding healthy tissue use hadrons (protons, light ions).
  - Accelerator-based hadrontherapy facilities.

Accelerator cancer therapy



Loma Linda Proton Treatment Centre Constructed at FNAL

Centers for HADRON Therapy
in operation end of 2010



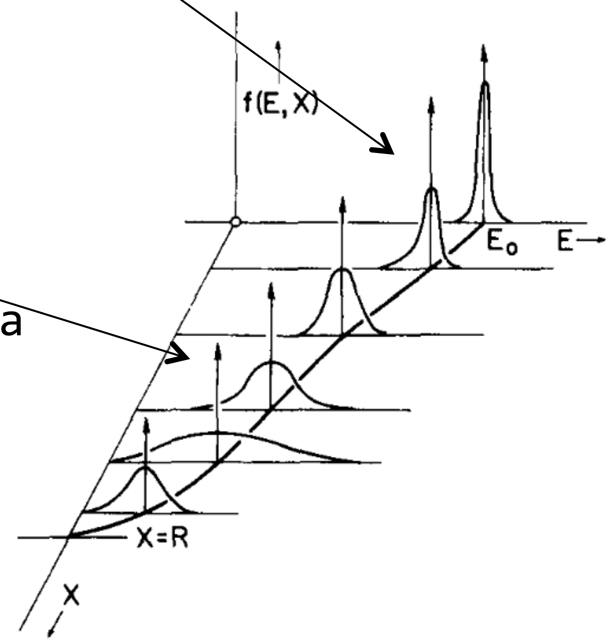
Hadron Therapy Patient Statistics facilities in operation end of 2010

Canada	152	Russia	6328
China	1204	South Africa	511
France	9425	Sweden	1000
Germany	2546	Switzerland	772
Italy	174	United Kingdom	2021
Japan	12124	USA	27797
Korea	648		

Worldwide: 30 centres (4 have C-ions): ~ 65'000 patients
Europe: 9 centres (with C-ions at GSI and Heidelberg): ~ 16'000 patients

Energy loss fluctuations

- Quando una particella attraversa un materiale di spessore L si hanno due casi estremi:
  - lo spessore del materiale e' molto minore del range medio, L<<R: la particella attraversa il mezzo perdendo solo una frazione piccola della sua energia $\Delta E \ll E_{inc}$. In tal caso, fissato lo spessore attraversato, si osservano fluttuazioni nell'energia ΔE depositata nel mezzo.
  - il materiale e' cosi' spesso che la particella vi deposita tutta la sua energia ($\Delta E = E_{inc}$) e si ferma. In tal caso si osserva, come abbiamo appena visto, una distribuzione nel range delle particelle a causa delle fluttuazioni statistiche del numero di urti e dell'energia persa in ciascun urto. Questo accade quando il range medio della particella e' molto minore dello spessore del materiale: R << L



Energy loss fluctuations

La perdita di energia dE/dx (Bethe Block) descrive la perdita media di energia per una particella nell'attraversare uno spessore dX di materiale. Cio' significa che in un fascio, una particella in media perde <dE/dX>

$$\left\langle -\frac{dE}{dx} \right\rangle = \rho K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\ln \frac{2mc^2\beta^2\gamma^2}{I} - \beta^2 - \frac{\delta(\gamma)}{2} \right]$$

In un mezzo omogeneo, dalla BB si ha <dE> ∝ X $\quad \langle dE \rangle \approx \rho K z^2 \frac{Z}{A} \frac{1}{\beta^2} X$

Se fissiamo l'attenzione su una singola particella, in generale

$-\frac{dE}{dx} \neq \left\langle -\frac{dE}{dx} \right\rangle$ ovvero la perdita totale in uno spessore ΔX, ΔE ≠ <ΔE>

a causa delle fluttuazioni statistiche che avvengono

i. nel numero di collisioni con gli elettroni del mezzo

ii. nella perdita di energia nelle singole collisioni

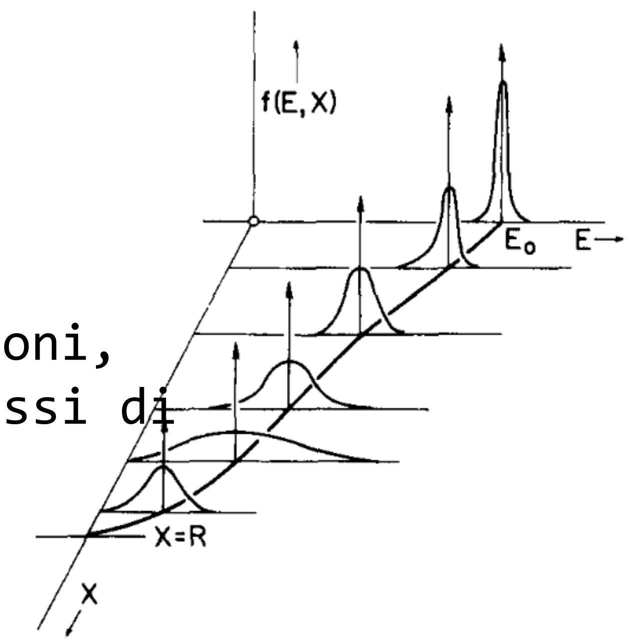
$$\Delta E = \sum_{n=1}^{N} \delta E_n$$

N : number of collisions
δE : energy loss in a single collision

δE distribuite statisticamente

Energy loss fluctuations for R<<L

Un fascio di particelle monoenergetiche con $E = E_0$ mostrera' uno sparpagliamento σ intorno a $E = E_0 - \langle dE \rangle$, dopo aver attraversato un spessore (costante) di materiale e non sara' piu' monoenergetico (una delta di Dirac) con $E = E_0 - \langle dE \rangle$, ma mostrera' una distribuzione di perdite intorno a quella media.



La forma della distribuzione dipende dallo spessore attraversato

Nel trattare il problema delle fluttuazioni, tradizionalmente si hanno due grandi classi di assorbitori:

1. assorbitori spessi
2. assorbitori sottili

(nel limite di cui sopra)

Energy loss fluctuations for R<<L

- La distinzione tra assorbitori spessi e sottili non e' netta.
- L'assorbitore diventa spesso quando il # di collisioni N $\rightarrow \infty$ ma nella pratica e' sempre finito e non e' facile distinguere sempre bene i due casi (anche perche' siamo nell'ipotesi che la particella perda una frazione piccola dell'energia, cioe' R<<L)

Energy loss fluctuations for R<<L

There is another way to define thick and thin absorbers:
Roughly, for a MIP the energy loss in a thickness X g/cm^2 of absorber is $\Delta E = \langle dE/dX \rangle X$ and also $\Delta E = \sum_0^N \delta E_i$
The δE_i are distributed statistically according to their probability between δE_{min} and δE_{max}

If $N \rightarrow \infty$, $\Delta E \gg \delta E_i$, for any i and thus also for the max cinematically allowed δE_{max}: $\Delta E \gg \delta E_{max}$

$$\langle dE/dX \rangle X \gg \delta E_{max}$$

Therefore a detector can be said "thick" if

$$X \gg \delta E_{max} / \langle dE/dX \rangle$$

In other words, the detector is thick if single collisions with large energy transfers are not important for the AVERAGE total energy loss

- Is a 1 cm scintillator thick or thin for 1 GeV muon?

1 GeV μ

$W_{max} = 2 m_e c^2 \beta^2 \gamma^2$

$\beta = p/E \quad \gamma = E/m \quad E^2 = p^2 + m^2$

$E^2 = 1^2 + (.105)^2 \Rightarrow E = 1.0055$

$\beta = 1/1.0055 = .9945$

$\gamma = 1.0055/.105 = 9.576$

$W_{max} = 2(.511)\cancel{c^2}/\cancel{c^2} \, (.9945)^2 (9.576)^2$

$= 92.69$ MeV

$\Delta E = (dE/dX)(\rho x)$

$= 1.956 \cdot 1.032 \cdot 1 = 2$ MeV

$W_{max} >> \Delta E$
So 1 cm plastic scintillator is NOT thick

Actually, even 1 cm of Pb is not thick since $\Delta E = (dE/dX)(\rho x) = 1.1*11.35*1 = 12.5$ MeV!!

Energy loss probability

- Trascurando lo spin dell'e- (ie scattering di Coulomb), la probabilita' di espellere un e- di energia fra E ed E + dE e' :

$$P(E)dE = k\frac{X}{E^2}dE \quad \text{where} \quad k = 2\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A}\frac{1}{\beta^2} \qquad (1)$$

Si ricordi la distribuzione del # di e- espulsi con E fra E ed E+dE in una collisione $n(E) = -\frac{dN}{dEdx} = \pi N_e (\frac{e^4 Z^2}{8\pi^2 \epsilon_o^2 m_e v^2})\frac{1}{E^2}$

Infatti il # di interazioni e' dN = ndV.
dV e' il volume di interazione disponibile: per definizione di sezione d'urto, tutte gli atomi in dV = L$d\sigma$ interagiscono con la particella incidente, quindi nL$d\sigma$ e' il # di interazioni.

Poiche' osserviamo la distribuzione di energia, $d\sigma = (d\sigma/dE)dE \equiv P(E)dE$ e quindi la (1)

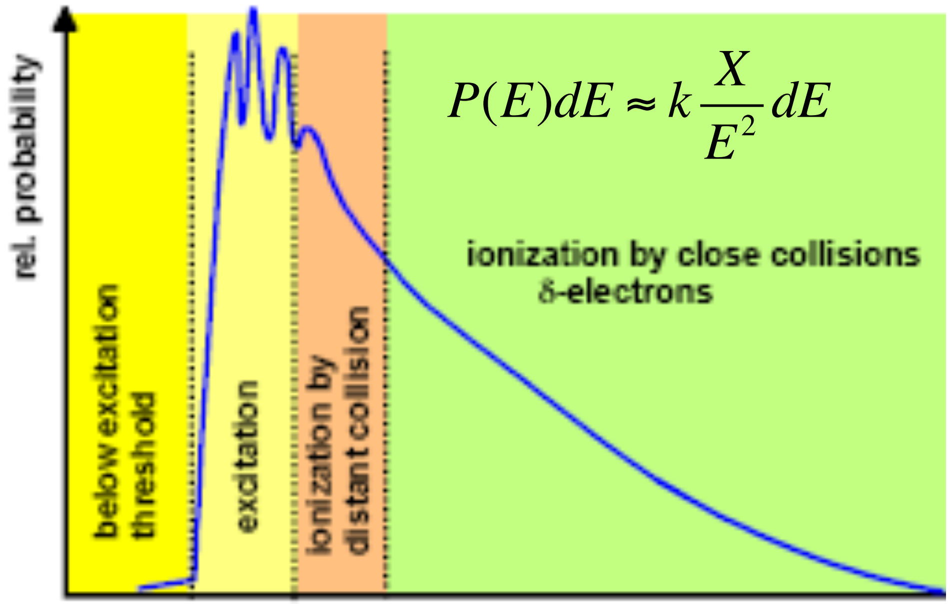
La distribuzioni esatta per elettroni e' $P(E)dE = k\frac{X}{E^2}(1-\beta^2\frac{E}{E_{\max}})dE$

La distribuzioni esatta per particelle pesanti di spin ½ e'

$$P(E)dE = k\frac{X}{E^2}(1-\beta^2\frac{E}{E_{\max}} + \frac{1}{2}\frac{E}{E+mc^2})dE$$

La probabilita' di emissione (cosi' come lo spettro) e' limitata dalla cinematica fra $E_{min} < E < E_{max}$

Energy loss probability



Andamento schematico della probabilita' di perdere un'energia W in una collisione.

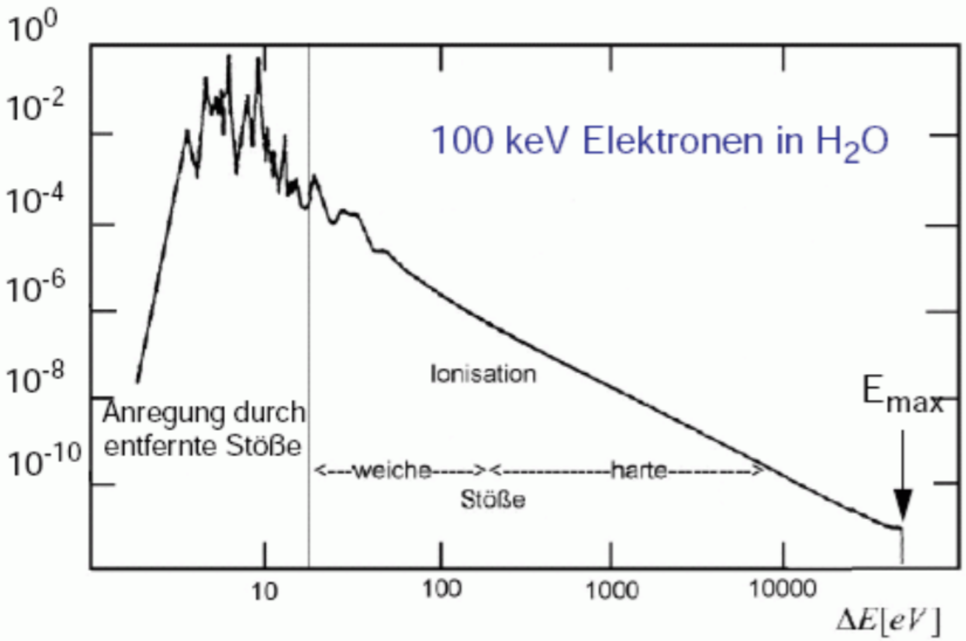
La distribuzione descrive bene la probabilita' finche' il trasferimento di energia e' grande rispetto all'energia di legame atomico (regione verde).

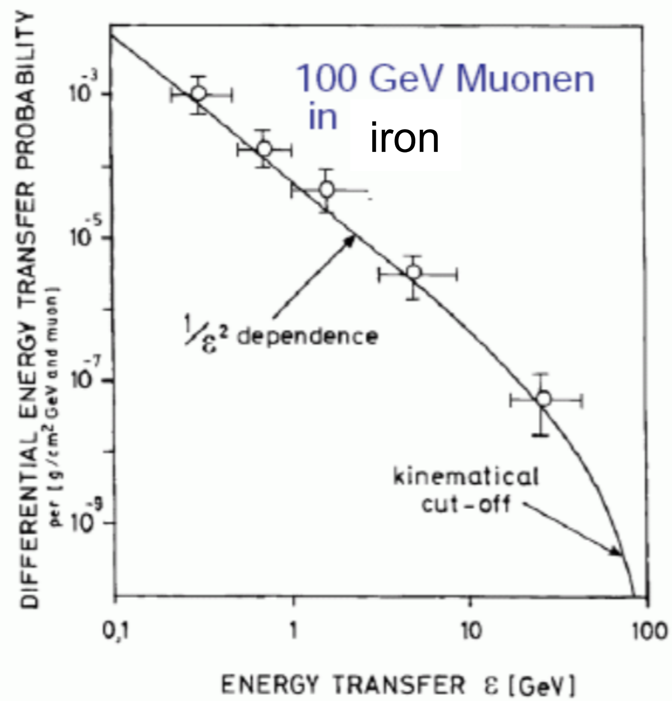
A basse energie (es. per collisioni distanti) entrano in gioco altri effetti (screening/ polarizzazione del mezzo, eccitazione di elettroni atomici, oscillazioni atomiche).

δ Electrons

Energy distribution of secondary electrons: $$\frac{dN^2}{dT\,dx} = \frac{1}{2} k\, z^2 \frac{Z}{A} \frac{1}{\beta^2} \frac{F(T)}{T^2}$$

$F(T) = 1 - \beta^2 T / T_{max}$ (Spin 0)





Example: 500 MeV pion in 300 μm Si: 5% produce an electron with T>166 keV
important background in Cherenkov counter
Number of δ-electrons proportional z^2/β^2

Energy loss fluctuations for R<<L

Il numero di collisioni nel mezzo determina la distribuzione delle perdite di energia intorno al valore medio <dE>

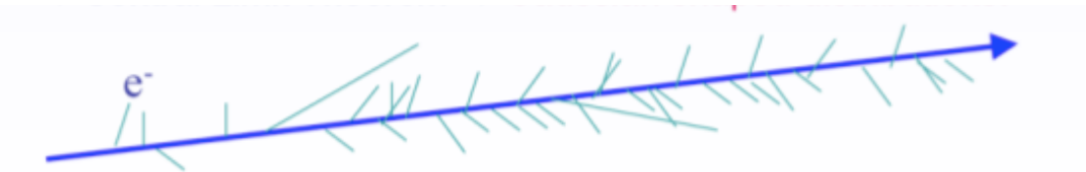
Ciò deriva direttamente dal teorema del limite centrale: la somma di N variabili casuali, che seguono tutte la stessa distribuzione statistica, diventa una gaussiana nel limite di $N \rightarrow \infty$.

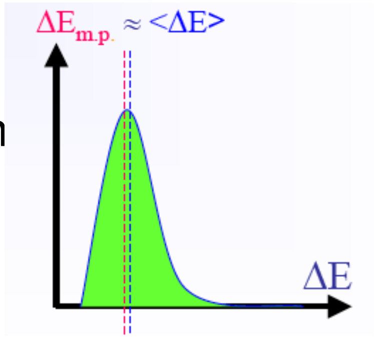
Se consideriamo come variabile casuale la dE, cioè l'energia persa in una collisione singola, che le collisioni sono statisticamente indipendenti e che in ogni collisione la velocità β del proiettile non è cambiata in maniera apprezzabile in modo che $d\sigma(E)/dE$ è costante → l'energia totale persa è la somma di tutte le dE, tutte con la stessa distribuzione ed e' quindi gaussiana.

Per assorbitori relativamente spessi la distribuzione della perdita di energia è gaussiana.

Energy loss probability (1)

- Real detector measures the energy ΔE deposited in a layer of finite thickness δx.
- For thick layers and high density materials:
  - Many collisions
  - Central limit theorem: distribution → Gaussian





$$P(E)dE = k\frac{X}{E^2}dE \qquad \text{con} \qquad k = 2\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A}\frac{1}{\beta^2}$$

La probabilita' e' piccata a piccoli valori di E, cioe' per valori ~ Emin: la maggior parte delle collisioni comporta piccole perdite di E, ma c'e' una probabilita' finita che in una singola collisione avvenga una perdita "grande", vicina a Emax
Se N e' sufficientemente elevato, in media le perdite di energia saranno simmetriche intorno al valore medio: singole collisioni con alta E sono compensate da tante collisioni a bassa E, in modo che la distribuzione tenda a una gaussiana, in cui il valore medio <dE> e la perdita di energia piu' probabile dE_{mp} coincidono

Assorbitori spessi: limite gaussiano.

Se il materiale è spesso (ma non troppo) o denso ➔ N è grande quindi vale il teorema del limite centrale e la perdita totale di energia E è distribuita secondo una gaussiana

$$f(X,E) \propto \exp\left(-\frac{\left(E - \bar{E}(X)\right)^2}{2\sigma^2(X)} \right)$$

Con X spessore del materiale, E perdita di energia nell'assorbitore, $\bar{E}$ perdita di energia media = (dE/dX)X, e σ la sua deviazione standard.

Assorbitori spessi: limite gaussiano.

Bohr ha calcolato la deviazione standard σ_0 per particelle non relativistiche:

$$\sigma_0^2 = 4\pi Z_{inc}^2 Z_{target} n_a e^4 x = 4\pi r_e^2 N_A (m_e c^2)^2 Z_{inc}^2 \frac{Z_t}{A_t} \rho x = 0.157 Z_{inc}^2 \frac{Z}{A} \rho x \left[MeV^2\right]$$

Dove N_A è il numero di Avogadro, r_e il raggio classico dell'elettrone, ρ la densità, A_t il peso atomico (in kg/mol) e Z_t il numero atomico dell'assorbitore.

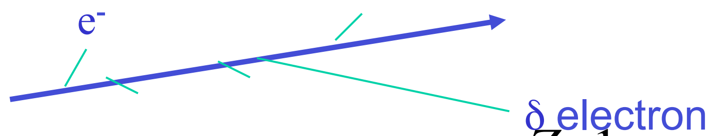
Estesa a particelle relativistiche diventa:

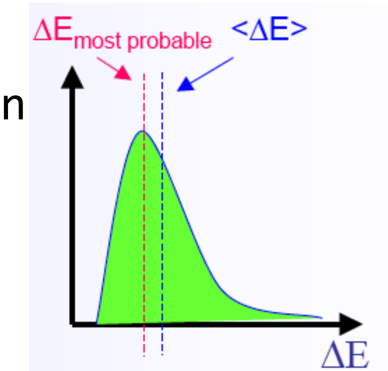
$$\sigma^2 = \sigma_0^2 \frac{\left(1 - \frac{1}{2}\beta^2\right)}{1 - \beta^2}$$

- For a Gaussian distribution resulting from N random events the ratio of the width/mean $\propto 1/\sqrt{N}$
- Increasing the thickness of the detector decreases the relative width of the Gaussian peak

Energy loss probability (2)

- For thin layers or low density materials:
  - Few collisions, some with high energy transfer.
  - ⇒ Energy loss distributions show large fluctuation towards high losses
  - ⇒ Long Landau tails





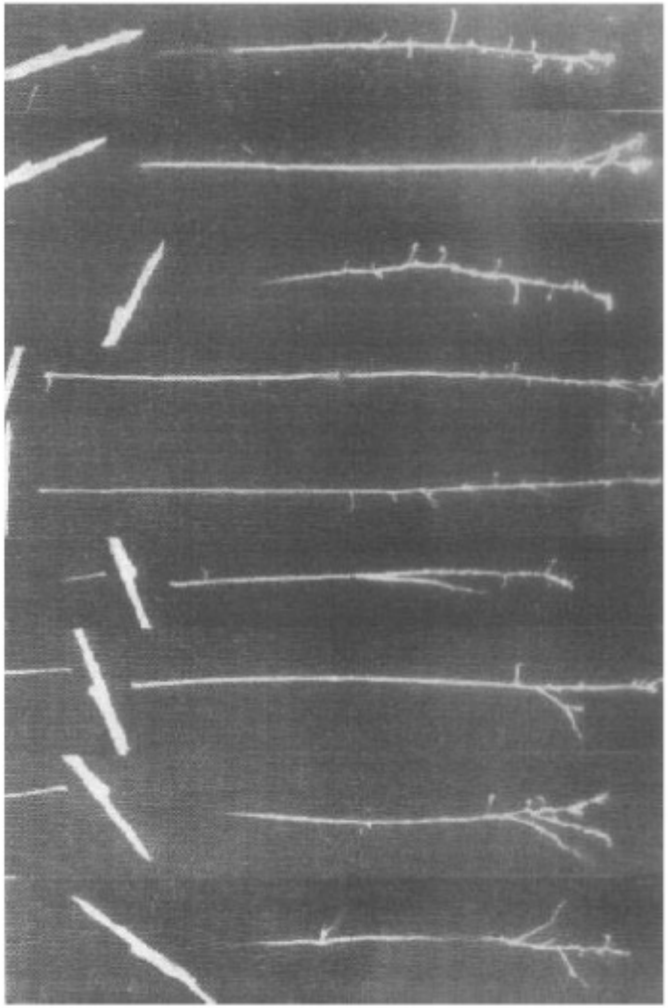
$$P(E)dE = k\frac{X}{E^2}dE \quad \text{con} \quad k = 2\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A}\frac{1}{\beta^2}$$

La probabilita' e' piccata a piccoli valori di E, cioe' per valori ~ Emin: la maggior parte delle collisioni comporta piccole perdite di E, ma c'e' una probabilita' finita che in una singola collisione avvenga una perdita "grande", vicina a Emax

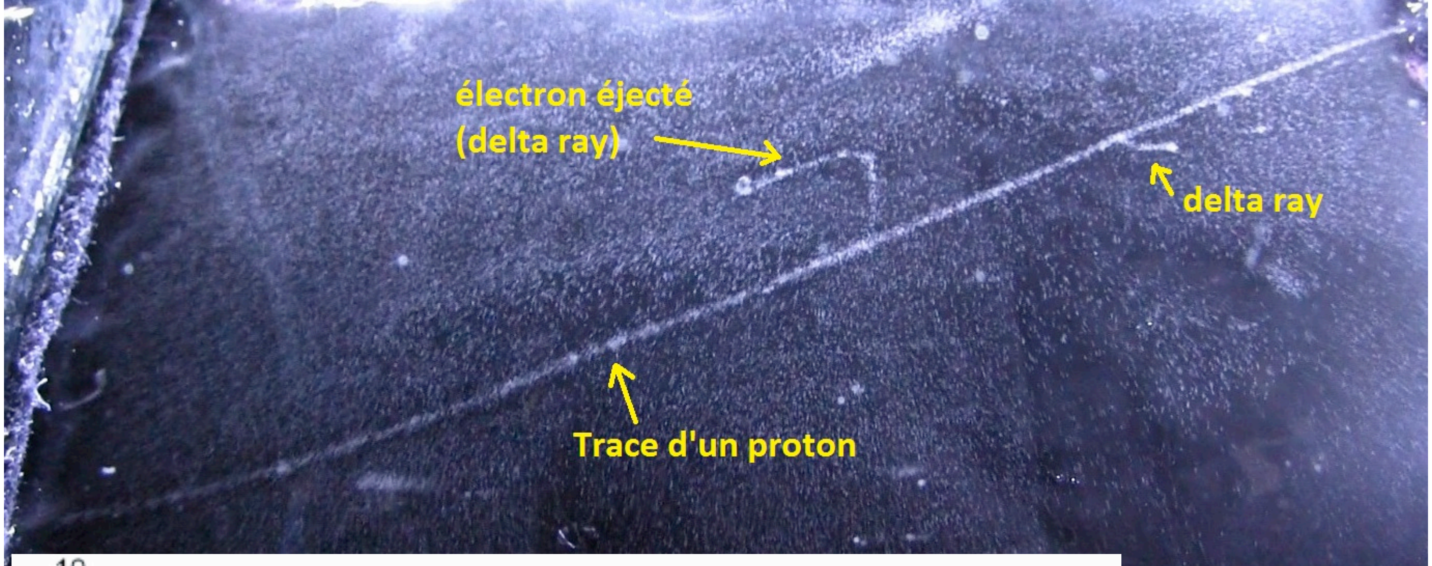
Se N non e' sufficientemente elevato, singole collisioni con grande E non sono compensate da tante collisioni con piccola E: la distribuzione delle perdite non e' simmetrica intorno al valore medio. Le singole collisioni spostano il valore medio verso alti valori di E, mentre la maggior parte delle perdite e' con piccola E, cioe' il valore piu' probabile rimane piccolo

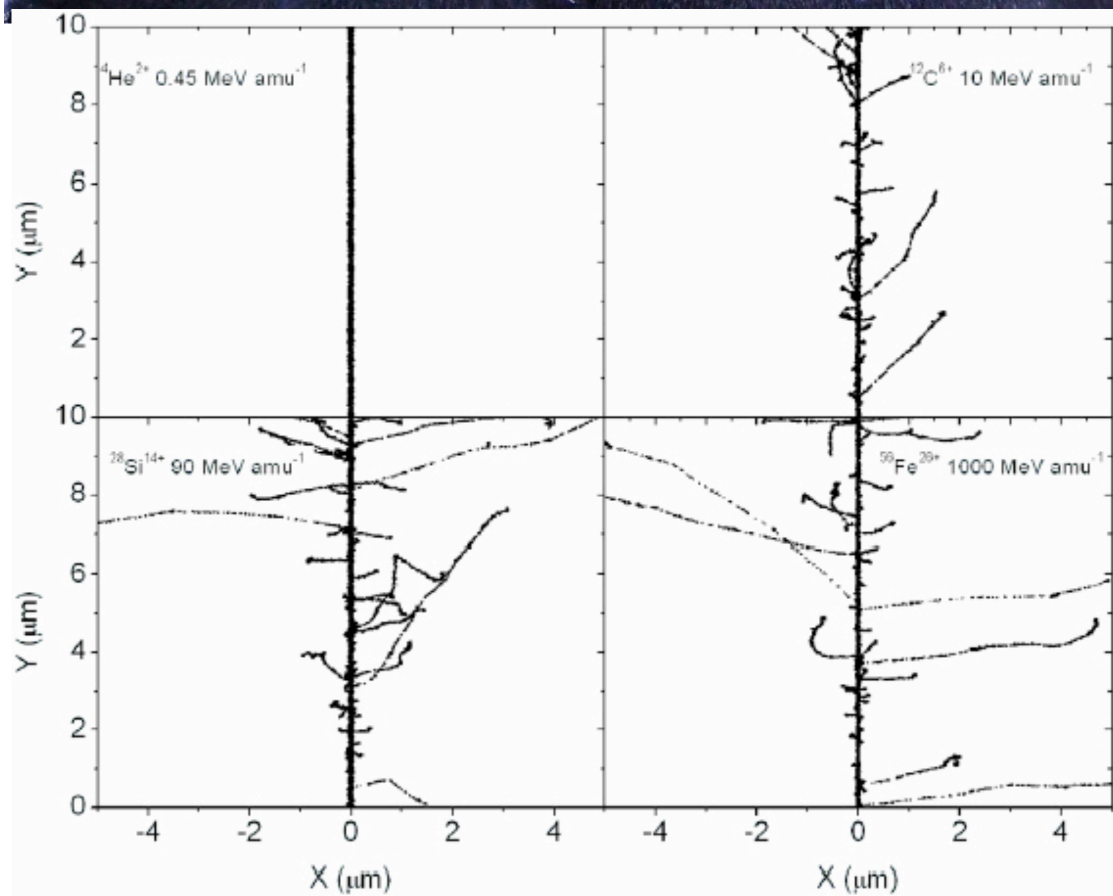
Delta Rays

Le perdite di energia appaiono come energia cinetica degli elettroni espulsi. I delta appaiono come sottili “rami” che dipartono dal tronco



From: http://www.lateralscience.co.uk/cloud/diff.html



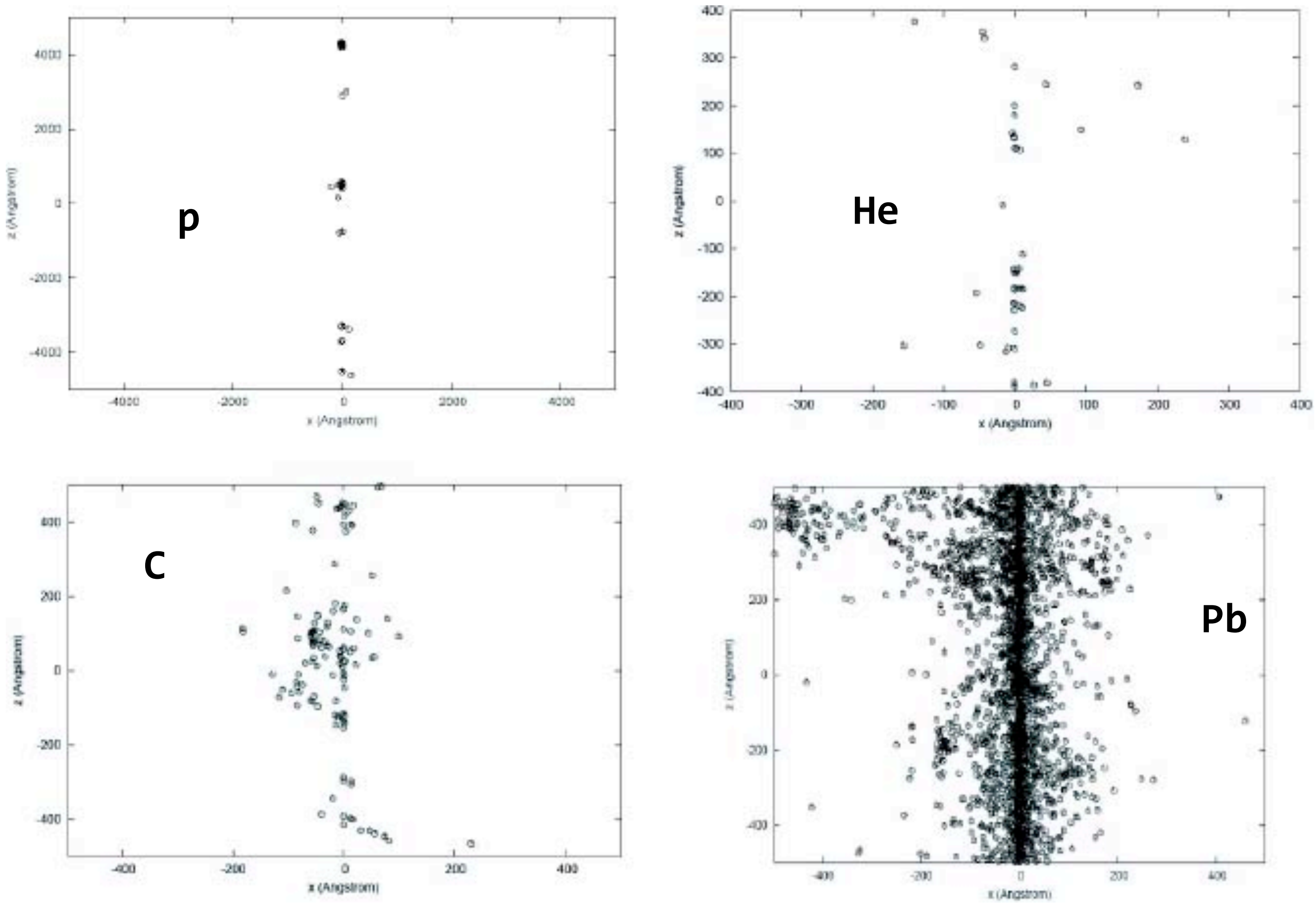


Delta rays can make additional secondary ionization trails that:

-may escape the active volume, so their energy can not be collected→potential bias to energy deposit

-confuse the main track, making difficult the pattern recognition, that is the task of assigning a energy deposit to a given track.

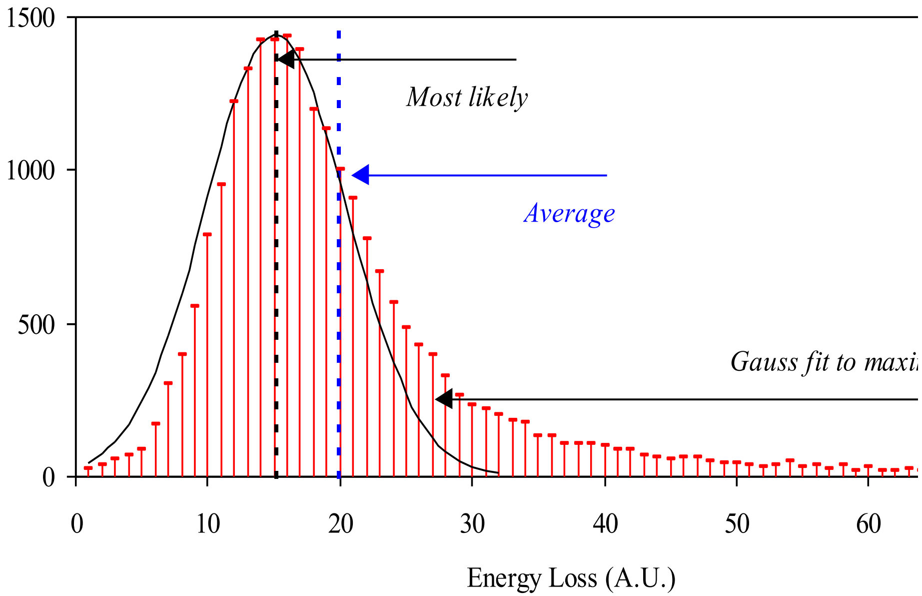
That's why we hate δ-rays



Delta rays grow quadratically with charge.
They worsen the capability to know the particle trajectory, i.e. position resolution and therefore

Energy loss probability

Energy loss distribution differs a lot from a gaussian.
Only around the peak the E loss distribution is approximated by a gaussian: the width is large and asymmetric, does not fall as $1/\sqrt{X}$.
This has important consequences from a practical point of view: it makes the average E loss, <dE/dX>, useless for measuring the energy deposit in the medium: it will fluctuate in a non-gaussian way due high energy deposit fluctuations



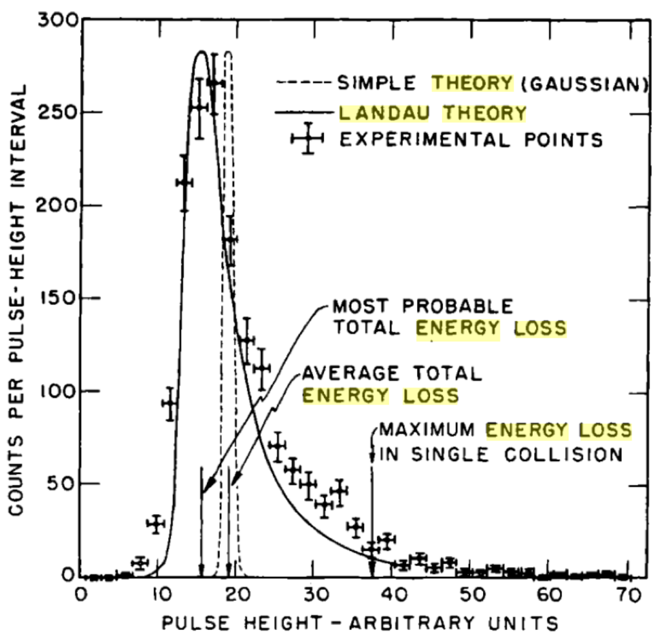
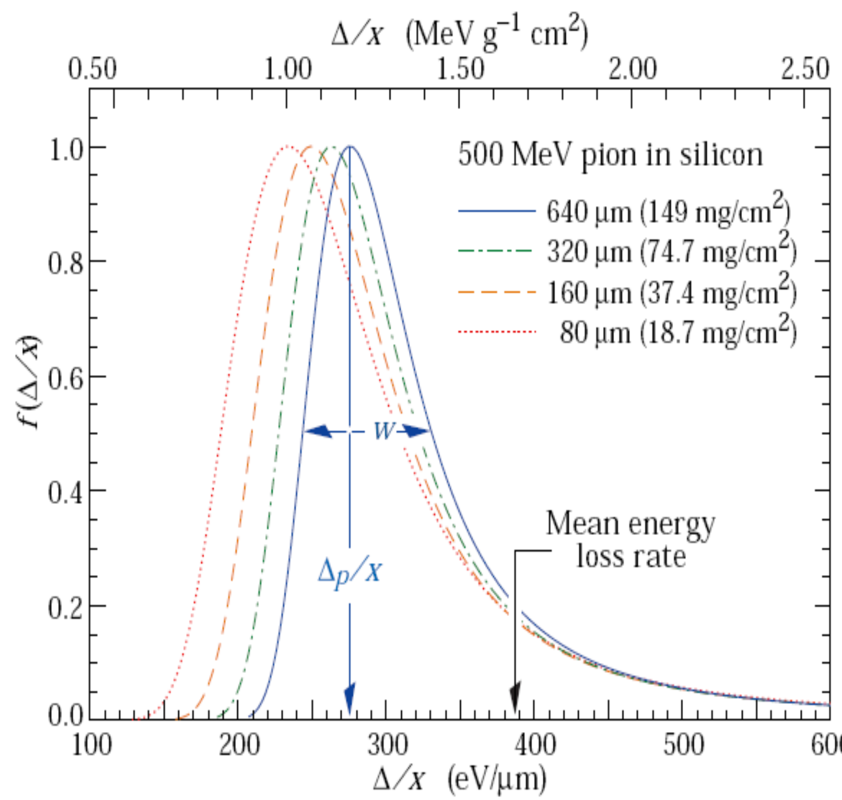


Fig. 2. Frequency distribution of energy losses of 31.5 MeV protons traversing a proportional counter filled with 96% Ar and 4% CO_2. The histogram of experimental points is compared to the theoretical Landau distribution and a gaussian distribution based on ion-electron pair statistics. From Igo *et al.* [3]

Delta rays, most probably energy loss

While <dE/dX> is affected by fluctuations, peak position (most probable energy loss) is not affected; therefore it is used when sampling dE/dx loss.

- It is important to keep in mind that the m.p. value and the average value do not coincide

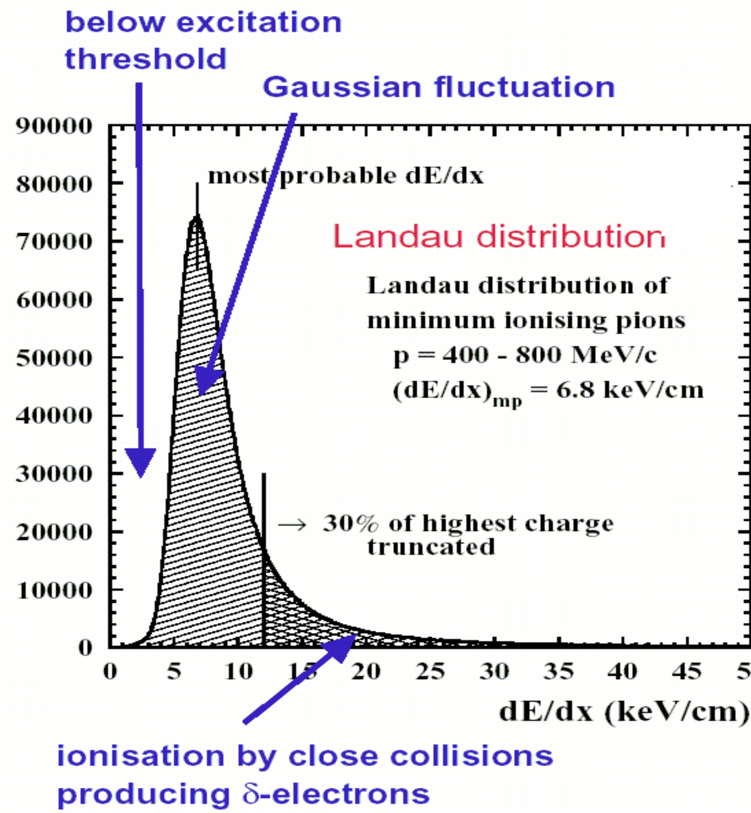


Straggling functions in silicon for 500 MeV pions, normalized to unity at the most probable value Δ/x. The width w is the FWHM.

Bibliografia Fernow (Introduction to experimental particle physics)
http://pdg.lbl.gov

Landau Tails

- Restricted dE/dX: mean of truncated distribution excluding energy transfers above some threshold T_{cut}
  - Useful for thin detectors in which δ-electrons can escape the detector (energy loss not measured)

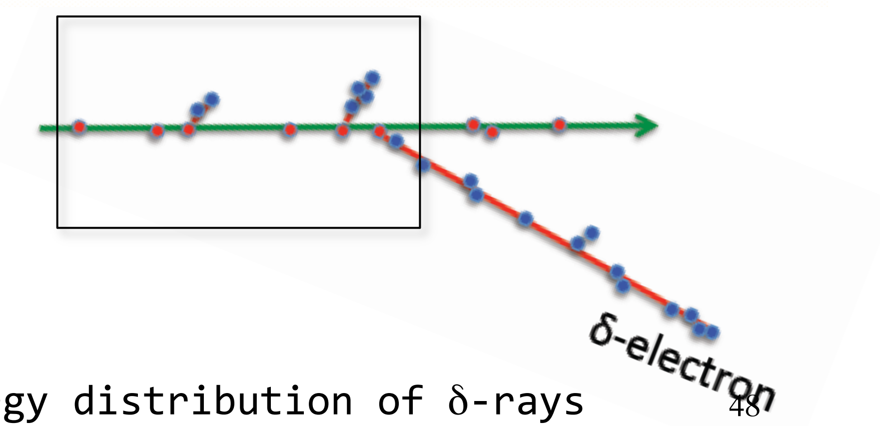


$$-\frac{dE}{dx}\bigg|_{T<T_{\rm cut}} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2m_ec^2\beta^2\gamma^2T_{\rm cut}}{I^2} - \frac{\beta^2}{2}\left(1+\frac{T_{\rm cut}}{T_{\rm max}}\right) - \frac{\delta}{2}\right].$$

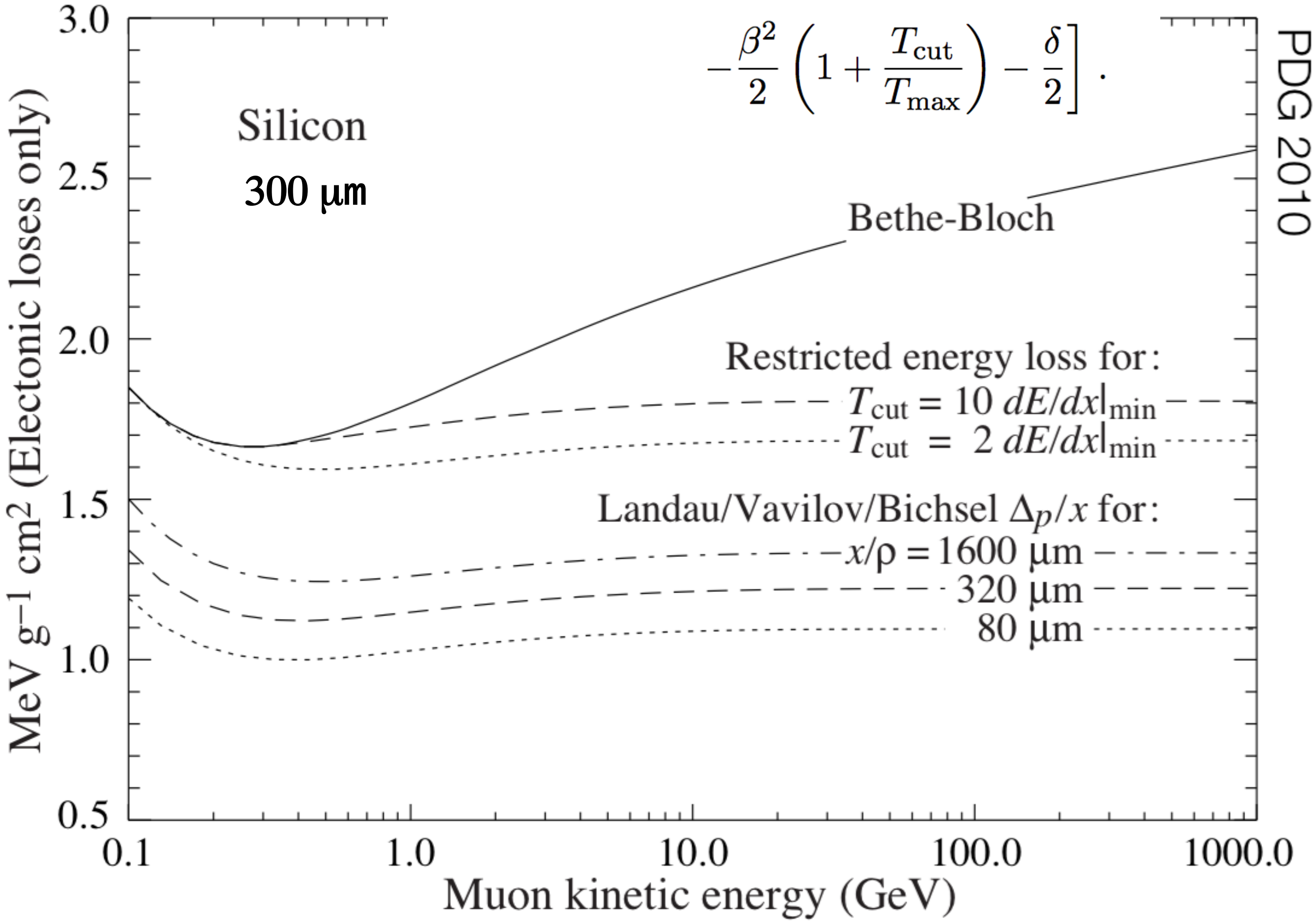
This form approaches the normal Bethe-Bloch function as Tcut→Tmax. It can be verified that the difference between restricted and full loss is equal to

$$\int_{T_{cut}}^{T_{\max}} T\,\frac{dn}{dTdx}dT$$

Where dn/dTdX is energy distribution of δ-rays

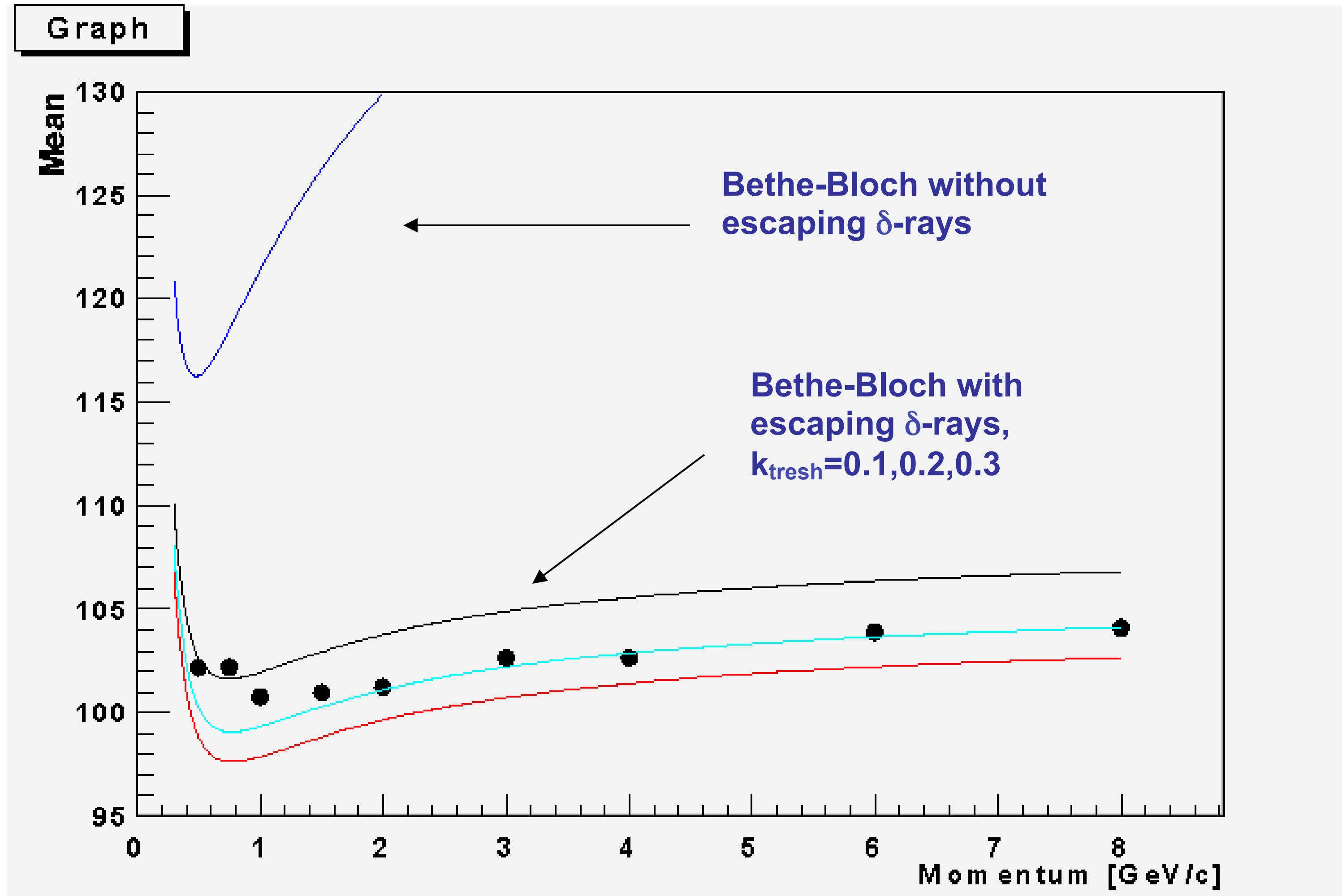


$$-\frac{dE}{dx}\bigg|_{T<T_{\mathrm{cut}}} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2m_ec^2\beta^2\gamma^2T_{\mathrm{cut}}}{I^2} - \frac{\beta^2}{2}\left(1+\frac{T_{\mathrm{cut}}}{T_{\mathrm{max}}}\right) - \frac{\delta}{2}\right].$$



Bethe-Bloch dE/dx, two examples of restricted energy loss, and the Landau most probable energy per unit thickness in silicon. The change of Δ_p/x with thickness x illustrates its a ln x + b dependence. Minimum ionization ($dE/dx|_{\mathrm{min}}$) is 1.664 MeV g^{-1} cm^2. Radiative losses are excluded. The incident particles are muons.

Data + Bethe-Bloch - Silicon



Continuous curves are MonteCarlo simulations with different max δ-rays energy cuts

Assorbitori sottili infinitesimi

La probabilità che una particella incidente di energia E perda energia compresa fra W e W+dW attraversando un dx infinitesimo è:

$$\Phi(W)dWdx = n_a \frac{d\sigma(W)}{dW} dWdx$$

Dove $n_a = N_A \rho / A$= numero di atomi per unità di volume, $d\sigma/dW$= sezione d'urto differenziale per la particella incidente di perdere energia W in una <u>singola</u> collisione con un atomo.

La probabilità totale di una collisione di perdere qualunque W nell'infinitesimo dx sarà:

$$qdx = \left(n_a \int_{W\min}^{W\max} \frac{d\sigma}{dW} dW \right) dx$$

q si chiama rate di <u>ionizzazione primaria.</u>

Assorbitori sottili

Semplice se dx è infinitesimo, ma complicato per x finito.

Consideriamo un fascio di N particelle di energia E. Sia $\chi(W,x)dW$ la probabilità che una particella perda un'energia fra W e $W+dW$ dopo avere attraversato uno spessore x.

La forma di χ può essere determinata considerando come varia il # di particelle che hanno perso W a x, quando le particelle attraversano un altro spessore dx:

variazione del # di part che subiscono perdite tra W, W+dW a x, x+dx=

part "in" (W, W+dW) a (x, x+dx) – part "out" da (W, W+dW) a (x, x+dx)

- Il numero di particelle con perdita di energia fra W e W+dW cresce perché qualcuna che ad x aveva perso meno energia di W colliderà e perderà un'energia fra W e W+dW in dx = N × [prob di perdere W - ε fino a x] × [Prob di perdere ε in dx], sommate su tutti i possibili valori di ε.
- Il numero di particelle con perdita fra W e W+dW diminuisce perché alcune particelle che avevano già perso l'energia giusta prima del tratto dx ne perderanno ancora e quindi ne perdono di più di W+dW = N × [prob di perdere W fino a x] × [Prob di perdere un'energia qualunque in dx]

Assorbitori sottili

Se assumiamo che le collisioni che avvengono successivamente siano statisticamente indipendenti, che il mezzo assorbitore sia omogeneo e che la perdita totale di energia sia piccola rispetto all'energia della particella incidente:

$$N\chi(W,x+dx)dW - N\chi(W,x)dW =$$

$$= N\int_{W\min}^{W\max} \chi(W-\varepsilon,x)\Phi(\varepsilon)\,d\varepsilon dW dx - N\chi(W,x)dWq\,dx$$

Cioè:

$$\frac{\partial\chi(W,x)}{\partial x} = \int_{W\min}^{W\max} \Phi(\varepsilon)\chi(W-\varepsilon,x)\,d\varepsilon - q\chi(W,x)$$

Equazione integro-differenziale molto difficile da risolvere. Le differenze nelle soluzioni derivano essenzialmente dalle assunzioni fatte sulla probabilità $\Phi(W)$ cioè dal trasferimento di energia per <u>collisione singola</u>.

Assorbitori sottili

- The description of ionisation fluctuations is characterised by the significance parameter k, which is proportional to the ratio of mean energy loss to the maximum allowed energy transfer in a single collision with an atomic electron, **$k = \xi/E_{max}$**
- ξ represents the energy above which at least a δ-ray will be emitted in the thickness X

$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z \rho \delta x}{m_e \beta^2 c^2 A} = 153.4 \frac{z^2}{\beta^2} \frac{Z}{A} \rho \delta x \quad \text{keV},$$

NB: emission prob of δ with energy between E,E+dE is $P(E)dE = (\xi/E^2)dE$

κ measures the contribution of the collisions with energy transfer close to $E_{\max}$. For a given absorber, κ tends towards large values if δx is large and/or if β is small. Likewise, κ tends towards zero if δx is small and/or if β approaches 1. The value of κ distinguishes two regimes which occur in the description of ionisation fluctuations :

Assorbitori sottili: teoria di Landau (cenni)

1. A large number of collisions involving the loss of all or most of the incident particle energy during the traversal of an absorber.

   As the total energy transfer is composed of a multitude of small energy losses, we can apply the central limit theorem and describe the fluctuations by a Gaussian distribution. This case is applicable to non-relativistic particles and is described by the inequality $\kappa > 10$ (i.e. when the mean energy loss in the absorber is greater than the maximum energy transfer in a single collision). **Already seen**

2. Particles traversing thin counters and incident electrons under any conditions.

   The relevant inequalities and distributions are $0.01 < \kappa < 10$, Vavilov distribution, and $\kappa < 0.01$, Landau distribution.

An additional regime is defined by the contribution of the collisions with low energy transfer which can be estimated with the relation ξ / I_0, where I_0 is the mean ionisation potential of the atom. Landau theory assumes that the number of these collisions is high, and consequently, it has a restriction $\xi / I_0 \gg 1$.

Assorbitori sottili: teoria di Landau (cenni)

Valida per **k=** $\xi/E_{max}<0.01$ $\xi=\rho k z^2(Z/A)(1/\beta^2)x$

Assunzioni:

- Perdita di energia piccola rispetto al massimo possibile in una singola collisione ($\xi/E_{max} \ll 1$), in pratica si assume Emax $\rightarrow\infty$

- Perdita di energia grande se paragonata all'energia di legame degli elettroni (elettrone libero). Si trascurano quindi le piccole perdite di energia dovute alle collisioni lontane ($\xi/E_{bind} \gg 1$).
- E sottinteso $\xi/E_{inc} \ll 1$, cosi' che $E_{inc} \approx$ cost.
- For gaseous detectors, typical energy losses are a few keV which is comparable to the binding energies of the inner electrons. In such cases a more sophisticated approach which accounts for atomic energy levels is necessary to accurately simulate data distributions.

Assorbitori sottili: teoria di Landau (cenni)

Con queste assunzioni χ può essere fattorizzata come segue:

$$\chi(W,x) = \frac{1}{\xi} f_L(\lambda)$$

$$\text{con} \quad \lambda = \frac{1}{\xi}\left[W - \bar{W} - \xi \ln\frac{\xi}{\varepsilon'} + 1 - c_E\right];$$

$$\ln\varepsilon' = \ln\frac{\left(1-\beta^2\right)I^2}{2mv^2} + \beta^2;$$

$$c_E = 0.577 \quad \text{(costante di Eulero)}$$

ε' è il taglio sulla minima energia trasferibile nel singolo urto, in accordo all'assunzione $\xi/E_{bind} >> 1$ (e in modo tale che la perdita media sia quella della bethe-bloch.

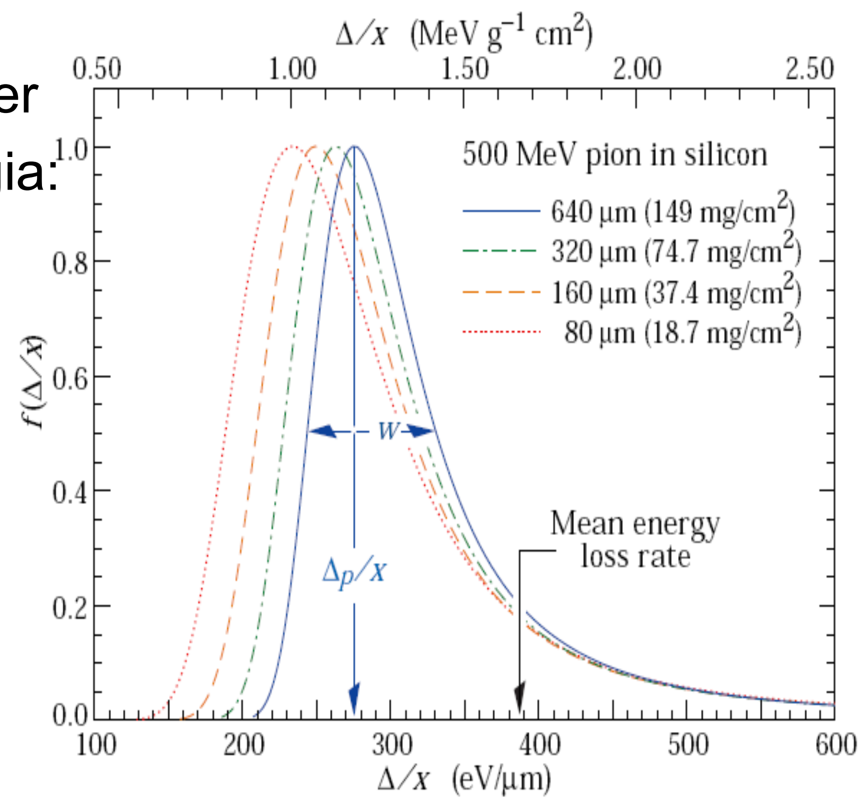
Assorbitori sottili: teoria di Landau (cenni)

La funzione universale $f_L(\lambda)$ può essere espressa come segue:

$$f_L(\lambda) = \frac{1}{\pi}\int_0^{\infty} e^{-u(\ln u + \lambda)} \sin(\pi u)\, du$$

Valutando numericamente $f_L(\lambda)$ si ottiene per il valore più probabile per la perdita di energia:

$$W_{mp} = \xi\left[\ln \xi/_{\varepsilon'} + 0.198 - \delta\right]$$



δ= correzione per effetto densità

e FWHM=4.02ξ

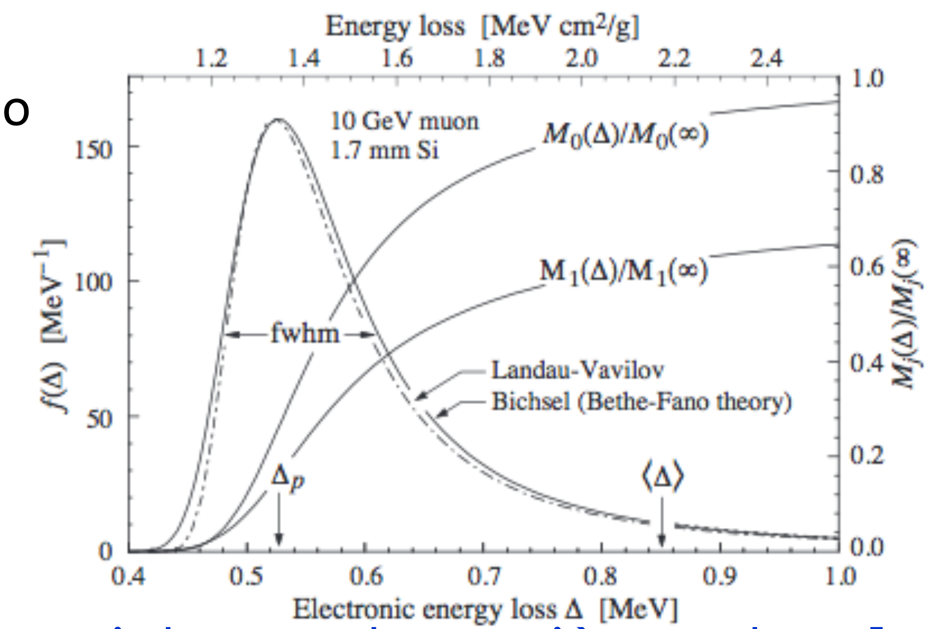
$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z \rho \delta x}{m_e \beta^2 c^2 A} = 153.4 \frac{z^2}{\beta^2}\frac{Z}{A}\rho\delta x \quad \text{keV},$$

Assorbitori sottili: teoria di Vavilov (cenni)

Il modello di Landau non funziona bene in gas (ex. TPC) e silicio (tracciatori) dato che in questi materiali le perdite tipiche di energia sono paragonabili alle energia di legame, per cui l'assunzioni che $\xi/E_{bind} >> 1$ non e' piu' valida

Valida per k>0.01, tiene conto che Emax e' finito

Caratterizzata da code un po' meno asimmetriche.



Osserviamo:

Anche se il limite gaussiano si ha per k≥10 già per k≥1 la distribuzione assomiglia ad una gaussiana.

Vavilov → landau per k → 0 ed ad una gaussiana per k → ∞.

Assorbitori sottili: teoria di Vavilov (cenni)

Vavilov[5] derived a more accurate straggling distribution by introducing the kinematic limit on the maximum transferable energy in a single collision, rather than using $E_{\max} = \infty$. Now we can write[2]:

$$f(\epsilon, \delta s) = \frac{1}{\xi}\phi_v\left(\lambda_v, \kappa, \beta^2\right)$$

where

$$\begin{aligned}
\phi_v\left(\lambda_v, \kappa, \beta^2\right) &= \frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty}\phi(s)\, e^{\lambda s} ds \qquad\qquad c \geq 0 \\
\phi(s) &= \exp\left[\kappa(1+\beta^2\gamma)\right] \exp\left[\psi(s)\right], \\
\psi(s) &= s\ln\kappa + (s+\beta^2\kappa)\left[\ln(s/\kappa) + E_1(s/\kappa)\right] - \kappa e^{-s/\kappa},
\end{aligned}$$

and

$$E_1(z) = \int_{\infty}^{z} t^{-1}e^{-t}dt \qquad \text{(the exponential integral)}$$

Assorbitori sottili: teoria di Vavilov (cenni)

$$\lambda_v = \kappa \left[\frac{\epsilon - \bar{\epsilon}}{\xi} - \gamma' - \beta^2 \right]$$

The Vavilov parameters are simply related to the Landau parameter by $\lambda_L = \lambda_v/\kappa - \ln\kappa$. It can be shown that as $\kappa \to 0$, the distribution of the variable λ_L approaches that of Landau. For $\kappa \leq 0.01$ the two distributions are already practically identical. Contrary to what many textbooks report, the Vavilov distribution *does not* approximate the Landau distribution for small κ, but rather the distribution of λ_L defined above tends to the distribution of the true λ from the Landau density function. Thus the routine **GVAVIV**

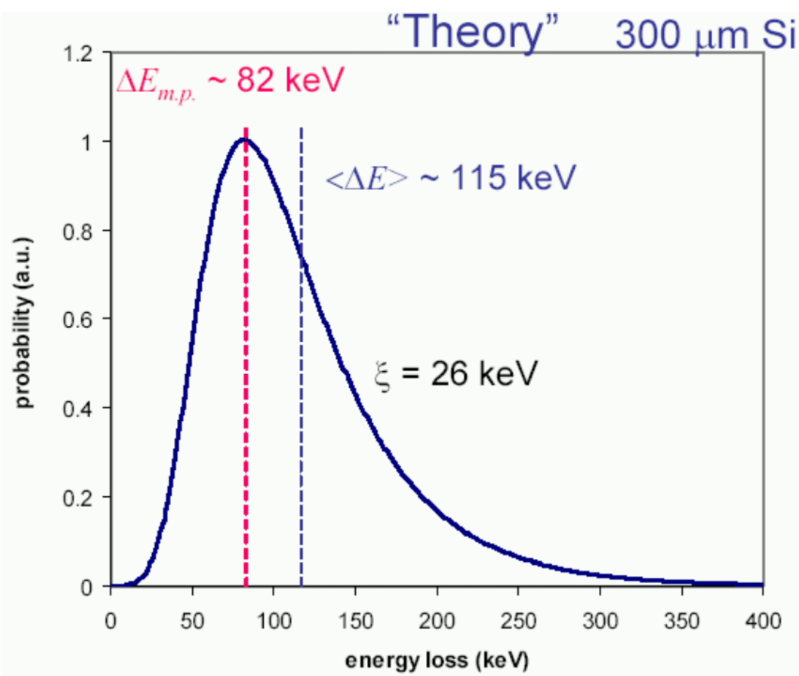
The Landau/Vavilov distribution depends on two parameters:

- the most probable energy loss W_{mp} and
- the FWHM of the distribution ~ξ

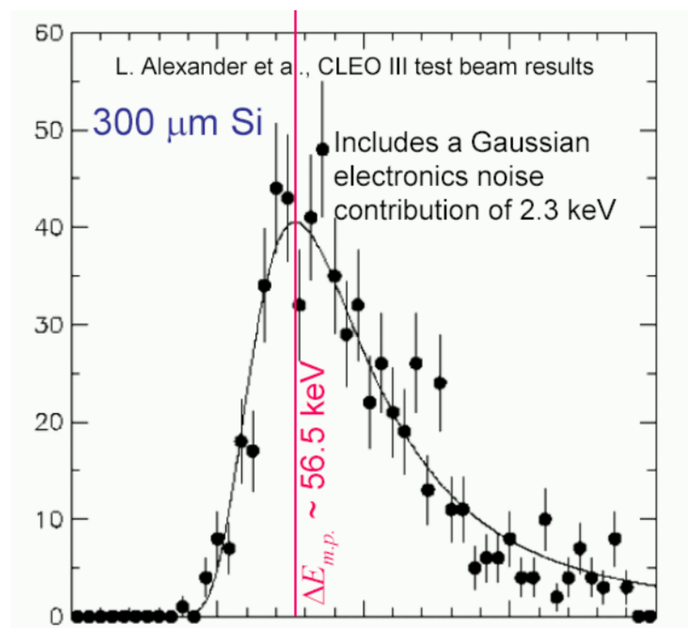
Typical application of the Landau distribution is the CHARGE Identification of the incident particle, since $W_{mp} = W_{mp}(Z_{inc},\beta)$ and $\langle W \rangle = W(Z_{inc},\beta)$

The particle crosses N active layers in which it deposits dEi, i=1,…,N according to the Landau distribution.

Then by a numerical fit, one finds the best parameters W_{mp},FWHM that describes the measured set of dEi. Knowing Wmp, and the width it is possible to get Z_{inc}



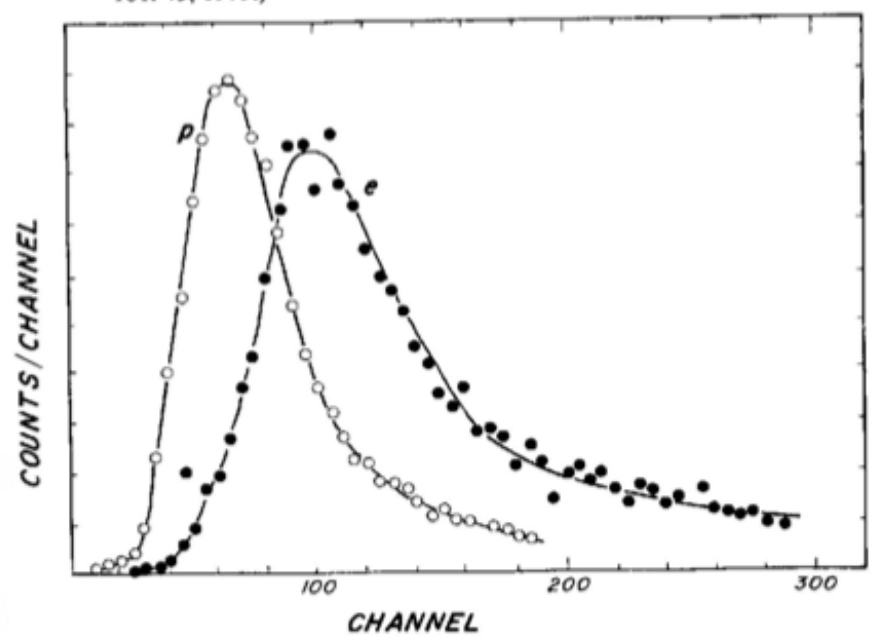
Measurement



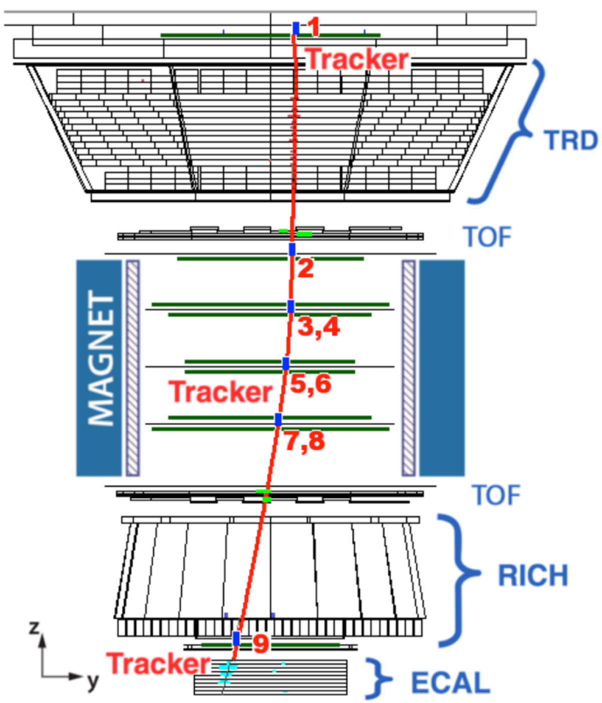
Charge ID

- Conseguenze sperimentali:
  - ID di carica fatta tramite la misura del deposito di energia (noto β e/o p): necessita' di avere piu' misure di dE/dX per campionare la distribuzione di Landau/Vavilov da cui ricavare Wmp e quindi il valore assoluto della carica dato che $W_{mp} \propto Z_{inc}^2$

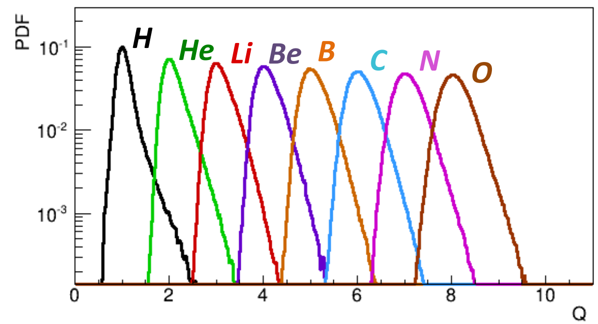
Figure 2.7 Measured pulse height distributions for 3-GeV/*c* protons and 2-GeV/*c* electrons in a 90% Ar + 10% CH_4 gas mixture. (After A. Walenta, J. Fischer, H. Okuno, and C. Wang, Nuc. Instr. Meth. 161: 45, 1979.)



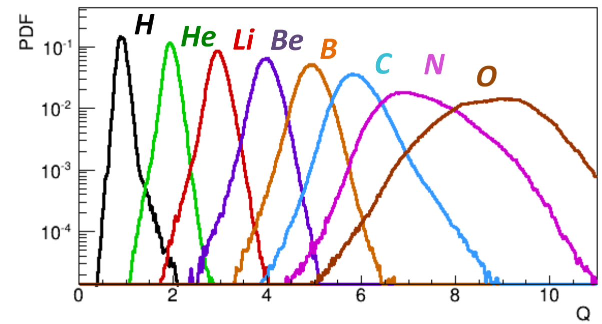
Charge ID: Multiple measurements of charge



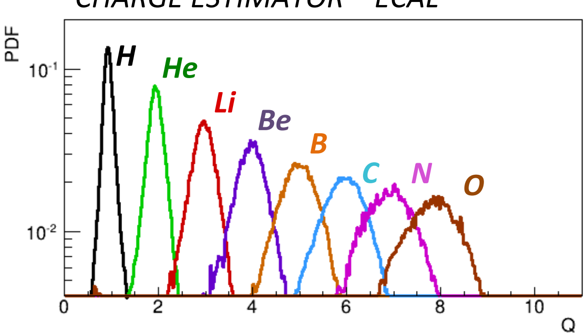
CHARGE ESTIMATOR – TRK LAYER1



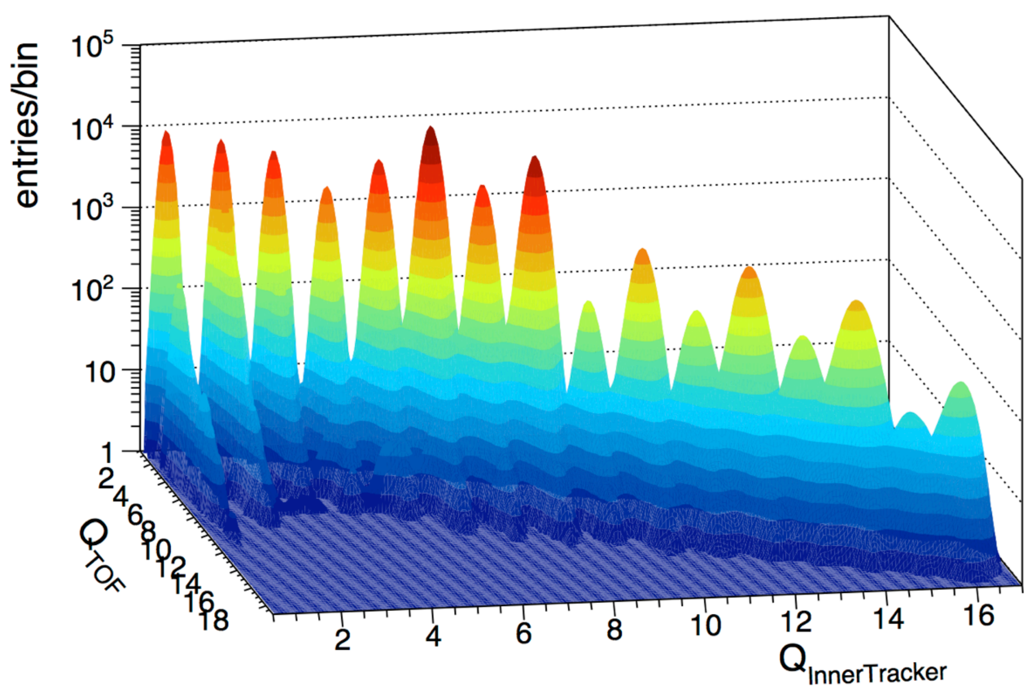
CHARGE ESTIMATOR – TRD



CHARGE ESTIMATOR – ECAL



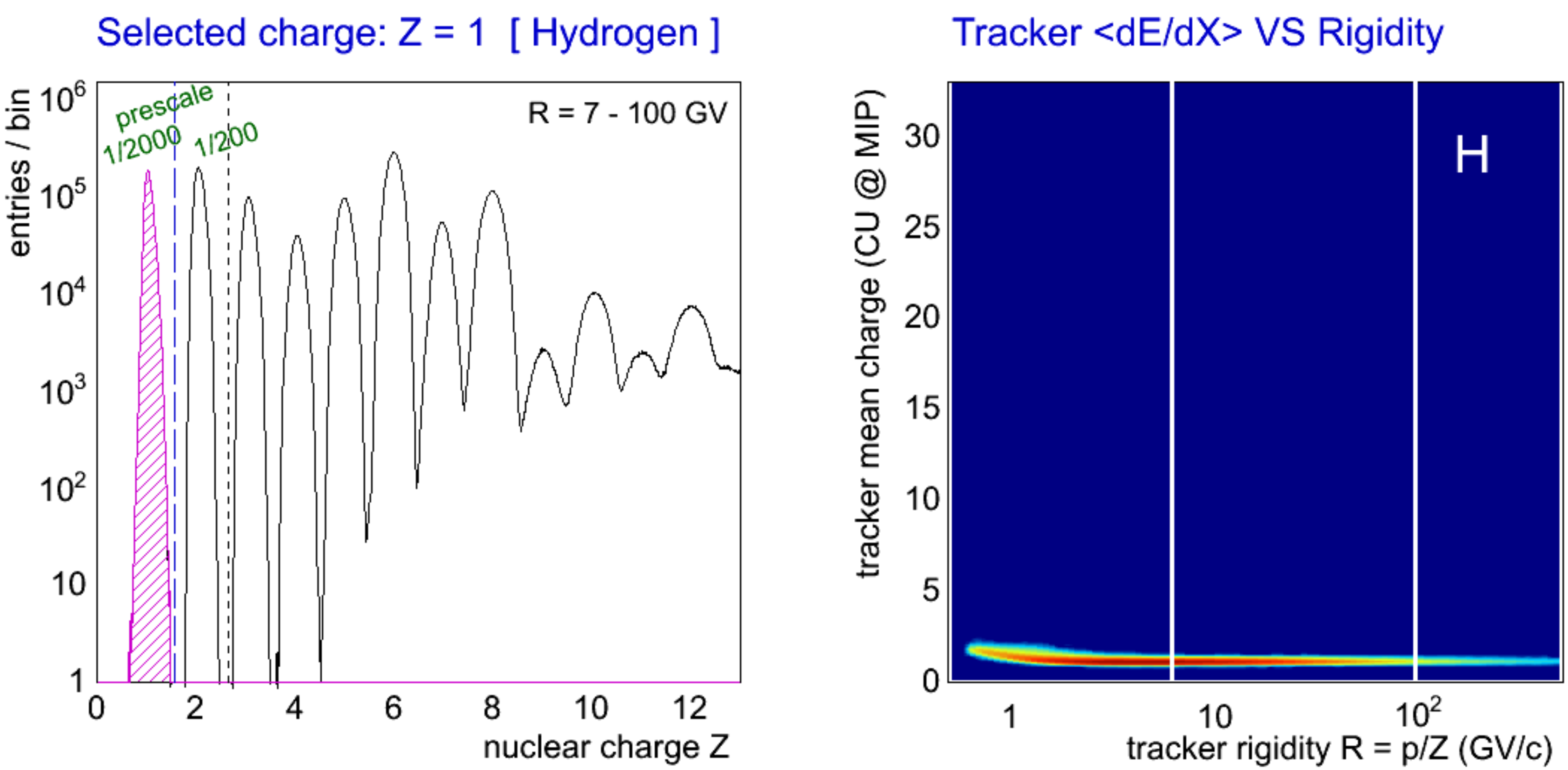
TOF & TRK charge estimators



Charge measurement using Tracker

Identification of Light Cosmic Ray Nuclei

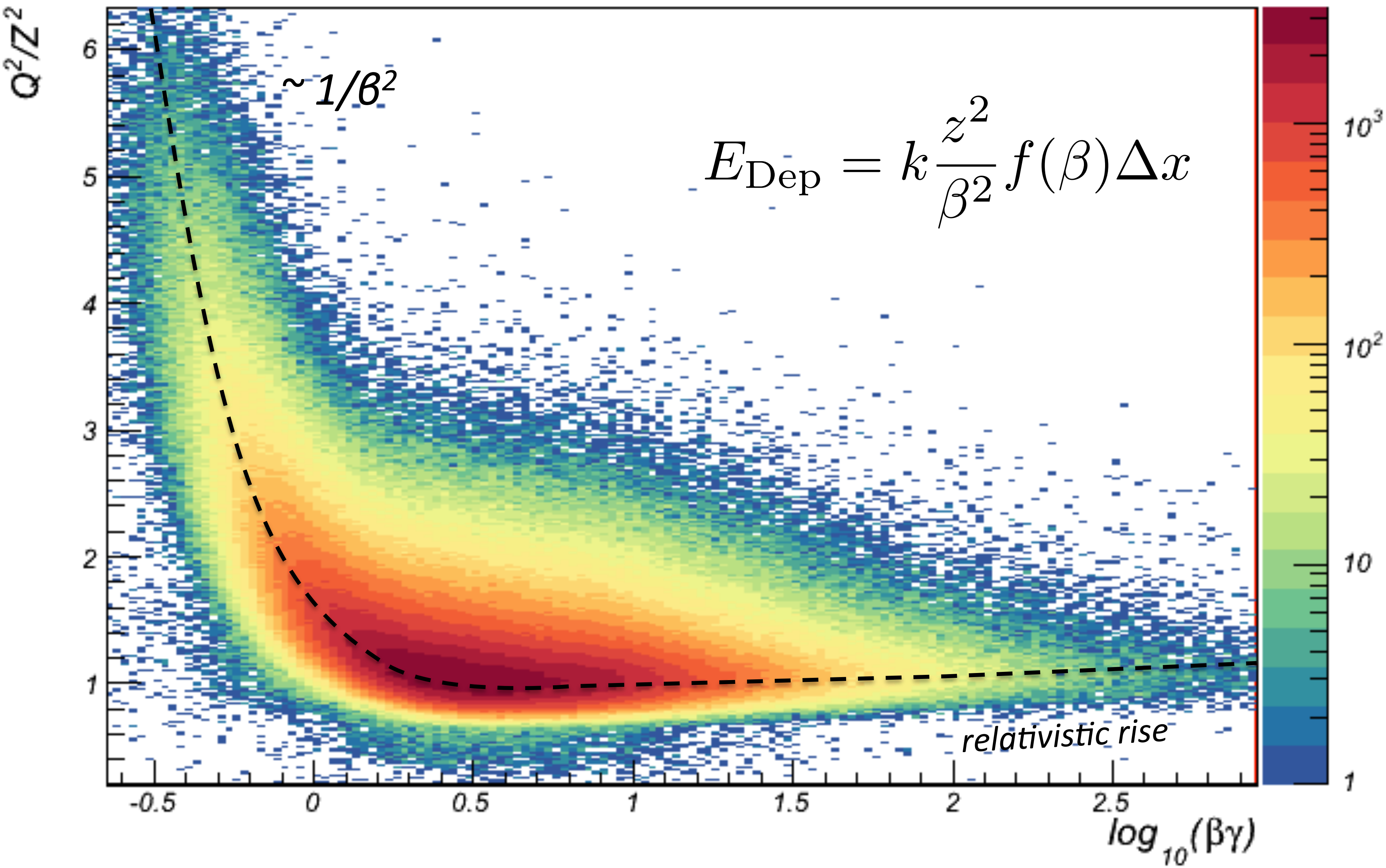
with the AMS Silicon Tracker



animated slide – set full screen

AMS Collaboration

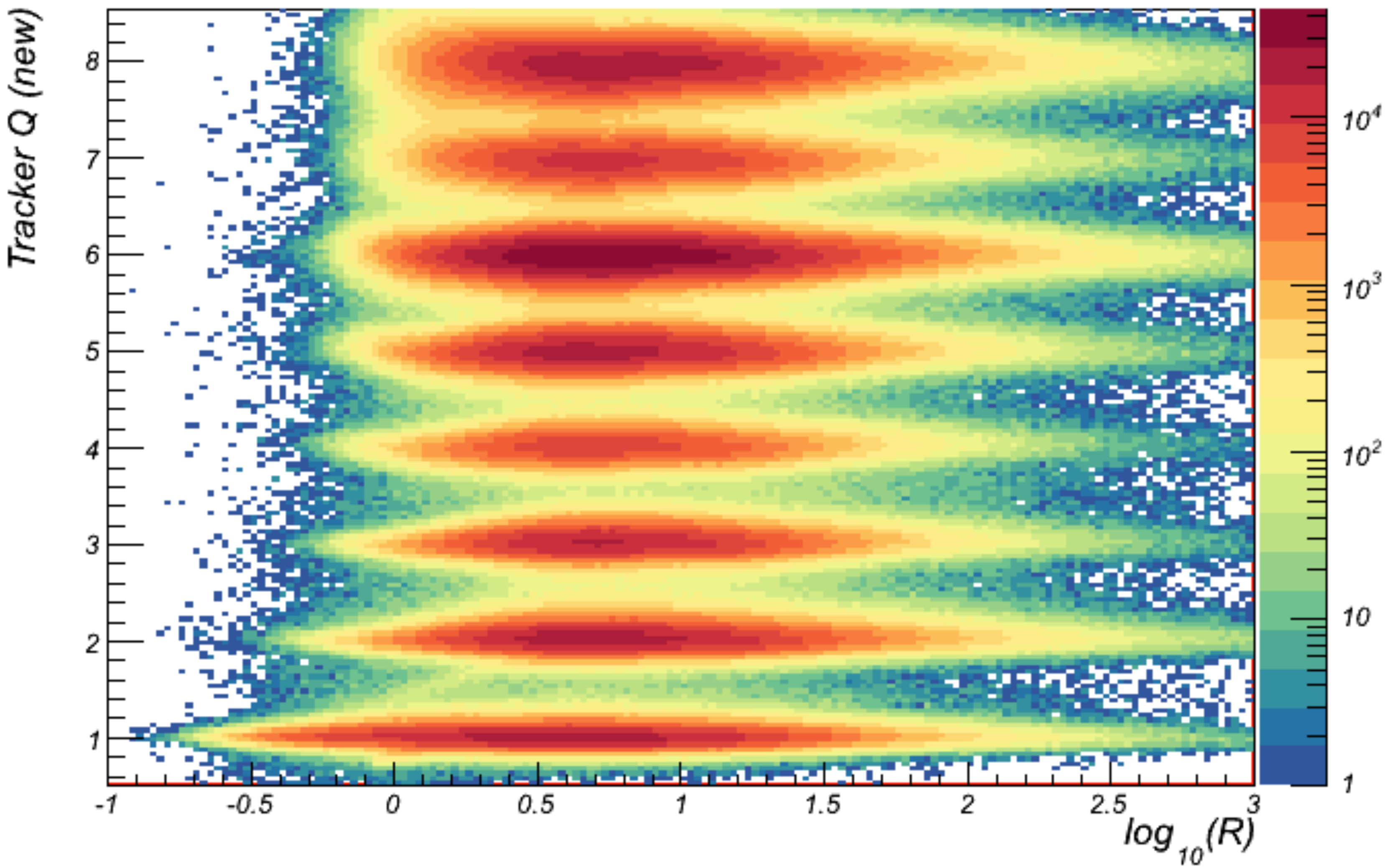
Bethe-Bloch for Protons



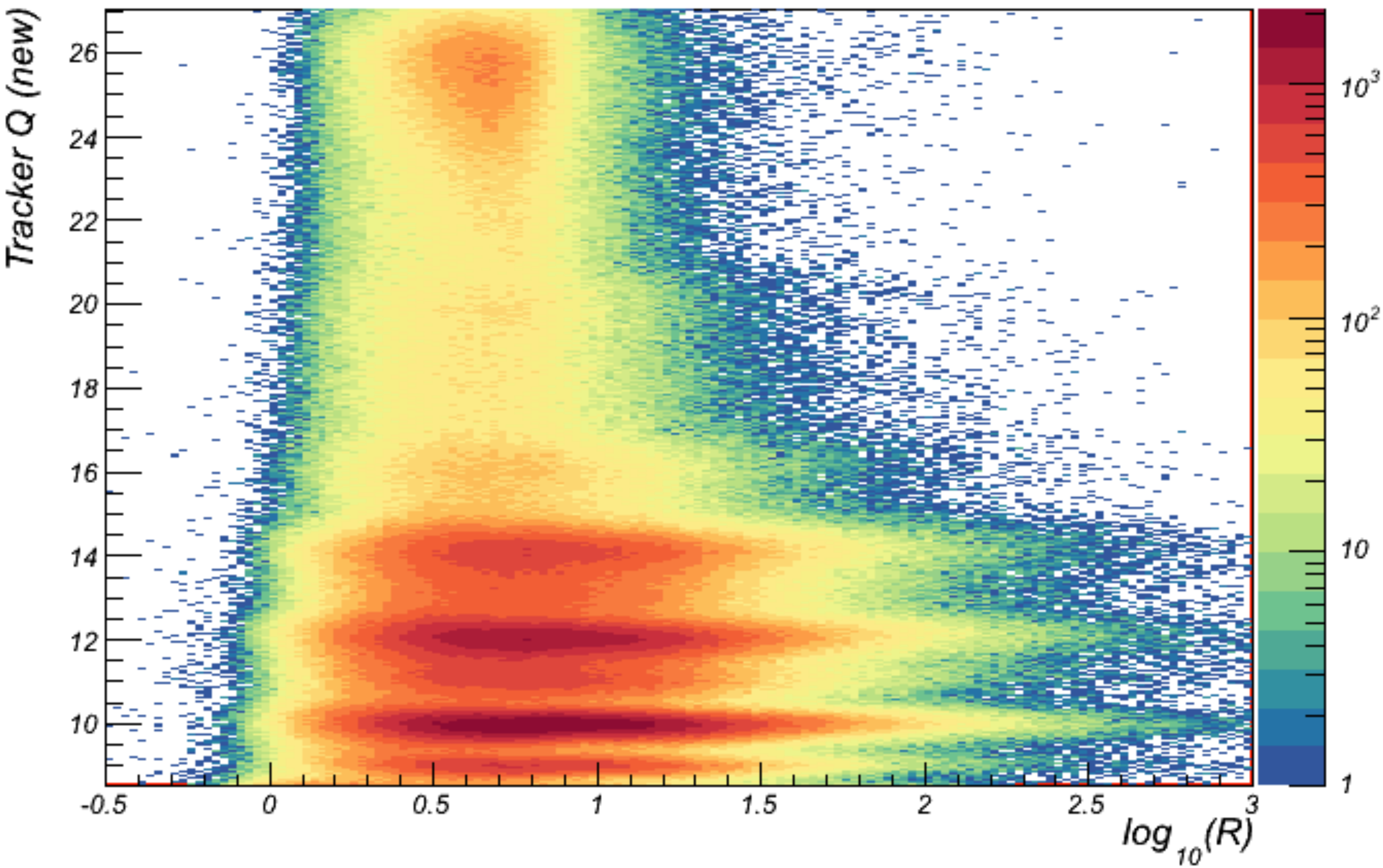
Velocity corrections to the measured dE/dX

To eliminate the velocity dependence of the charge, usually a velocity correction is applied to the measured energy depositions

$$\Delta E_{corr} = \Delta E_{meas} * \beta/F(\beta)$$



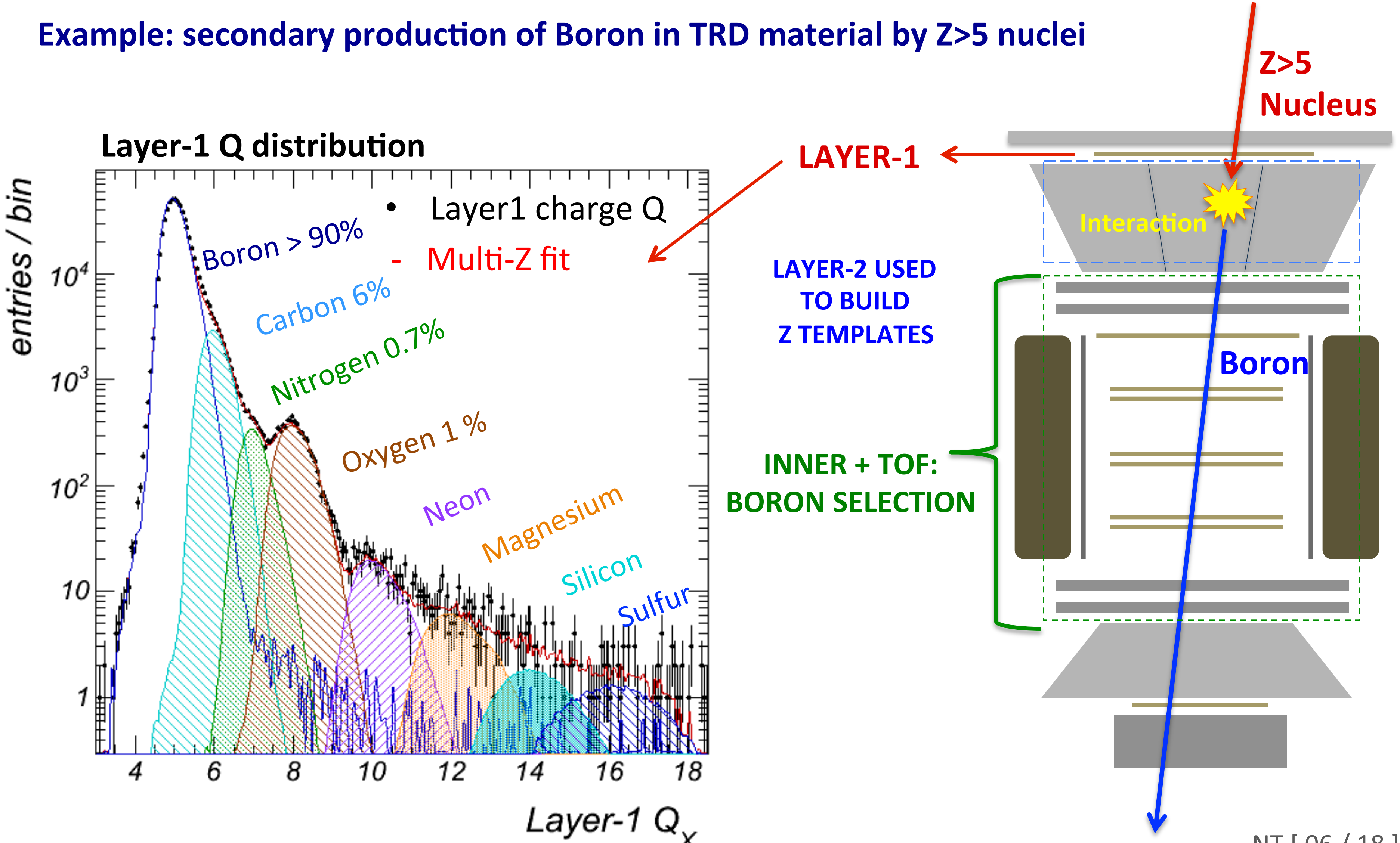
After Rigidity Correction (high-Z)



Fragmentation studies in the detector

Redundancy in Z-measurements allows us to study different fragmentation processes appearing at different levels in the detector.

Example: secondary production of Boron in TRD material by Z>5 nuclei

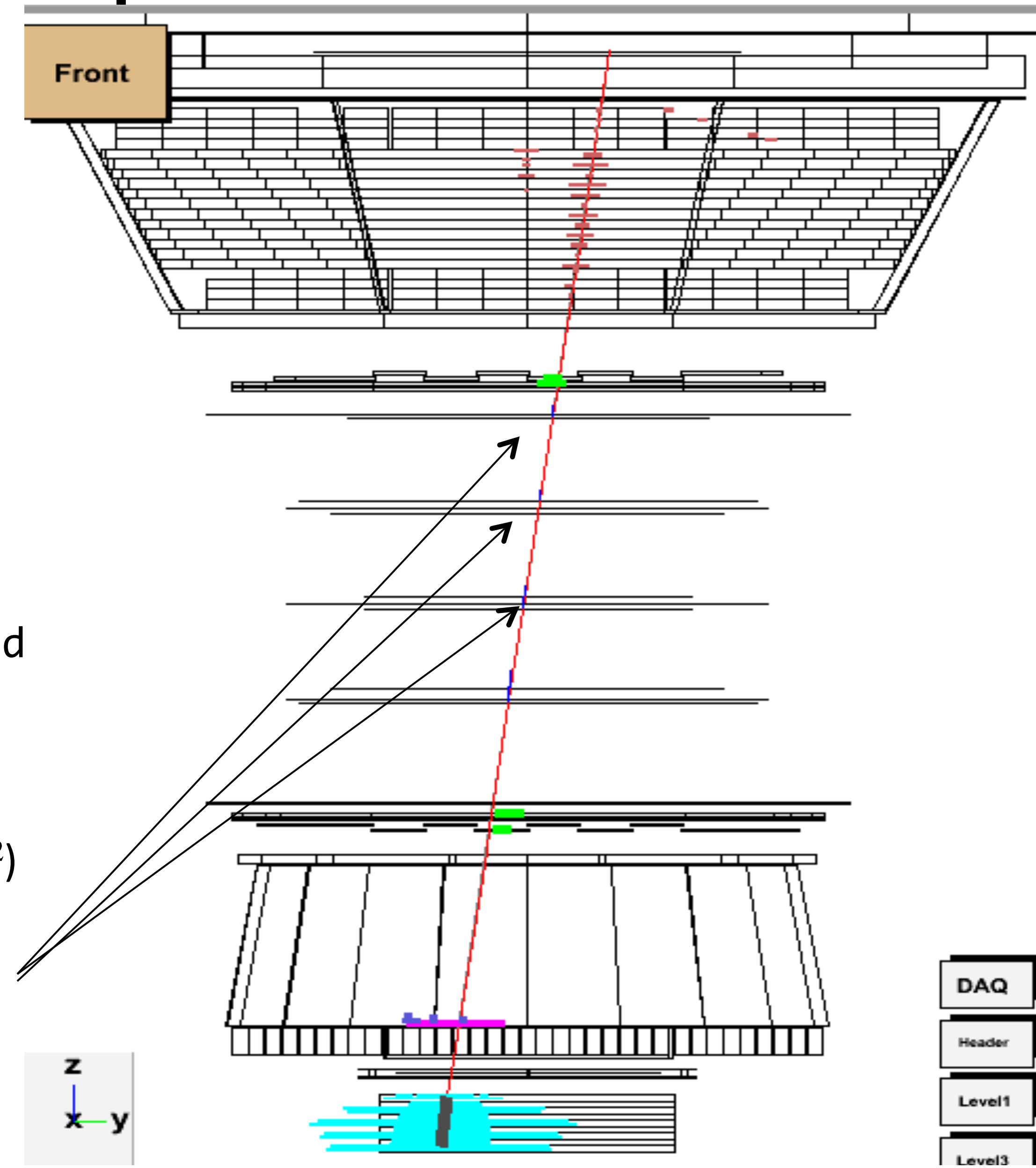


dE/dX and particle ID

dE/dX can (and it IS used) to signal the passage of a charged particle through an active medium, as the active planes (where the energy is deposited and converted in an electrical signal) of a tracking device (eg immersed in a magnetic field)

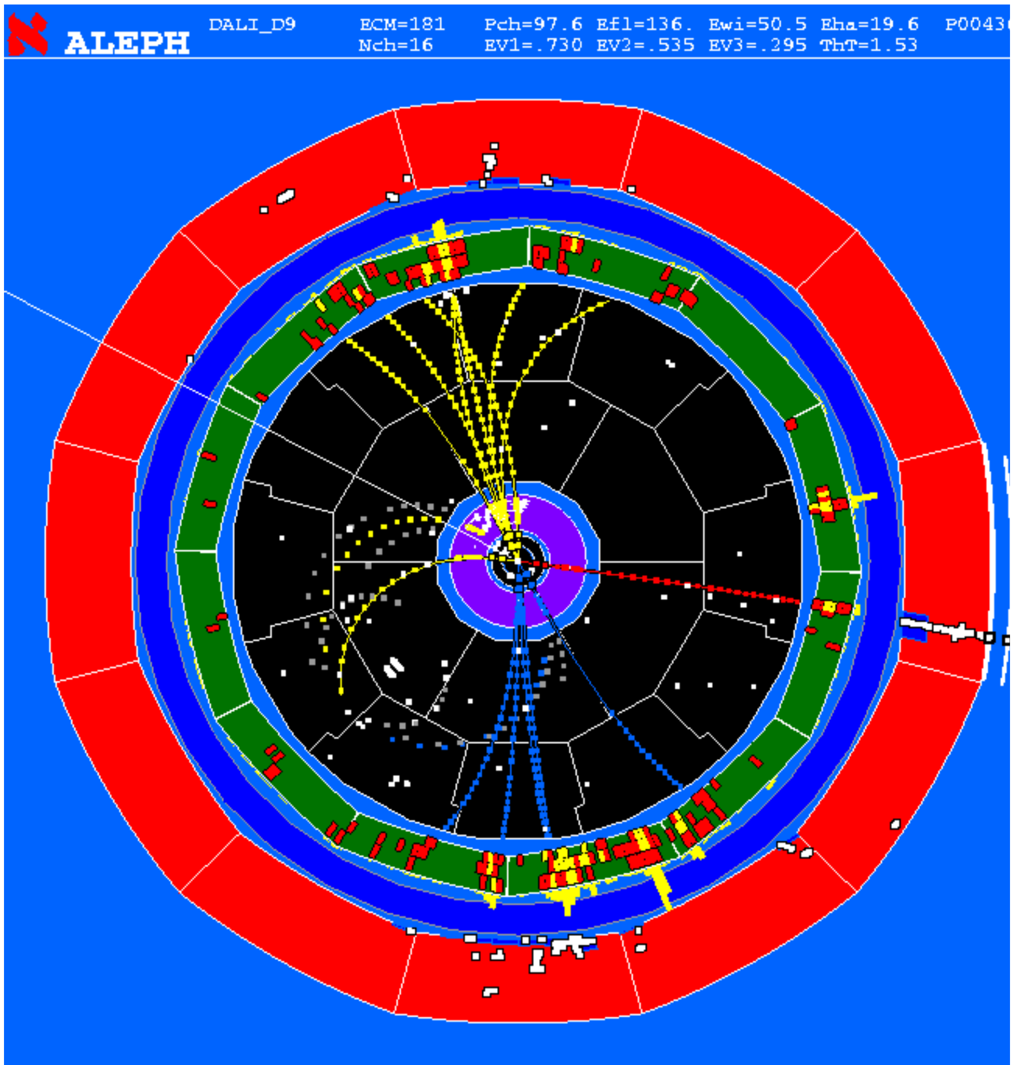
Each energy deposit is ∝ z^2 and therefore they can be used to measure the particle $|z|$, provided that an indipendent measurement of β is available (since energy deposit ∝ z^2/β^2)

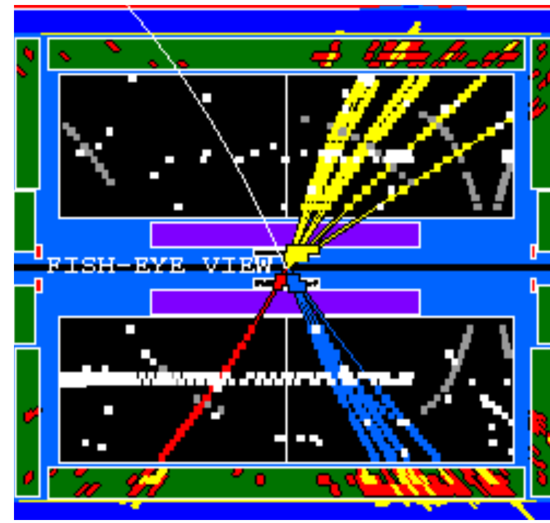
AMS Event display for a high energy electron: the passage at a given point is signalled by the energy deposits in the tracking planes



A W^+W^- decay in ALEPH (a LEP experiment)

e^+e^- ($\sqrt{s}$=181 GeV)
→ W^+W^- → $qq\mu\nu_\mu$
→ 2 hadronic jets + μ + missing momentum

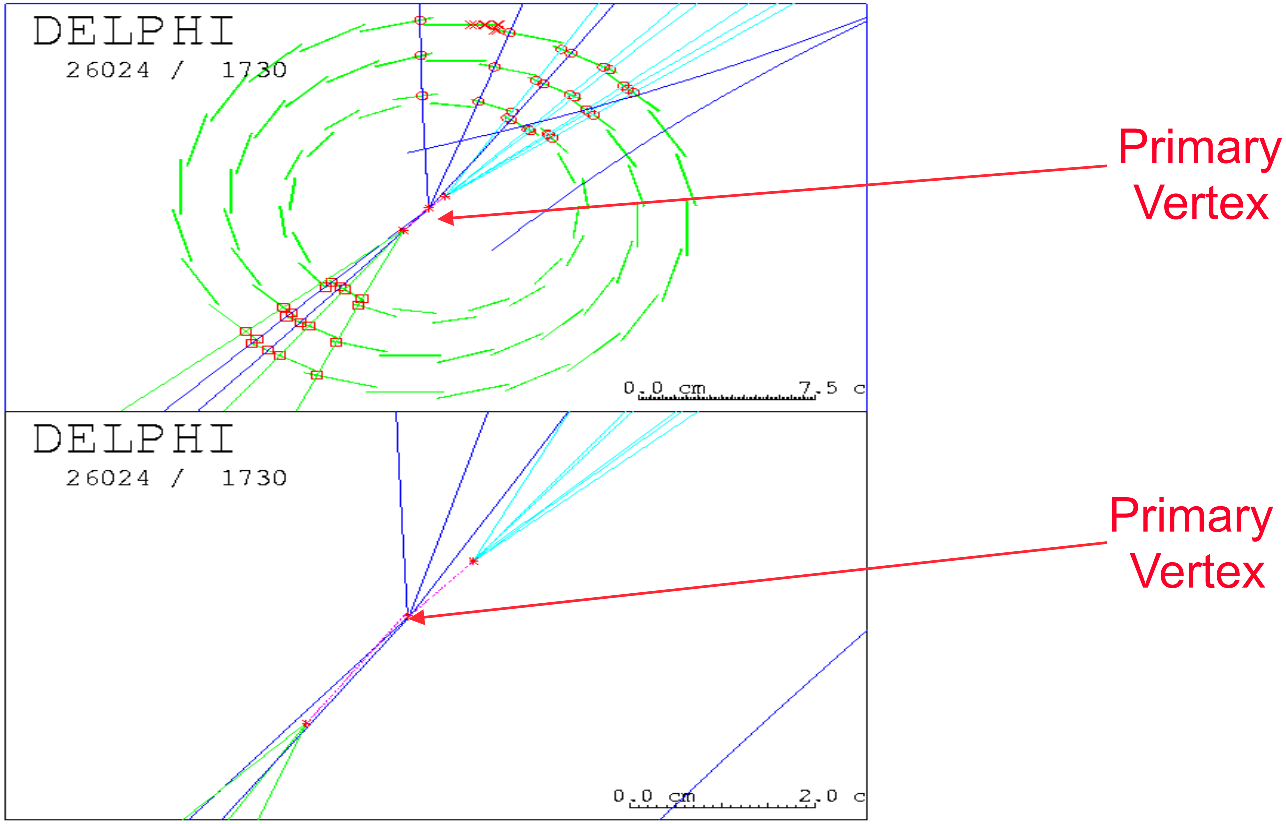


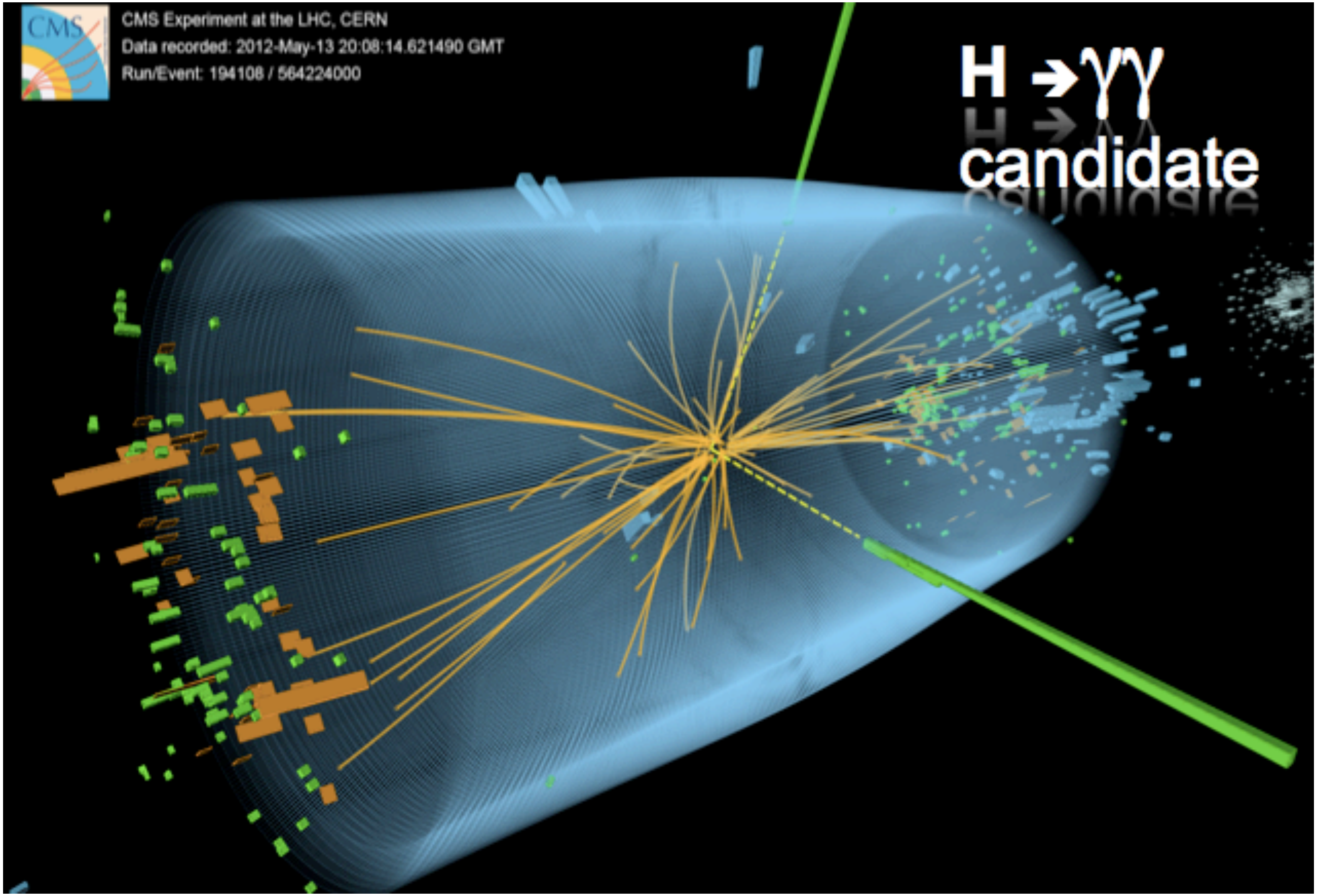


Most of position sensitive detectors are based on the energy deposit in the active medium by ionization, as tracking detectors used to reconstruct trajectories the particles

Reconstructed B-mesons in the DELPHI micro vertex detector

$\tau_B \approx 1.6$ ps $\quad l = c\tau\gamma \approx 500\ \mu m \cdot \gamma$

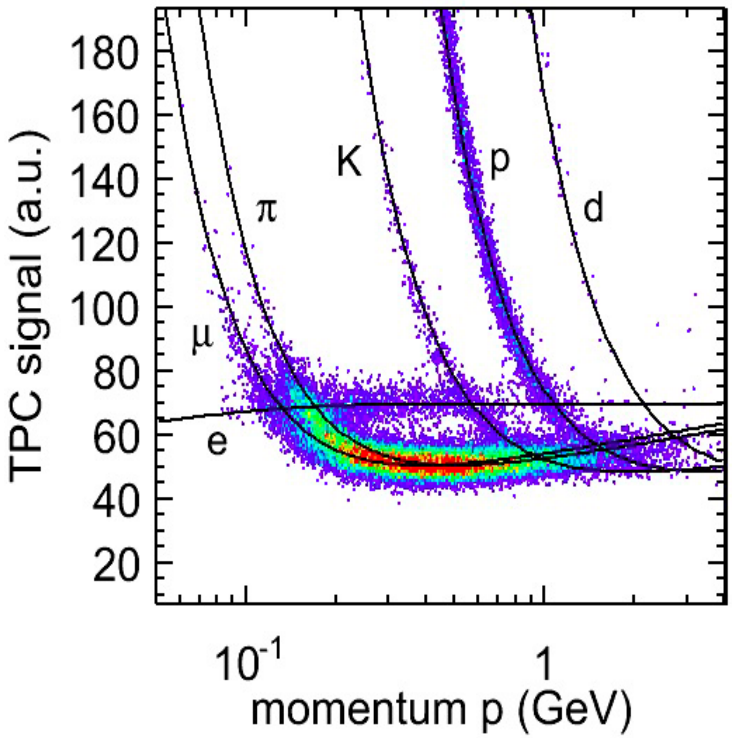


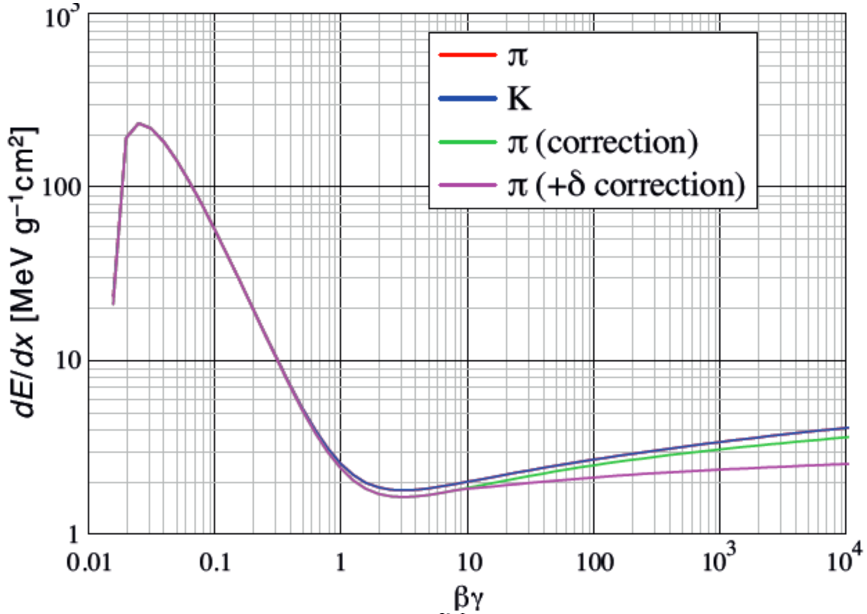

CMS Experiment at the LHC, CERN
Data recorded: 2012-May-13 20:08:14.621490 GMT
Run/Event: 194108 / 564224000
H → γγ
candidate

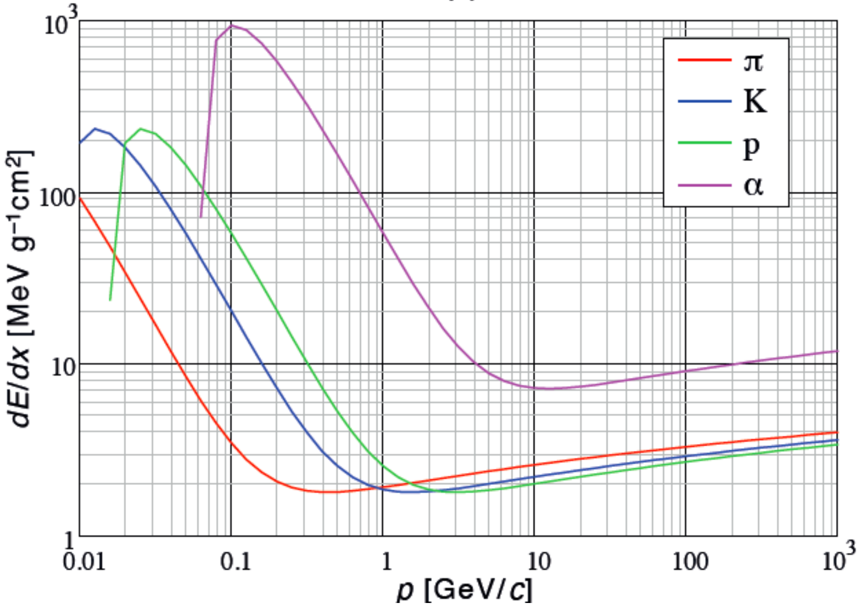
dE/dx and particle ID

For a given medium, dE/dX depends only on β

But for a given momentum, $\beta\gamma$ and hence dE/dX are different for particles with different masses:





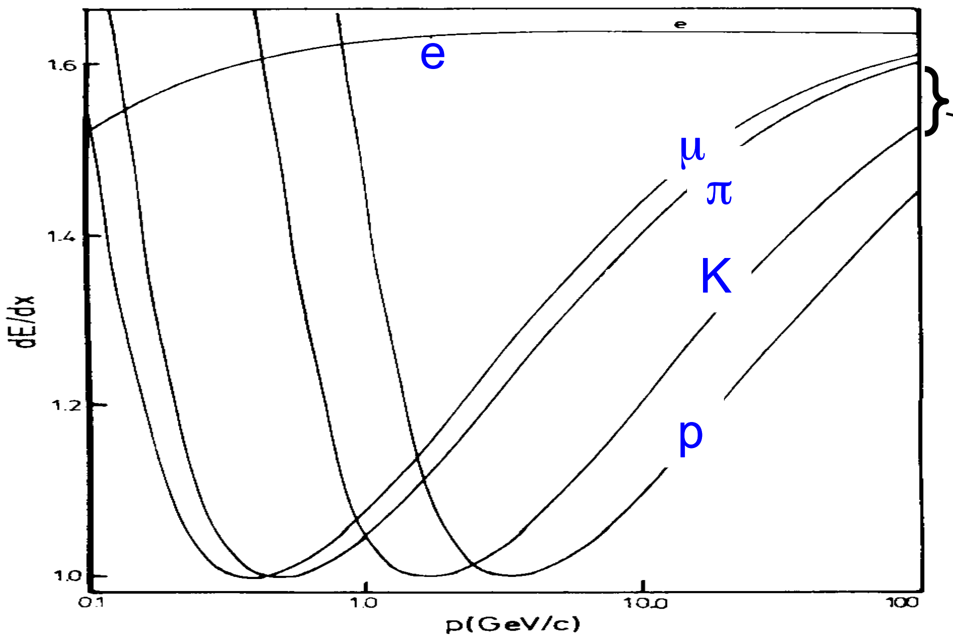


Particle ID using the specific energy loss dE/dx

$$p = m_0 \beta\gamma c$$

$$\frac{dE}{dx} \propto \frac{1}{\beta^2}\ln\left(\beta^2\gamma^2\right)$$

Simultaneous measurement of p and dE/dx defines mass m, hence the particle identity



π/K separation (2σ) requires a dE/dx resolution of < 5%

Average energy loss for e,μ,π,K,p in 80/20 Ar/CH_4 (NTP)

(J.N. Marx, Physics today, Oct.78)

But: Large fluctuations + Landau tails !

When β→1, there only a ln(βγ) dependence, all the particles tend to have similar loss: difficult to separate particles from single dE/dX → very good resolution both in p and dE/dX measurements needed

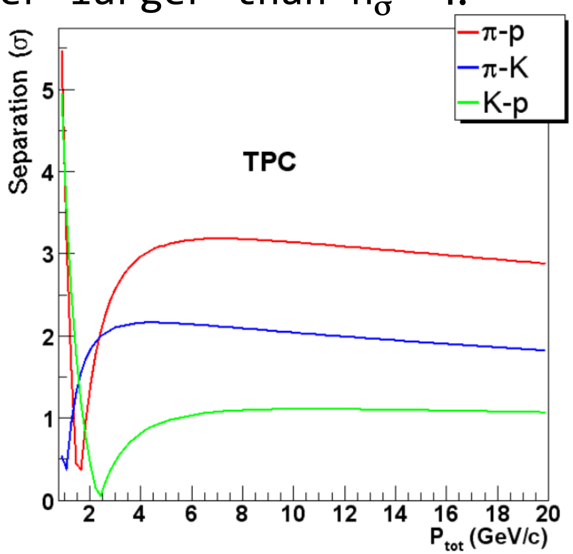
dE/dx and particle ID

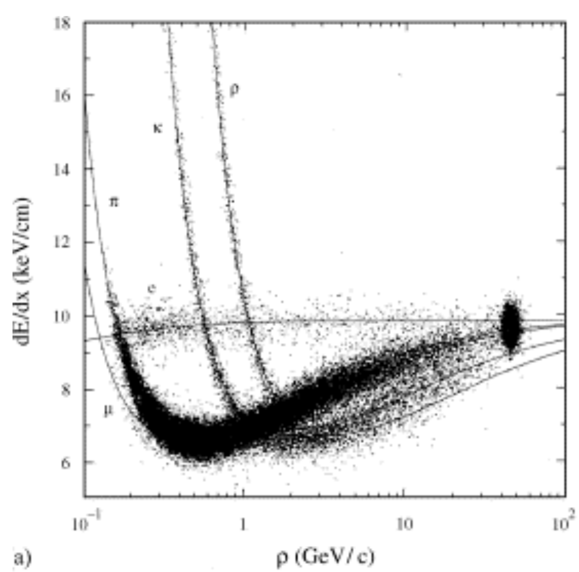
Any PID technique is based on the fact that the detector response is different "enough" to different types of particles. Therefore it is worthwhile to introduce the concept of "separation power" of a PID system, defined by the significance of the measurement of the detector response, herein generically indicated with S. The detector response could actually be, according to the method employed, either time (in the case of TOF detectors), the ionization energy loss or the Cherenkov angle.

If S_A and S_B are the mean values of such a quantity measured for particles of type A and B, respectively, and σ_{AB} is the average value between the standard deviations of the measured distributions, the separation power $\mathbf{n_\sigma}$ is given by:

$$\mathbf{n_\sigma = (S_A - S_B)/\sigma_{AB}}$$

By assuming that the measured quantities S_A and S_B are distributed according to a Gaussian, a particle contamination smaller than 1% requires a separation power larger than $n_\sigma = 4$.

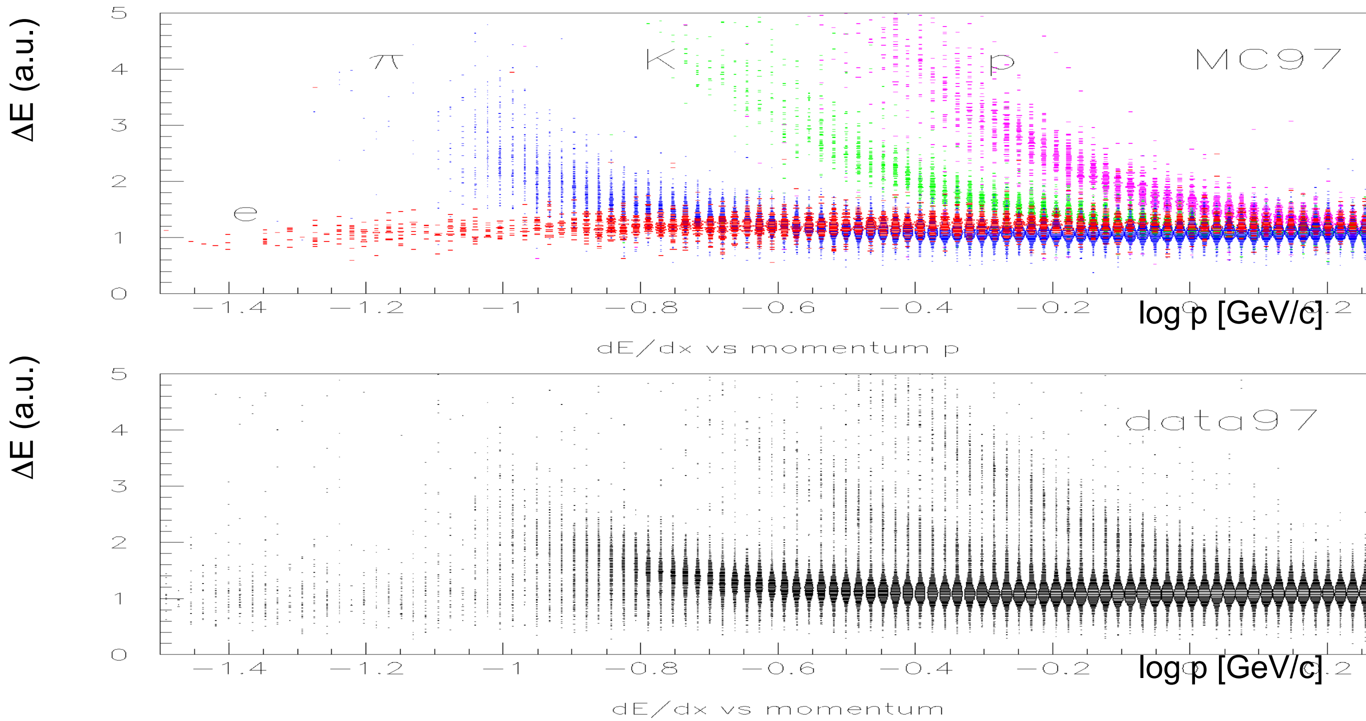




dE/dx and particle ID

dE/dx can also be used in
Silicon detectors

Example DELPHI microvertex detector (3 x 300 μm Silicon)



Time of flight

Particle ID using Time Of Flight (TOF)

$$t = \frac{L}{\beta c}$$

$$(p = m_0 \beta \gamma)$$

Combine TOF with momentum measurement

$$m = p\sqrt{\frac{c^2 t^2}{L^2} - 1}$$

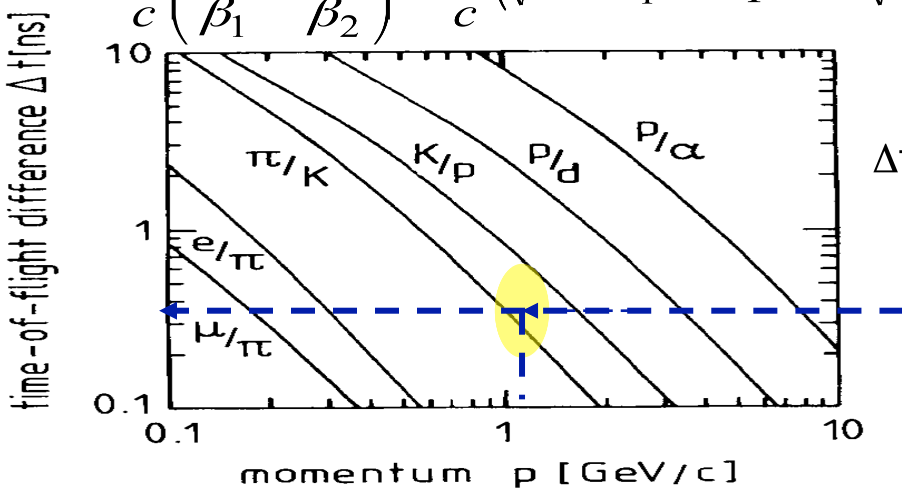
Mass resolution

$$\frac{dm}{m} = \frac{dp}{p} \oplus \gamma^2 \left(\frac{dt}{t} + \frac{dL}{L} \right)$$

Quadratic sum

TOF difference of 2 particles at a given momentum

$$\Delta t = \frac{L}{c}\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) = \frac{L}{c}\left(\sqrt{1 + m_1^2 c^2 / p^2} - \sqrt{1 + m_2^2 c^2 / p^2}\right) \approx \frac{Lc}{2p^2}\left(m_1^2 - m_2^2\right)$$



Δt for L = 1 m path length

σ_t = 300 ps

π/K separation up to 1 GeV/c

Collisioni fra particelle cariche: Multiple Scattering

Riprendiamo una slide gia' vista

Una particella di massa >> dell'elettrone in moto (veloce) in un materiale collide con:

- **Nuclei** → poca energia rilasciata al nucleo, ma angolo di scattering della particella incidente significativo.
- **Elettroni atomici** → gli elettroni (leggeri) si prendono energia dalla particella incidente, ma questa fa uno scattering trascurabile → Bethe-Bloch, Landau,…

Fatto

Collisioni fra particelle cariche: Multiple Scattering

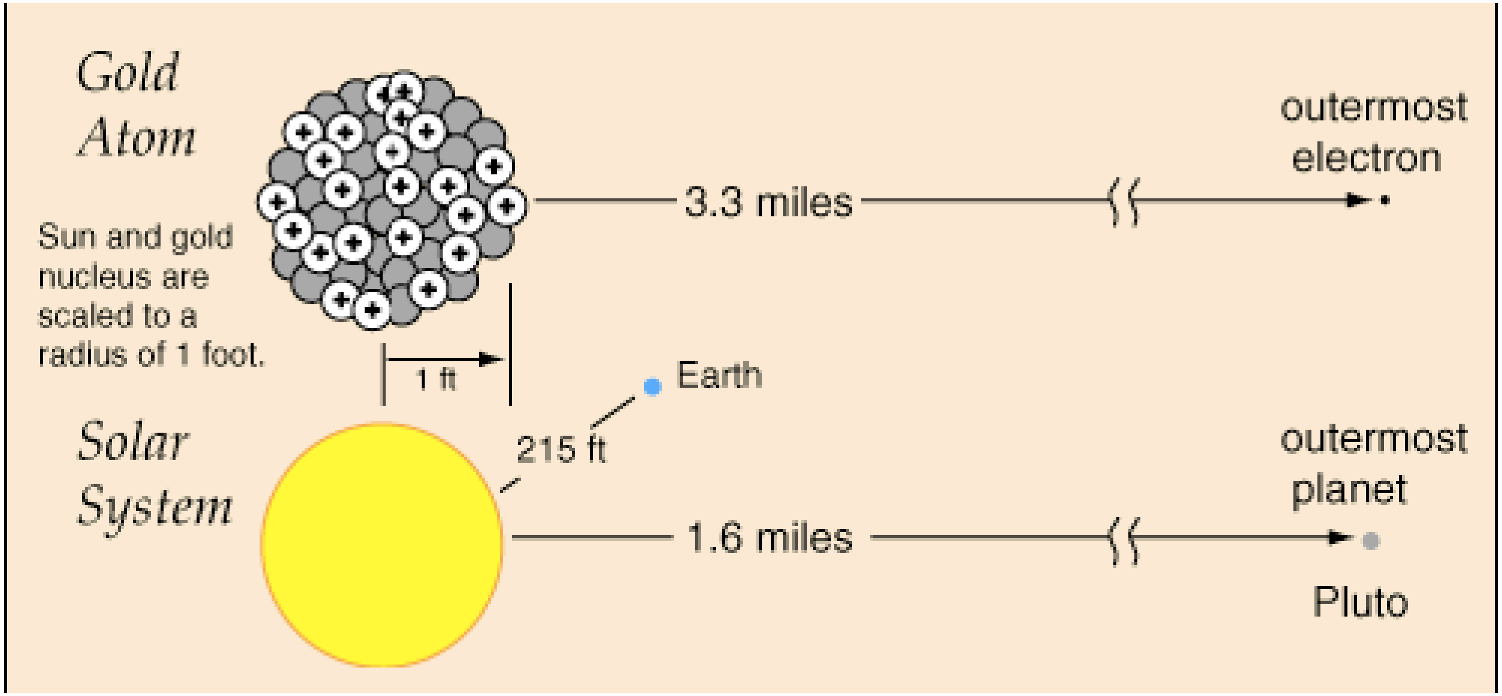
- Quindi quando una particella attraversa un mezzo continuo siamo interessati a conoscere:
  - quanta E deposita → Bethe-Bloch
  - **che deviazione subisce**
- Tratteremo i due processi in maniera indipendente (anche se perdite di energia e deviazione NON sono in realta' indipendenti)

Multiple Scattering

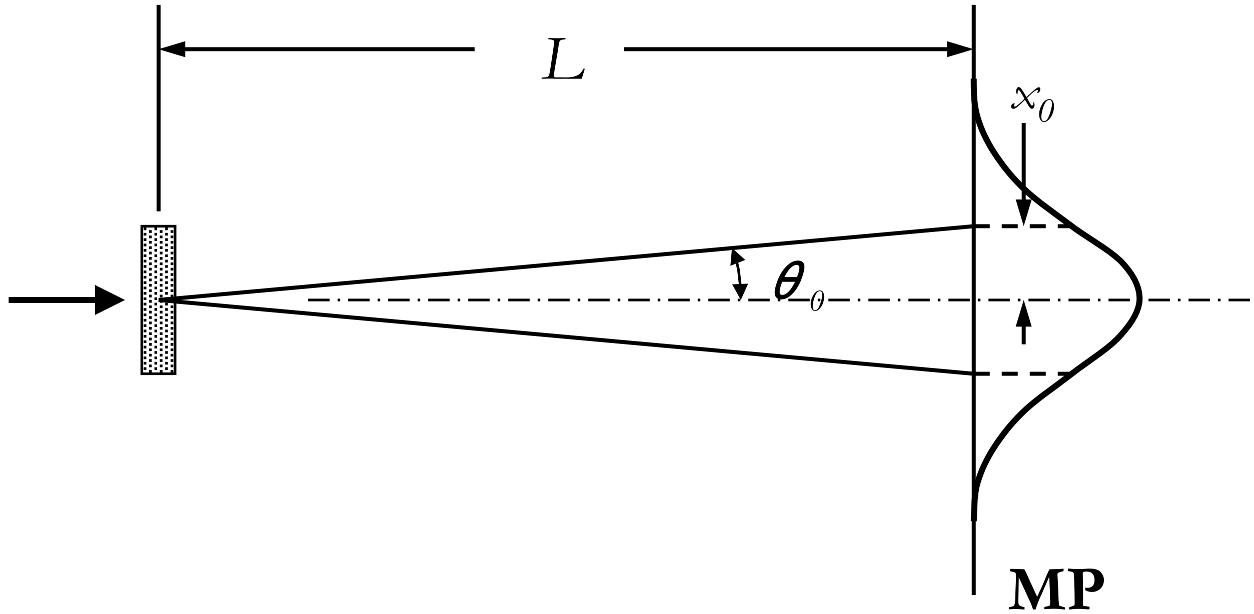
- Looking at dE/dx from ionization, ignore nuclei.
  - Energy transfer small compared to scattering from (lighter) electrons.
- However, scattering from nuclei does change the **direction** of the particles momentum, if not its **magnitude**.
  - Deflection of particle's path limits the accuracy with which the curvature in a magnetic field can be determined, and hence the momentum measured.

Size of Nuclei

Atomic radius of aluminum = 1.3 x 10^{-10} m
Nuclear radius aluminum = 3.6 x 10^{-15} m
The nucleus occupies a tiny fraction of atom volume.



Multiple Scattering



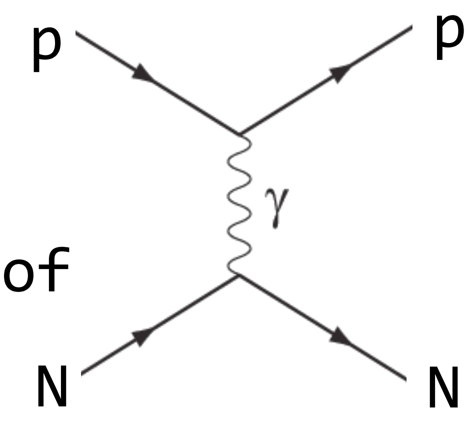
An initially monodirectional beam of particles shows a spread in directions after it crossed a slab of material. This is known as direction straggling or multiple scattering.

When particles, as protons, pass through a slab of material they suffer many collisions with atomic nuclei. The statistical outcome is a *multiple scattering angle* whose distribution is approximately Gaussian. The width parameter of the angular distribution is θ_0.

The task of multiple scattering theory is to predict θ_0 given the scattering material and thickness, and the incident particle energy.

Multiple Scattering

Basic process is the scattering p + N → p + N with a probability to scatter in $d\Omega$ around θ, ϕ as $dP = (d\sigma/d\Omega)d\Omega$



Each collision deviates the incident particle direction

We want to know the cumulative effect of many collisions, ie the amount of deviation after crossing a given thickness of material

As usual, two approaches are possible:

classical and quantistic

Classical Coulomb cross section

- Il primo passo e’ conoscere la probabilita' di diffusione ad un dato angolo in una collisione singola su un nucleo
- Quindi derivare l’effetto statistico delle collisioni dopo un dato spessore x di materiale attraversato

- For **repulsive scattering** (what we mainly look at here) the situation is as shown in the figure:

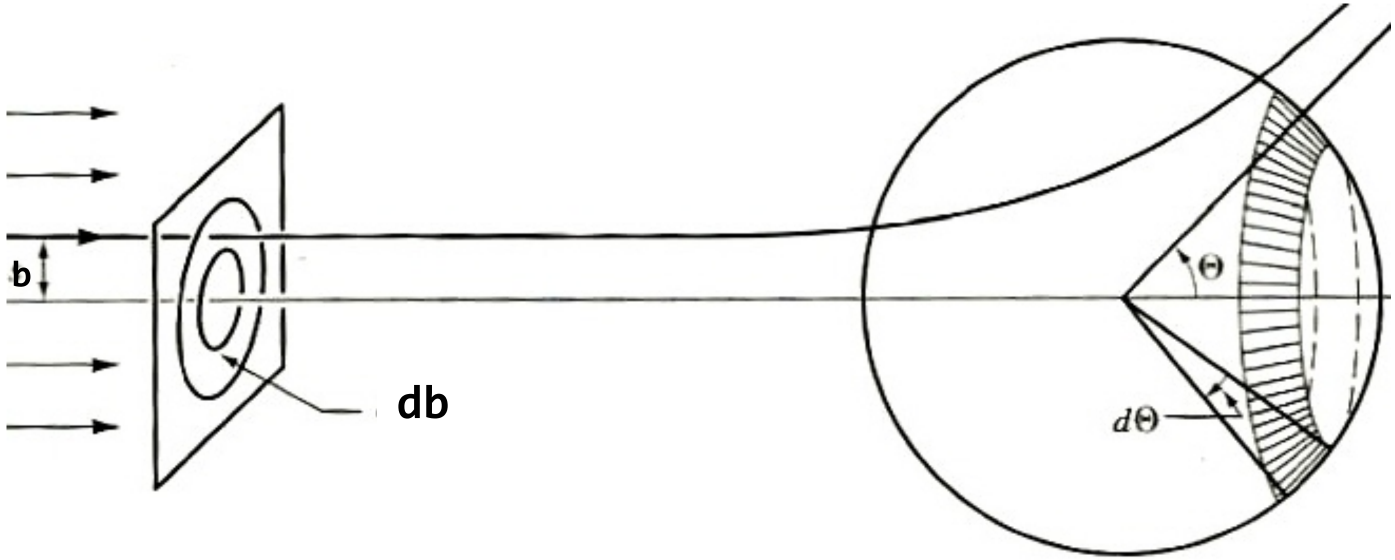


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- ***Define*: Differential Cross Section for Scattering** in a given direction (into a given solid angle **dΩ**):

$\sigma(\Omega)d\Omega \equiv (N_s/I)$. With **I** = incident intensity

N_s = # particles/time scattered into solid angle **dΩ**

- **Scattering Cross Section:**

$$\sigma(\Omega)d\Omega \equiv (N_s/I)$$

I = incident intensity

N_s = # particles/time

scattered into angle $d\Omega$

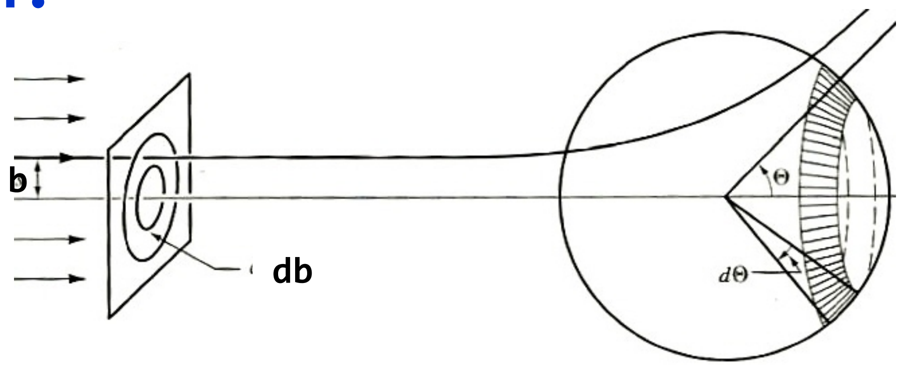


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- In general, the solid angle Ω depends on the spherical angles Θ, Φ. However, for central forces, there must be **symmetry** about the axis of the incident beam

$\Rightarrow \sigma(\Omega)\ (\equiv \sigma(\Theta))$ is independent of azimuth angle Φ

$$\Rightarrow d\Omega \equiv 2\pi \sin\Theta d\Theta\ ,\ \ \sigma(\Omega)d\Omega \equiv 2\pi\ \sigma(\Theta) \sin\Theta d\Theta,$$

$\Theta \equiv$ Angle between incident & scattered beams, as in the figure.

$\sigma \equiv$ "***cross section***". It has units of area

Also called the differential cross section.

- As in all Central Force problems, for a given particle, the orbit, & thus the amount of scattering,

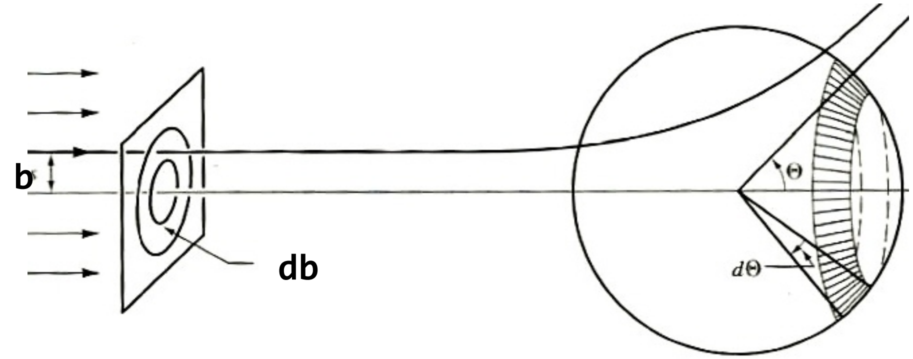


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

is determined by the energy **E** & the angular momentum ℓ

- ***Define*: Impact parameter**, **b**, & express the angular momentum ℓ in terms of **E** & **b**.
- Impact parameter **b** ≡ the ⊥ distance between the center of force & the incident beam velocity (fig).
- ***GOAL:*** **Given** the energy **E**, the impact parameter **b**, & the force **f(r)**, what is the cross section **σ(Θ)**?

- **Beam**, intensity **I**. Particles, mass **m**, incident speed (at **r** → ∞) = v_0.

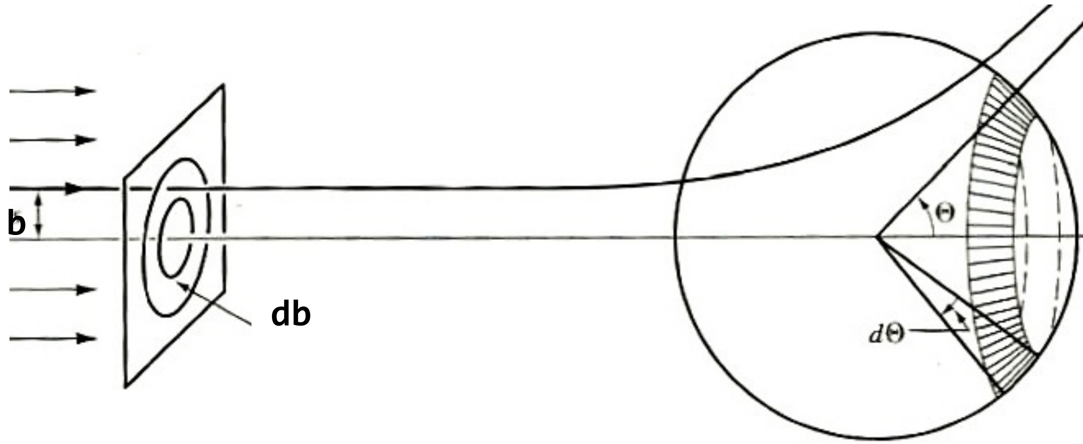


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- **Energy conservation:**

$$E = T + V = \tfrac{1}{2} mv^2 + V(r) = (\tfrac{1}{2})mv_0^2 + V(r\to\infty)$$

- Assume $V(r\to\infty) = 0 \Rightarrow E = \tfrac{1}{2} m(v_0)^2$

$$\Rightarrow v_0 = (2E/m)^{1/2}$$

- **Angular momentum:** $\ell \equiv mv_0 b \equiv b(2mE)^{1/2}$

ℓ (vector) is conserved for central forces

- Angular momentum $\ell \equiv mv_0 b \equiv b(2mE)^{½}$

  Incident speed v_0.

- $N_s \equiv$ # particles scattered into solid angle $d\Omega$ between Θ & $\Theta + d\Theta$. Cross section definition

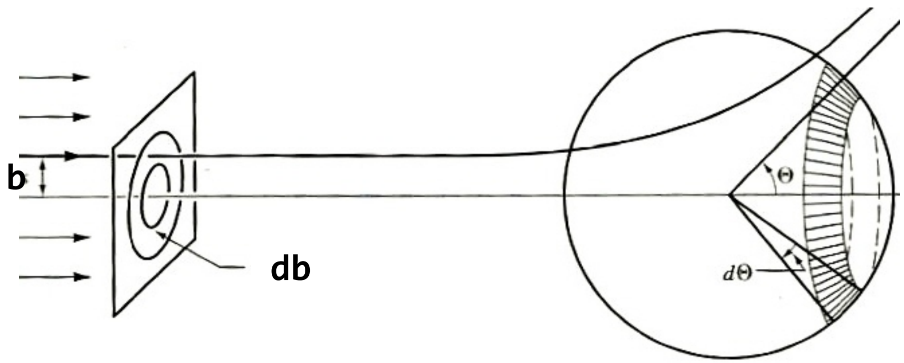


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$$\Rightarrow \qquad N_s \equiv I\sigma(\Theta)d\Omega = 2\pi I\sigma(\Theta)\sin\Theta d\Theta$$

- $N_i \equiv$ # incident particles with impact parameter between b & b + db . $N_i = 2\pi Ibdb$

- Conservation of particle number

$$\Rightarrow N_s = N_i \text{ or: } \quad 2\pi I\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi Ib|db|$$

$2\pi I$ cancels out! (Use absolute values because N's are always > 0, but db & $d\Theta$ can have any sign.)

$$\sigma(\Theta)\sin\Theta|d\Theta| = b|db| \qquad (1)$$

- **b** = a function of energy **E**
- & scattering angle **Θ**:

$$b = b(\Theta,E)$$

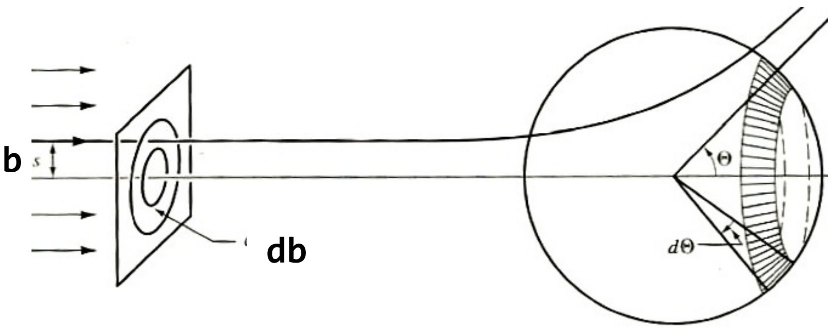


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$$(1) \quad \Rightarrow \sigma(\Theta) = (b/\sin\Theta)\,(|db|/|d\Theta|) \qquad (2)$$

- To compute **σ(Θ)** we clearly need **b = b(Θ,E)**
- Get **Θ = Θ(b,E)** from the orbit eqtn. For general central force (**θ** is the angle which describes the orbit **r = r(θ); θ ≠ Θ**)

$$\theta(r) = \int(\ell/r^2)(2m)^{-½}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-½}dr$$

- Orbit eqtn. **General central force:**

$$\theta(r) = \int(\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}dr \quad (3)$$

- Consider purely **repulsive scattering**. See figure:

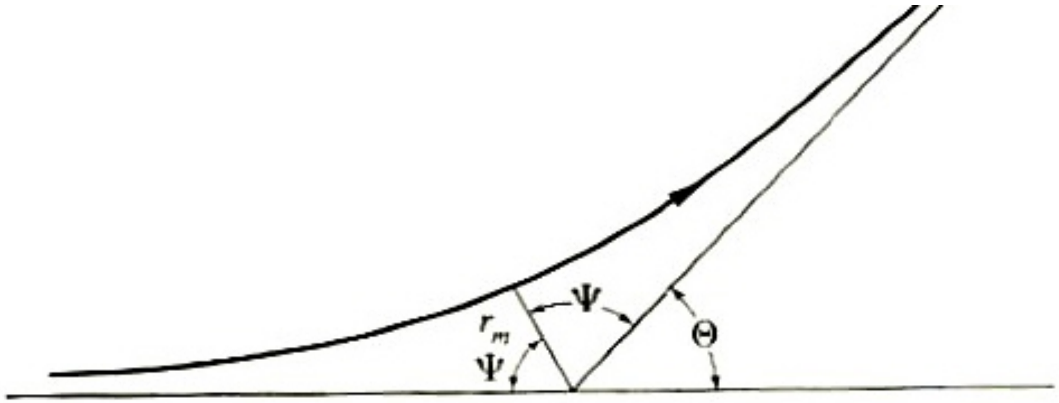


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- **Closest approach distance** $\equiv \mathbf{r_m}$. Orbit must be symmetric about $\mathbf{r_m} \Rightarrow$ Sufficient to look at angle Ψ (see figure): Scattering angle $\Theta \equiv \boldsymbol{\pi - 2\Psi}$. Also, orbit angle

$\boldsymbol{\theta = \pi - \Psi}$ ***in the special case*** $\mathbf{r = r_m}$

$\Rightarrow$ After some **manipulation** can write **(3)** as:

$$\Psi = \int (dr/r^2)[(2mE)/(\ell^2) - (2mV(r))/(\ell^2) - 1/(r^2)]^{-1/2} \quad (4)$$

- Integrate from $\mathbf{r_m}$ to $\mathbf{r} \to \infty$
- Angular momentum in terms of impact parameter

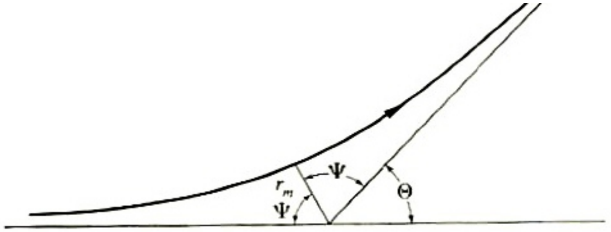


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

b & energy **E**: $\boldsymbol{\ell \equiv mv_0b \equiv b(2mE)^{1/2}}$. Put this into **(4)** & get for scattering angle $\boldsymbol{\Theta}$ (after **manipulation**):

$$\Theta(b) = \pi - 2\int dr(b/r)[r^2\{1 - V(r)/E\} - b^2]^{-1/2} \qquad (4')$$

Changing integration variables to $\mathbf{u = 1/r}$:

$$\Theta(b) = \pi - 2\int b\,du\,[1 - V(r)/E - b^2u^2]^{-1/2} \qquad (4'')$$

- Integrate from $\mathbf{u = 0}$ to $\mathbf{u = u_m = 1/r_m}$

- ***Summary:*** Scattering by a ***general central force***:
- **Scattering angle Θ** = **Θ(b,E)** (**b** = impact parameter, **E** = energy):

$$\Theta(b) = \pi - 2\int b\,du\,[1 - V(r)/E - b^2u^2]^{-1/2} \qquad (4'')$$

Integrate from $u = 0$ to $u = u_m = 1/r_m$

- **Scattering cross section:**

$$\sigma(\Theta) = (b/\sin\Theta)\,(|db|/|d\Theta|) \qquad (2)$$

- **"Recipe":** To solve a scattering problem:
  1. Given force **f(r)**, compute potential **V(r)**.
  2. Compute **Θ(b)** using **(4´´)**.
  3. Compute **$\sigma(\Theta)$** using **(2)**.

Coulomb Scattering

- ***Special case:*** **Scattering by $\mathbf{r^{-2}}$ repulsive forces**:
  - **For this case**, as well as for others where the orbit eqtn $\mathbf{r = r(\theta)}$ is known analytically, instead of applying **(4´´)** directly to get $\mathbf{\Theta(b)}$ & then computing $\mathbf{\sigma(\Theta)}$ using **(2)**, we make use of the known expression for $\mathbf{r = r(\theta)}$ to get $\mathbf{\Theta(b)}$ & then use **(2)** to get $\mathbf{\sigma(\Theta)}$.
- Repulsive scattering by $\mathbf{r^{-2}}$ repulsive forces =

  Coulomb scattering by like charges
  - Charge **+Ze** scatters from the center of force, charge **Z´e**

$$\Rightarrow \quad \mathbf{f(r) \equiv (ZZ'e^2)/(r^2) \equiv -k/r^2}$$

  - Gaussian E&M units! Not SI! For SI, multiply by $\mathbf{(1/4\pi\varepsilon_0)}$!

  $\Rightarrow$ For $\mathbf{r(\theta)}$, in the previous formalism for $\mathbf{r^{-2}}$ attractive forces, make the replacement $\mathbf{k \rightarrow -ZZ'e^2}$

- **Like charges:** $\mathbf{f(r) \equiv (ZZ'e^2)/(r^2)}$

  $\Rightarrow$ $\mathbf{k \to -ZZ'e^2}$ in orbit eqtn $\mathbf{r = r(\theta)}$

- Orbit eqtn for $\mathbf{r^{-2}}$ force is a conic section:

$$[\alpha/r(\theta)] = 1 + \varepsilon\cos(\theta - \theta') \qquad (1)$$

With: **Eccentricity** $\boldsymbol{\equiv \varepsilon = [\,1 + \{2E\ell^2/(mk^2)\}]^{1/2}}$ &

$\mathbf{2\alpha = [2\ell^2/(mk)]}$. Eccentricity = $\boldsymbol{\varepsilon}$ to avoid confusion with electric charge **e**. $\mathbf{E > 0}$ $\Rightarrow$ $\boldsymbol{\varepsilon > 1}$ $\Rightarrow$ **Orbit is a hyperbola**, by previous discussion.

- Choose the integration const $\boldsymbol{\theta' = \pi}$ so that $\mathbf{r_{min}}$ is at $\boldsymbol{\theta = 0}$
- Make the changes in notation noted:

$$\Rightarrow \quad [1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

$$\mathbf{f(r) = (ZZ'e^2)/(r^2)}$$

$$\mathbf{[1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1)} \qquad \mathbf{(2)}$$

- **Hyperbolic orbit.**
- With change of notation, eccentricity is

$$\boldsymbol{\varepsilon} = \mathbf{[\,1 + \{2E\ell^2/(mZ^2Z'^2e^4)\}]^{1/2}}$$

- Using the relation between angular momentum, energy, & impact parameter, $\ell^2 = \mathbf{2mEb^2}$ this is:

$$\boldsymbol{\varepsilon} = \mathbf{[\,1 + (2Eb)^2/(ZZ'e^2)^2\,]^{1/2}}$$

$$[1/r(\theta,b)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

$$\varepsilon = [\,1 + (2Eb)^2/(ZZ'e^2)^2\,]^{1/2} \qquad (3)$$

- From **(2)** get $\boldsymbol{\theta(r,b)}$. Then, use relations between orbit angle $\boldsymbol{\theta}$ scattering angle $\boldsymbol{\Theta}$, & auxillary angle $\boldsymbol{\Psi}$ in the scattering problem, to get $\boldsymbol{\Theta = \Theta(b)}$ & thus **the scattering cross section.**

$$\Theta = \pi - 2\Psi$$

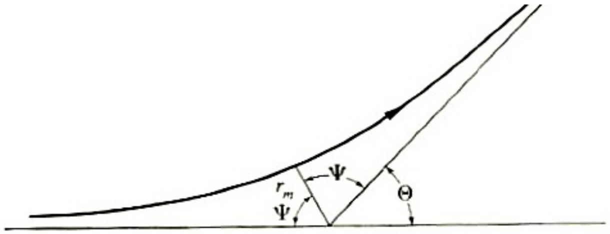


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- $\boldsymbol{\Psi}$ = direction of incoming asymptote. Determined by $\boldsymbol{r \rightarrow \infty}$ in **(2)** $\Rightarrow$ $\boldsymbol{\cos\Psi = (1/\varepsilon)}$.
- In terms of $\boldsymbol{\Theta}$ this is: $\boldsymbol{\sin(\frac{1}{2}\Theta) = (1/\varepsilon)}$. **(4)**

$$\varepsilon = [\ 1 + (2Eb)^2/(ZZ'e^2)^2\ ]^{1/2} \qquad (3)$$

$$\sin(\tfrac{1}{2}\Theta) = (1/\varepsilon) \qquad (4)$$

- Put (4) into left side of (3) and solve for b:

$$\Rightarrow \quad b = b(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta) \qquad (7)$$

(7), the impact parameter as function of **Θ** & **E** for Coulomb scattering is an **important result!**

$$\mathbf{b = b(\Theta,E) = (ZZ'e^2)/(2E)\ \cot(\tfrac{1}{2}\Theta)} \qquad \mathbf{(7)}$$

- Now, use **(7)** to compute the **Differential Scattering Cross Section for Coulomb Scattering.**
- We had: $\boldsymbol{\sigma(\Theta) = (b/\sin\Theta)\ (|db|/|d\Theta|)}$ **(8)**

  (7) & **(8)** (after using trig identities):

$$\Rightarrow \boldsymbol{\sigma(\Theta) = (\tfrac{1}{4})[(ZZ'e^2)/(2E)]^2(1/\sin^4(\tfrac{1}{2}\Theta))} \quad \mathbf{(9)}$$

(9) ≡ ***The Rutherford Scattering Cross Section***

- Get the same results in a (non-relativistic) QM derivation!
- Valid for spin 0 particles

Multiple Scattering

- Il fatto che il nucleo domini la diffusione angolare si vede anche dal fatto che la sez d'urto sul nucleo va come Z^2e^4
- e come e^4 per i singoli e- atomici;se ci sono Z elettroni, la sez d'urto sugli e- va come Ze^4
- quindi la diffusione sul nucleo e' Z volte quella degli elettroni

Single Scattering

- Individual collisions governed by Rutherford scattering formula:

$$\frac{d\sigma}{d\Omega} = z^2 Z^2 r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \frac{1}{4\sin^4(\theta/2)} \qquad **$$

  - Does not take into account spin effects or screening
- Although single large angle scattering can occur for very small impact parameter, probability that a single interaction will scatter through a significant angle is very small due to $1/\sin^4(\theta/2)$ dependence.
- For large impact parameter (much more probable), scattering angle is further reduced w.r.t. Rutherford formula due to partial screening of nuclear charge by atomic electrons.
- **NB: the form is equivalent to the previous one, provided that $Mv^2 = cp\beta$, M is the mass of incident particle and $r_e = e^2/m_e c$ is the classical electron radius

Scattering elastico

Nel limite di piccoli angoli $\theta \ll 1$, $\sin\theta \sim \theta \rightarrow$ la sezione d'urto si riduce a

$$\frac{d\sigma}{d\Omega} \approx 4z^2 Z^2 r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \frac{1}{\theta^4}$$

Questa formula ci dice che la sezione d’urto diverge a piccolo angolo.

Ma esiste un θ minimo ed un θ massimo ….

Scattering elastico: quick calc per $\theta \ll 1$

La sezione d'urto e' = all'elemento di area trasversa $d\sigma = 2\pi b db$
Il momento scambiato durante l'interazione e'
$P_t = 2zZe^2/bv$, perpendicolare all'impulso iniziale
La deflessione che la particella subisce e'
$\tan\theta \approx \theta = p_t/p$ per piccoli angoli
$\theta \approx 2zZe^2/bvp$ ovvero **$b = (2zZe^2/vp)(1/\theta)$**
La sezione d'urto infinitesima e'
$d\sigma = 2(2zZe^2/vp)^2 (d\theta/\theta^3)$
Percio' **$d\sigma/d\Omega = 2\pi (2zZe^2/vp)^2 (1/\theta^3)(d\theta/d\Omega)$**

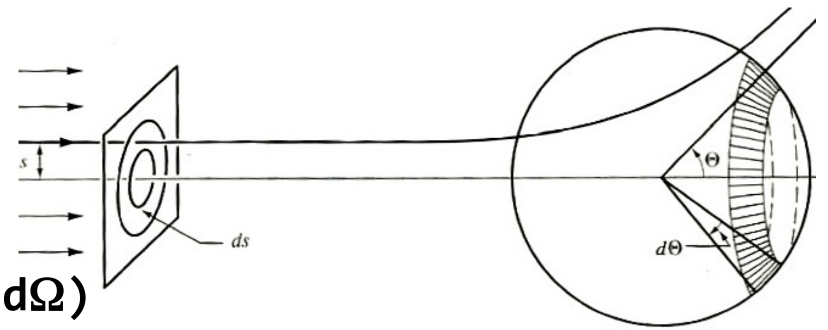


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$d\Omega = 2\pi\sin(\theta)d\theta \sim 2\pi\theta d\theta$ (per piccoli angoli) → **$d\theta/d\Omega = 1/2\pi\theta$**

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{vp}\right)^2 \frac{1}{\theta^4}$$

Calcolo classico, ma viene esattamente lo stesso risultato in meccanica quantistica.
NB: diverge a piccoli angoli, ma l'approx si applica fra un angolo minimo e uno massimo di deflessione

Scattering elastico: angolo **min** e max

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{vp}\right)^2 \frac{1}{\theta^4}$$

X-section diverges for $\theta \to 0$ because Coulomb interaction is long range for an single target-projectile pair.

But in a medium, also atomic electrons are usually present that screen nuclear charge for distances r (that is impact parameter b) > atomic radius **a** and the x-section remains finite for $\theta = 0$

The simplest classical approximation is to truncate the coulomb potential at r = a → The scattering angle reaches a minimum, θ_{min}, and the x-section tends to a constant value for diffusion angles below a mimimum deflection θ_{min}

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{\beta p}\right)^2 \frac{1}{\left(\theta^2 + \theta^2_{min}\right)^2}$$

θ_{min} is obtained from $\theta \approx$ **$2zZe^2/bvp$** with b = a

θ_{min} = **$2zZe^2/vpa$**

In terms of impact parameter, b is max (cioe' θ e' min) when $b_{max} = R_{atomo}$

- $b_{max} = a_0 = r_e/\alpha^2$ for hydrogen
- $b_{max} = r_a \sim [(h/2\pi\alpha mc)]\, Z^{-1/3}$ for materials heavier than H (Fermi-Thomas model)

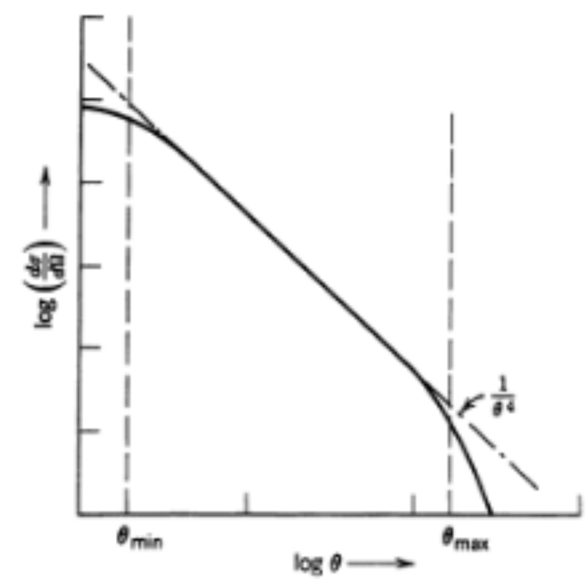
Scattering elastico: angolo min e max

Quantisticamente, affinche' lo scattering possa avvenire, la posizione della particella incidente deve essere localizzata entro $\delta x \sim R_{atomo}$ per avere una probabilita' apprezzabile di essere deviata.

Allora $\Delta p \approx h/(2\pi r_{atomo})$ e $\Delta\theta \approx \Delta p/p$ e quindi

$\theta_{min} \approx h/(2\pi r_{atomo} p) = (Z^{1/3}/192)\,(mc/p)$

Dei due minimi, classico e quantistico, si sceglie il piu' grande (analogamente al caso della BB per la scelta del b_{min}). Usualmente e' quello quantistico.



Scattering elastico: angolo min e **max**

Abbiamo anche un θ_{max} (che corrisponde a b_{min}).
Ad angoli grandi rispetto a θ_{min} (ma piccoli comunque in assoluto), la sezione 'urto devia dal valore previsto da Rutherford a causa della dimensione finita del nucleo.
Le cause sono diverse:
-per i leptoni e,μ,τ, l'influenza delle dimensioni nucleari e' di natura puramente elettromagnetica,
-per gli adroni (π,K,p,α,…) sono in genere presenti reazioni anelastiche dovute alle interazioni nucleari forti.
L'effetto netto complessivo e'quello di ridurre la la sezione d'urto rispetto a quella elastica di Rutherford.

Ci limitiamo a considerare solo l'aspetto elettromagnetico, non quello delle interazioni nucleari

Scattering elastico: angolo min e **max**

Abbiamo anche un θ_{max} (che corrisponde a b_{min}) (**$\theta \approx 2zZe^2/bvp$**).
Ad angoli grandi rispetto a θ_{min} (ma piccoli comunque in assoluto), la sezione 'urto devia dal valore previsto da Rutherford a causa della dimensione finita del nucleo.
Quando $\lambda = h/p \sim R_{nucleo}$, la diffusione diventa diffrattiva e confinata entro un angolo $\theta = \lambda/R = h/pR$
Alternativamente, con il principio di indeterm., $\Delta p = h/R$ e quindi $\theta \approx \Delta p/p = h/pR$
Una stima di R e' $R_n \sim 1.4 \times 10^{-15} A^{1/3}$

$$\Rightarrow \theta_{max} \cong \frac{274}{A^{1/3}} \frac{mc}{p}$$

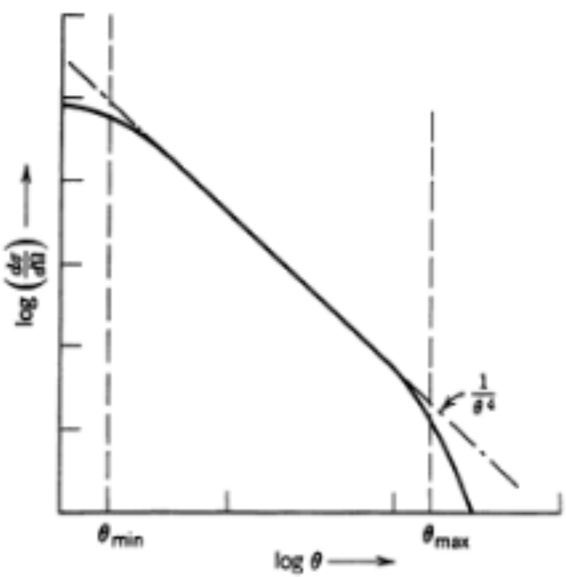


Figure 13.6 Atomic scattering, including effects of electronic screening at small angles and finite nuclear size at large angles.

Si noti che $1 \gg \theta_{max} \gg \theta_{min}$ per tutti valori fisici di A e Z

$\theta_{max}/\theta_{min} = 274 \times 192/(AZ)^{1/3} \approx 2000$ per l'Uranio

Scattering elastico: sezione d'urto totale

La sezione d'urto totale e' finita e puo' essere ottenuta integrando sull'angolo solido quella differenziale

$$\Rightarrow \sigma = \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\sigma}{d\Omega} 2\pi\vartheta d\theta \cong \left(\frac{2zZe^2}{\beta p}\right)^2 \frac{1}{\theta_{\min}^2}$$

$\theta_{min} \approx (Z^{1/3}/192)\ (mc/p)$

$$\Rightarrow \sigma = \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\sigma}{d\Omega} 2\pi\vartheta d\theta \cong \pi \left(\frac{2zZe^2}{\beta}\right)^2 \left(\frac{192}{Z^{1/3}}\right)^2 \frac{1}{(mc)^2}$$

La sezione d'urto totale di deflessione per singolo urto decresce aumentando β.

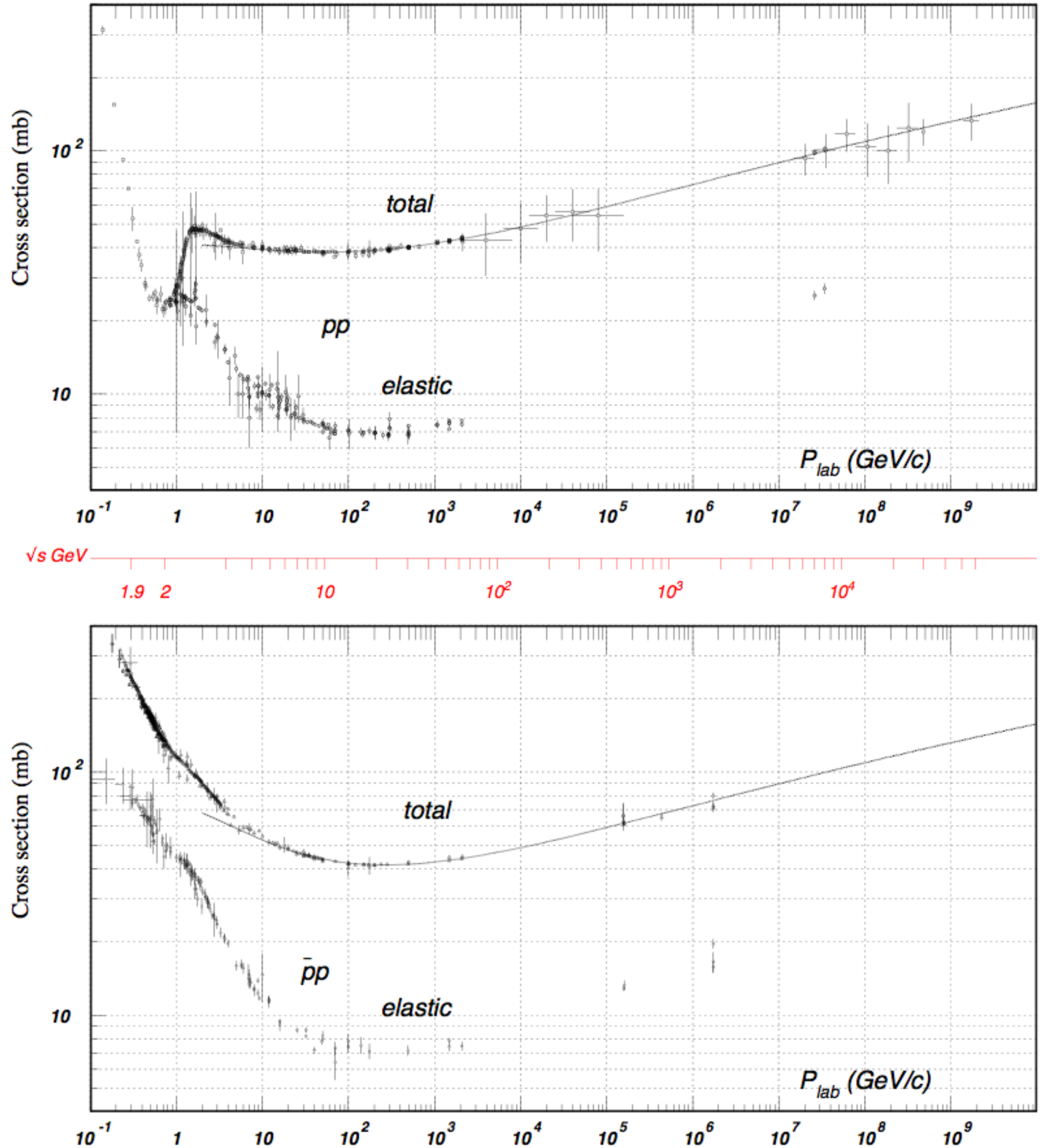


Figure 49.9D11: Total and elastic cross sections for pp and $\overline{p}p$ collisions as a function of laboratory beam momentum and total center-of-mass energy. Corresponding data files may be found at

Scattering multiplo

- La distribuzione di probabilita' $\Psi_{ms}(\theta,x)$ per gli angoli di diffusione obbedisce ad un'equazione analoga a quella per le fluttuazioni del deposito di energia:

Prob di deflessione di $\theta-\zeta$ a x

Prob di una deflessione qualsiasi in dx

$$\frac{\partial \psi_{ms}}{\partial x}(\theta, x) = \int \psi_{ms}(\theta - \zeta, x) n_a \frac{d\sigma}{d\Omega}(\zeta)\zeta\, d\zeta - \psi_{ms}(\theta, x) \int n_a \frac{d\sigma}{d\Omega}(\zeta)\zeta\, d\zeta$$

Prob di deflessione di ζ in dx

n_a = densita' di particelle del mezzo assorbitore
La distribuzione cumulativa dipende sia dalla "storia" della propagazione fino a x sia dalla distribuzione di probabilita' di deflessione per la singola particella.
Se si assume la diffusione di Rutherford, e' possibile trovare soluzioni analitiche (Moliere)

Scattering multiplo

$$\frac{\partial \psi_{ms}}{\partial x}(\theta, x) = \int \psi_{ms}(\theta - \zeta, x) n_a \frac{d\sigma}{d\Omega}(\zeta)\zeta\, d\zeta - \psi_{ms}(\theta, x) \int n_a \frac{d\sigma}{d\Omega}(\zeta)\zeta\, d\zeta$$

La soluzione puo’ essere espressa in forma integrale

to be the small angle Rutherford cross section, becomes [22, 23]

$$\psi_{ms}(\theta, x) = \frac{1}{\theta_c^2} \int_0^\infty y\, dy\, J_0\left(\frac{\theta y}{\theta_c}\right) \exp\left[\frac{y^2}{4}\left(-b + \ln \frac{y^2}{4}\right)\right] \quad (2.89)$$

where J_0 is a Bessel function and θ_c is a characteristic angle given by

$$\theta_c^2 = \frac{4\pi n_a e^4 Z_1^2 Z_2(Z_2 + 1)x}{(pv)^2} \quad (2.90)$$

The angle θ_c contains the dependence of ψ_{ms} on the macroscopic properties of the scattering medium. Note that θ_c^2 grows linearly with x. The quantity b is defined as

$$b = \ln(\theta_c/\theta_a)^2 + 1 - 2C_E \quad (2.91)$$

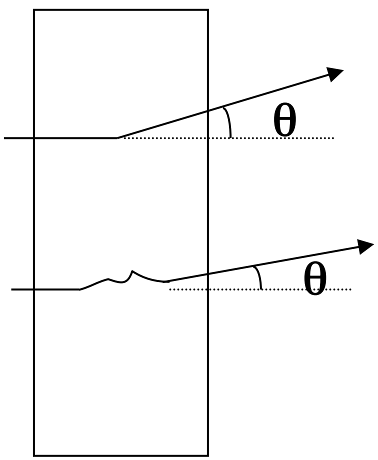
where C_E is Euler’s constant. The dependence of the scattering on the

Scattering multiplo

- Non risolveremo l'equazione ma tratteremo in maniera separata e approssimata le varie regioni di diffusione
  - Singolo scattering: se l'assorbitore e' sottile cosi' che la probabilita' di piu' un singolo scattering e trascurabile, la distribuzione angolare e' data dalla sezione d'urto di Rutherford.
  - Scattering "plurale": se il numero medio di collisioni non e' sufficientemente elevato, N<20, la distribuzione e' complicata perche' ne' la semplice formula di rutherford ne' metodi statistici possono essere applicati in modo semplice.
  - Scattering multiplo: se il numero medio di collisioni e' elevato,N>20, e la perdita di energia e' trascurabile, il problema puo' essere trattato statisticamente (cioe' possiamo applicare il teorema del limite centrale).

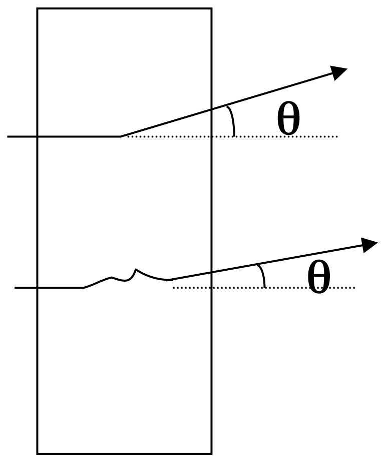
Scattering multiplo

La diffusione di Rutherford e' limitata ad angoli molto piccoli, anche per un campo coulombiano di carica puntiforme e per particelle veloci si ha sempre $\theta_{max} << 1$.
La prob di diff a piccoli angoli e' quindi molto elevata.
Una particella che attraversa uno spessore finito subira' un gran numero di piccole deviazioni ed emergera' dalla lastra ad un piccolo angolo rispetto alla direzione di incidenza.
L'angolo e' la sovrapposizione cumulativa statistica di un gran numero di piccole deflessioni singole.



Solo raramente la particella subira' una deviazione a grande angolo. Poiche' e' poco probabile, la particella non subira' praticamente mai piu' di una (o poche) di queste deviazioni (siamo nell'ipotesi che lo spessore sia piccolo rispetto al range)
Questa constatazione ci permette di dividere l'intervallo angolare delle deviazioni in due zone: una comprendente angoli relativamente grandi, in cui si presentano soltanto diffusioni singole e una zona comprendente angoli molto piccoli che contiene le diffusioni multiple o composte.

Scattering multiplo

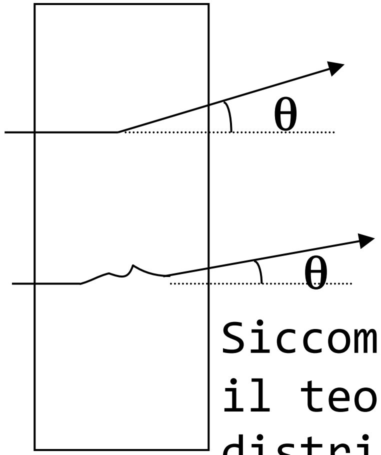


Tradizionalmente si divide l'intervallo angolare delle deviazioni in due zone: una ad angoli relativamente grandi, in cui si hanno diffusioni singole, ed una ad angoli molto piccoli, che contiene le diffusioni multiple.

Ovviamente, noi osserviamo la deviazione finale dopo uno spessore x. Una particella può subire un solo scattering, o fare molti scattering coulombiani .
Non possiamo dire se l'angolo all'uscita dal materiale sia dovuto a un singolo scattering a grande angolo o a molti scattering a piccolo angolo

Scattering multiplo: regione a piccoli angoli

La grandezza determinante nella zona di diffusione multipla, dove la particella subisce numerose deflessioni a piccoli angoli, distribuite simmetricamente intorno alla direzione di incidenza, e’ l’angolo quadratico medio di deflessione.
In ogni urto singolo le deflessioni angolari seguono la legge di Rutherford, tra θ_{min} e θ_{max} con valore medio nullo, $\langle\theta\rangle = 0$ (se considerate rispetto alla dir di incidenza), e con un angolo quadratico medio $\langle\theta^2\rangle \neq 0$.



$$\left\langle \theta^2 \right\rangle = \frac{\int \theta^2 \frac{d\sigma}{d\Omega} d\Omega}{\int \frac{d\sigma}{d\Omega} d\Omega} \cong \frac{\int \frac{d\theta}{\theta}}{\int \frac{d\theta}{\theta^3}} \cong 2\theta_{\min}^2 \left[\ln\left(\theta_{\max} / \theta_{\min} \right) \right]$$

Siccome gli urti successivi sono statisticamente indipendenti, il teorema del limite centrale ci assicura che la distribuzione dopo un gran numero N di urti e' una gaussiana con deviazione standard data dalla somma (in quadratura) delle larghezze nelle singole collisioni:

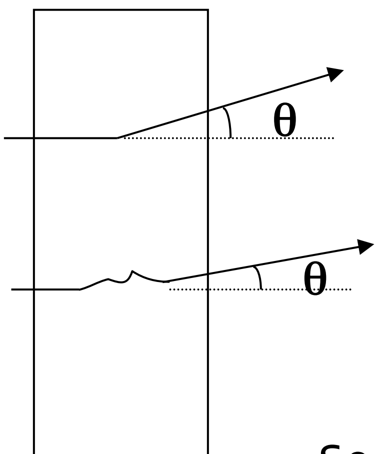
$$\langle\Theta^2\rangle = N\langle\theta^2\rangle$$

Scattering multiplo: regione a piccoli angoli

Il numero di urti che una particella subisce nell'attraversare uno spessore x e' N = nσx, dove s e' la sez. d'urto totale di deflessione ed n e’ la densita’ di particelle del mezzo, percio'

$$\Rightarrow \sigma \cong \left(\frac{2zZ\alpha}{\beta p}\right)^2 \frac{1}{\theta_{\min}^2} \qquad N \cong \pi n\left(\frac{2zZe^2}{pv}\right)\frac{x}{\theta_{\min}^2}$$

$$\left\langle \theta^2 \right\rangle = \frac{\int \theta^2 \frac{d\sigma}{d\Omega} d\Omega}{\int \frac{d\sigma}{d\Omega} d\Omega} \cong \frac{\int \frac{d\theta}{\theta}}{\int \frac{d\theta}{\theta^3}} \cong 2\theta_{\min}^2 \left[\ln\left(\theta_{\max}/\theta_{\min}\right)\right]$$



$$\Rightarrow \left\langle \Theta^2 \right\rangle = 2\pi n x \left(\frac{2zZe^2}{\beta p}\right)^2 \ln\left(\frac{\theta_{\max}}{\theta_{\min}}\right)$$

Non c’e’ quasi piu’ dipendenza da θ_{min}

Considerando che n = $N_A\rho/A$

$$\Rightarrow \left\langle \vartheta_{ms}^2 \right\rangle \cong \frac{N_A \rho x}{A} \cdot 2\pi \left(\frac{2zZe^2}{\beta p}\right)^2 \cdot \ln \frac{2}{\alpha^2 A^{1/3} Z^{1/3}}$$

Se A~2Z il termine logaritmico diventa 2ln($173Z^{-1/3}$).

Il fattore nel logaritmo cambia da autore ad autore (Le considerazioni svolte qui sono tratte dal Jackson, Elettrodinamica Classica, Cap. 13)

L’angolo quadratico medio cresce linearmente con il grammaggio attraversato ρx, avendo in mente che ρx e’ comunque tale che le perdite di energia della particella sono trascurabili

Scattering multiplo: regione a piccoli angoli

A piccoli angoli, N e' molto grande e quindi si puo' applicare il teorema del limite centrale.
A piccoli angoli quindi la probabilita' di avere uno scattering fra θ e $\theta+d\theta$ e' una gaussiana di larghezza

$$<\Theta_{ms}> = <\Theta^2>^{1/2}$$

$$P_M(\alpha)d\alpha \approx \frac{1}{\sqrt{\pi}} e^{-\alpha^2} d\alpha \qquad \alpha = \frac{\theta}{<\theta_{RMS}>}$$

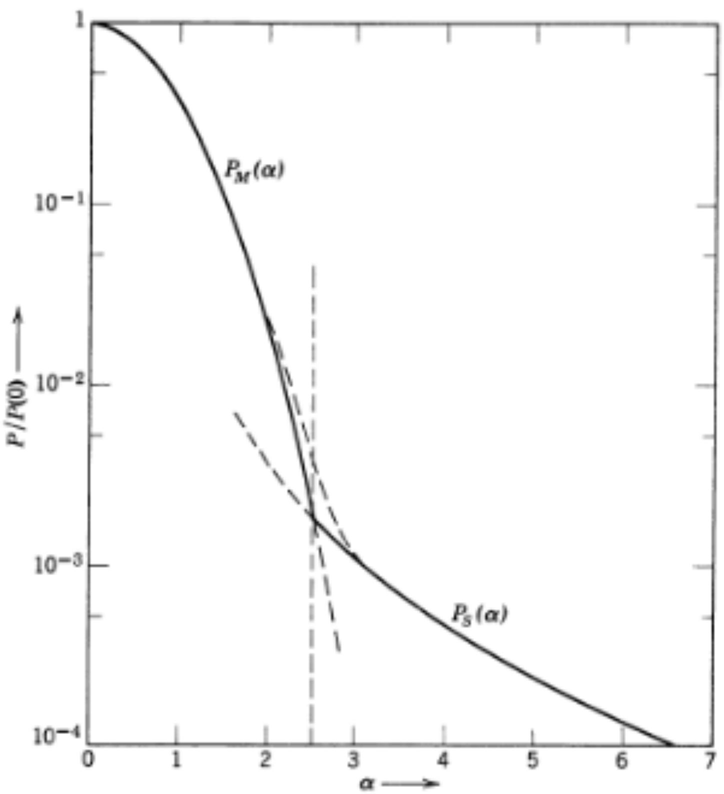


Figure 13.8 Multiple- and single-scattering distributions of projected angle. In the region of plural scattering ($\alpha \sim 2$–3) the dashed curve indicates the smooth transition from the small-angle multiple scattering (approximately Gaussian in shape) to the wide-angle single scattering (proportional to α^{-3}).

Scattering multiplo: regione a grandi angoli

La diffusione a grandi angoli e' dovuta a singoli (o pochi) scattering.
La distribuzione di probabilita' e' quella di singolo scattering

$$P_S d\Omega \approx \left(\frac{2zZe^2}{\beta p}\right)^2 \frac{d\Omega}{\left(\theta^2+\theta_{\min}^2\right)^2}$$

nel limite di piccoli angoli (ie $\theta<<1$, $d\Omega = 2\pi\theta d\theta$) e vicini a θ_{max}:

$$P_S d\theta \approx \left(\frac{2zZe^2}{\beta p}\right)^2 * \frac{2\pi\theta d\theta}{\theta^4} \qquad \alpha = \frac{\theta}{<\theta_{RMS}>}$$

$$P_S d\alpha \approx \frac{1}{8\ln(173Z^{-1/3})} * \frac{d\alpha}{\alpha^3}$$

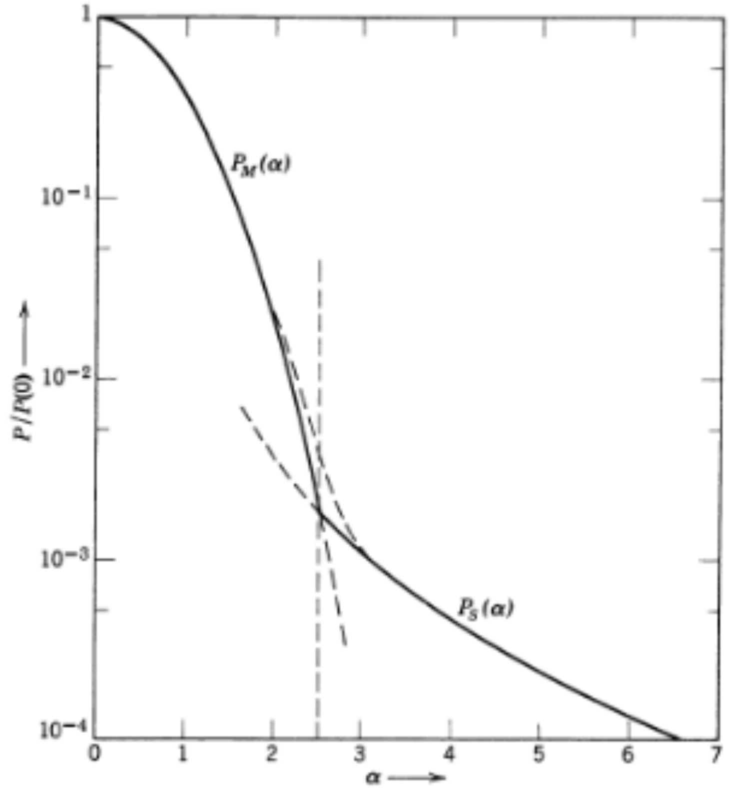


Figure 13.8 Multiple- and single-scattering distributions of projected angle. In the region of plural scattering ($\alpha \sim 2$–3) the dashed curve indicates the smooth transition from the small-angle multiple scattering (approximately Gaussian in shape) to the wide-angle single scattering (proportional to α^{-3}).

An Early Measurement with Electrons

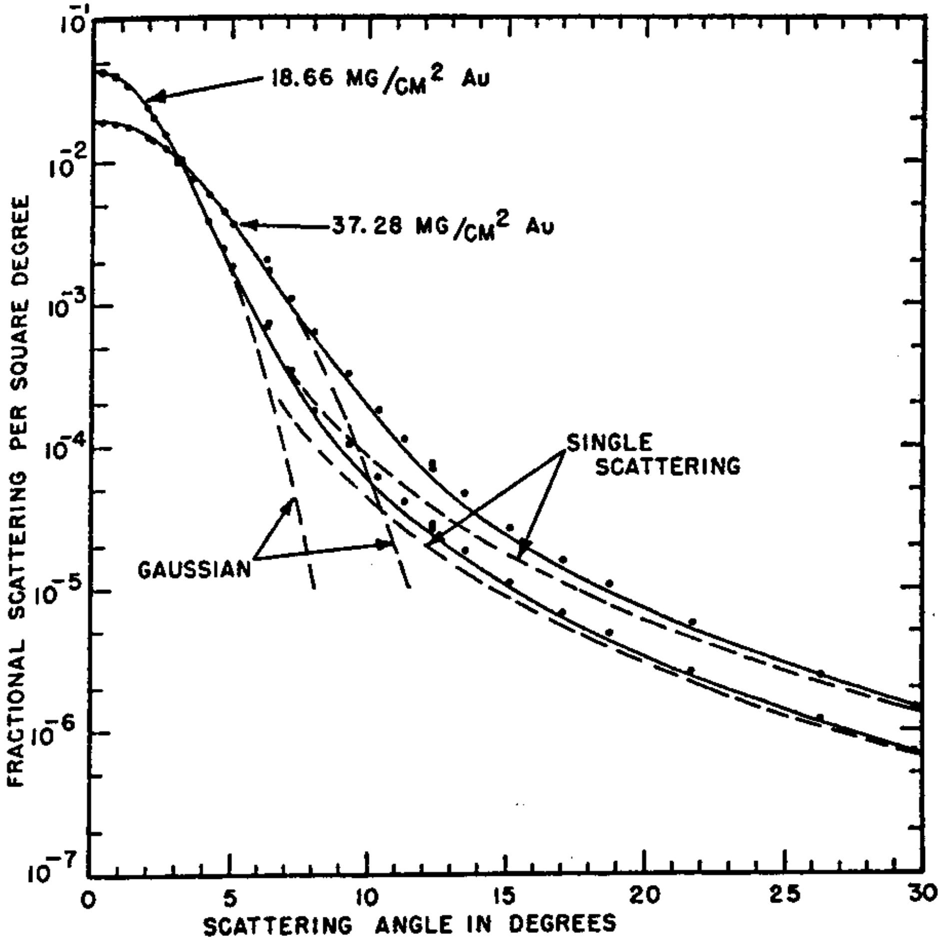


FIG. 3. Angular distribution of electrons from thick and thin gold foils from 0° to 30°. The solid line represents the theory of Molière extrapolated through the region where his small and large angle approximations give different values. The dotted lines at small angles represent the continuation of the gaussians of Fig. 1. At larger angles, the dotted line represents the single scattering contribution.

- Continuous line is the exact solution of the transport differential equation.
- At small angles the gaussian approximation is very good
- The single scattering distribution approximates well the data only at very large angles

Scattering multiplo

Tradizionalmente si scrive l'angolo di scattering in termini della lunghezza di radiazione X_0.

Attenzione X_0 è definita per processi radiativi. Lo scattering multiplo non è un processo radiativo → θ_{ms} e' espressa in funzione di X_0 solo per fini pratici poiche' X_0 e' facilmente misurabile e nota per tutti i materiali

$$\langle \theta_{ms}^2 \rangle = z^2 \frac{x}{X_0} \frac{2\pi}{\alpha} \frac{m_e^2 c^4}{\beta^2 p^2} \quad \text{ed in termini di energia}$$

$$\theta_{ms} = z \frac{E_s}{\beta c p} \sqrt{x / X_0} \qquad \left(E_s = \sqrt{\frac{4\pi}{\alpha}} \cdot mc^2 \approx 21\ MeV \right) \tag{1}$$

Radiation Length (cfr. Bremsstrahlung)

- Define **Radiation Length, X_0:**

$$-\left.\frac{dE}{dX}\right|_{rad} = \frac{E}{X_0} \quad \text{where} \quad \frac{1}{X_0} = 4\alpha N_A r_e^{\,2} \frac{Z^2}{A} \ln \frac{183}{Z^{1/3}}$$

$$\Rightarrow \quad E = E_0 e^{-x/X_0}$$

Units of X_0: g cm^{-2}
Divide by density ρ to get X_0 in cm

- Radiation length is the the mean distance over which a high-energy electron loses all but 1/e of its energy by bremsstrahlung.

- e.g. Pb: Z=82, A=207, ρ=11.4 g/cm^3:
  - $X_0 \approx 5.9$ g/cm^2
  - Mean penetration distance: $x = X_0/\rho = 5.9/11.4 = 5.2$mm

Scattering multiplo

$$\left\langle \theta_{ms}^2 \right\rangle = z^2 \frac{x}{X_0} \frac{2\pi}{\alpha} \frac{m_e^2 c^4}{\beta^2 p^2}$$ ed in termini di energia

$$\theta_{ms} = z \frac{E_s}{\beta c p} \sqrt{x / X_0} \qquad \left(E_s = \sqrt{\frac{4\pi}{\alpha}} \cdot mc^2 \approx 21\ MeV \right)$$

Valida solo se attraverso molte lunghezze di radiazione, altrimenti è una sovrastima di $\theta_{ms.}$. Più accurata:

$$\theta_{ms} = z \frac{19.2}{\beta c p} [MeV] \sqrt{x/X_0} \left(1 + 0.038 \ln \left(x / X_0 \right) \right)$$

Formule valide per piccoli angoli. Non tengono conto delle code non gaussiane dovute ai singoli scattering

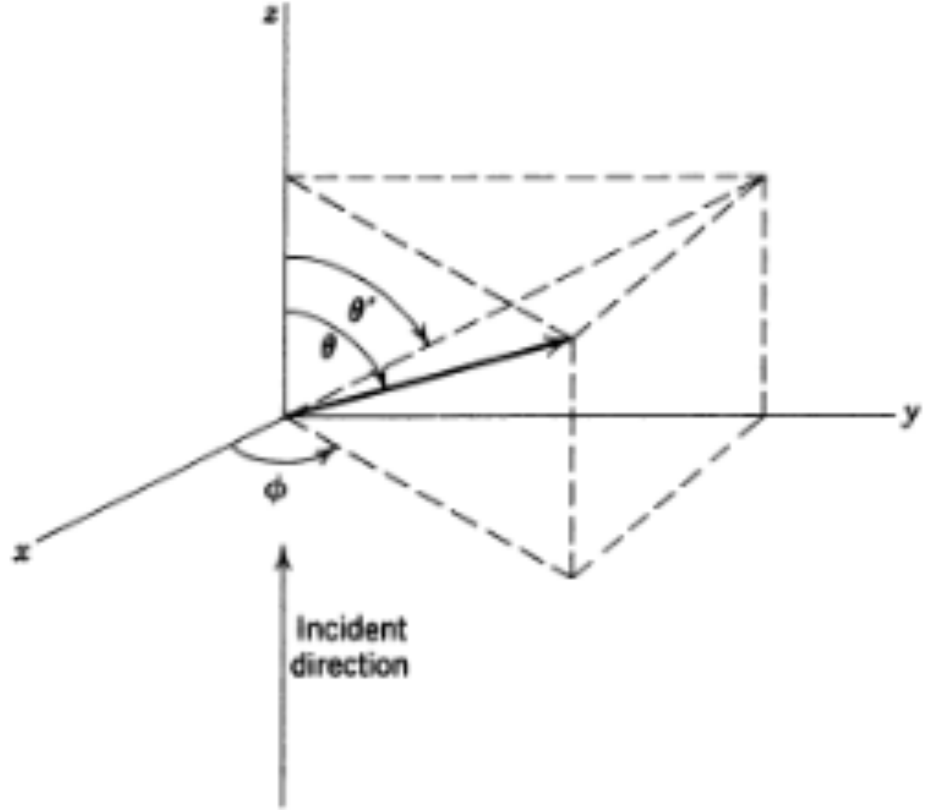


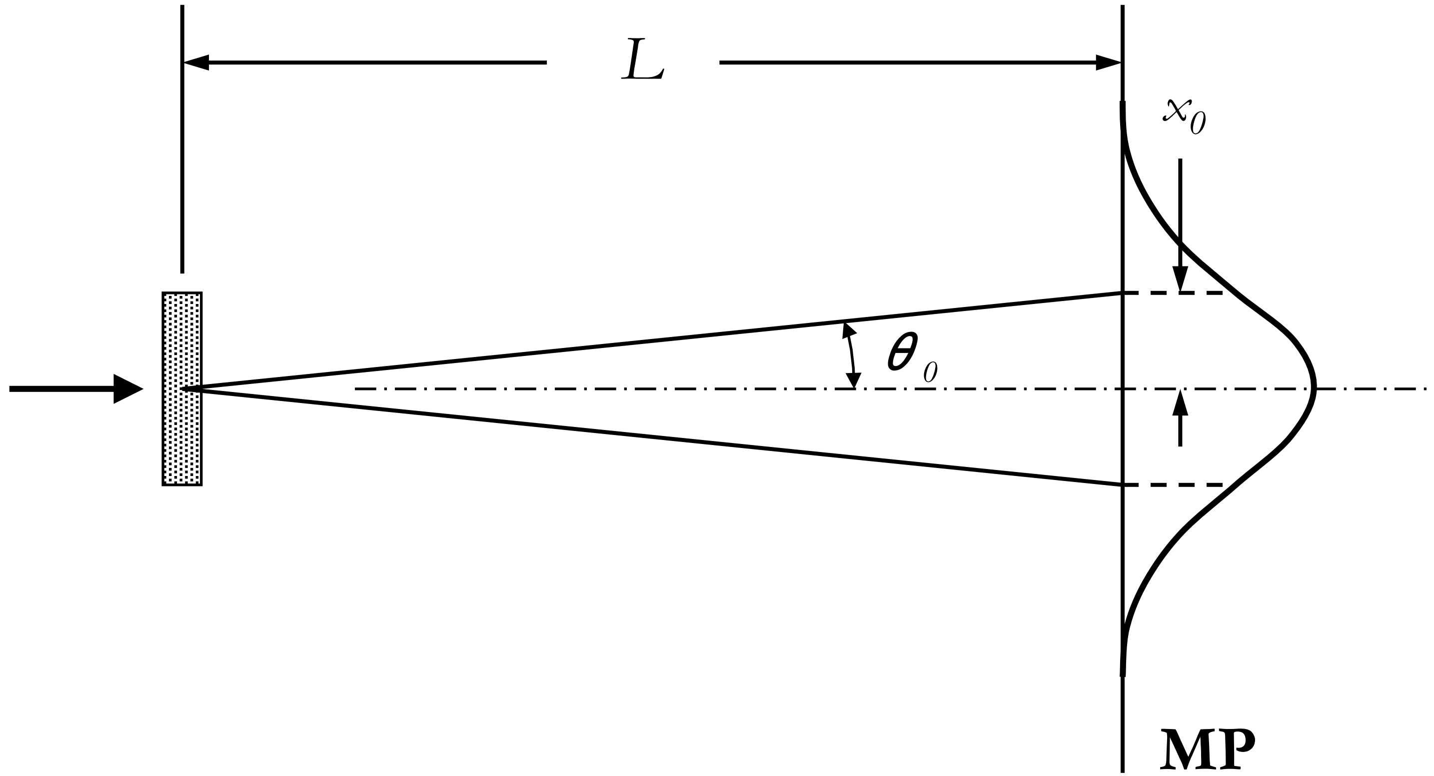
Figure 13.7

It is often desirable to use the projected angle of scattering θ', the projection being made on some convenient plane such as the plane of a photographic emulsion or a bubble chamber, as shown in Fig. 13.7. For small angles it is easy to show that

$$\langle \theta'^2 \rangle = \tfrac{1}{2}\langle \theta^2 \rangle \qquad (13.63)$$

Infatti, $\tan\theta' = \tan\theta \sin\Phi \rightarrow \theta' \approx \theta \sin\Phi$, percio' $\langle\theta'^2\rangle = 1/2 \langle\theta^2\rangle$

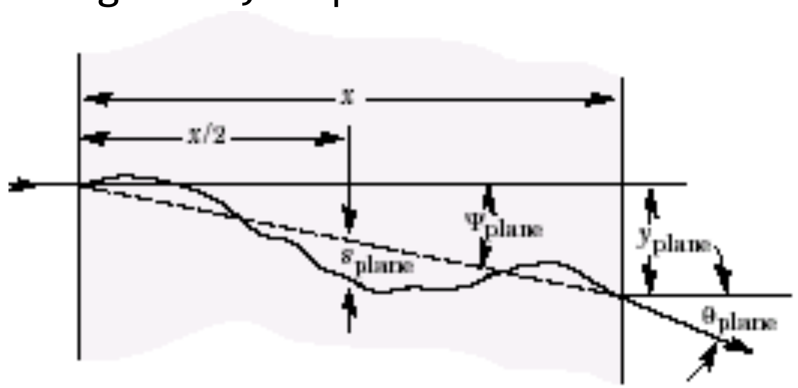
Multiple Scattering



When protons pass through a slab of material they suffer millions of collisions with atomic nuclei. The statistical outcome is a *multiple scattering angle* whose distribution is approximately Gaussian. For protons, this angle is always small so the projected displacement in any measuring plane **MP** is also Gaussian. The width parameter of the angular distribution is θ_0 . The corresponding displacement, x_0 , can easily be measured by scanning a dosimeter across the MP. The task of multiple scattering theory is to predict θ_0 given the scattering material and thickness, and the incident proton energy.

Scattering multiplo

La deviazione angolare causata dallo scattering multiplo produce anche uno spostamento laterale della posizione di uscita della particella rispetto al punto di ingresso, dopo aver attraversato uno spessore x di materiale.



La deviazione quadratica media della proiezione su un piano (che e' di solito quella che si misura) è data da :

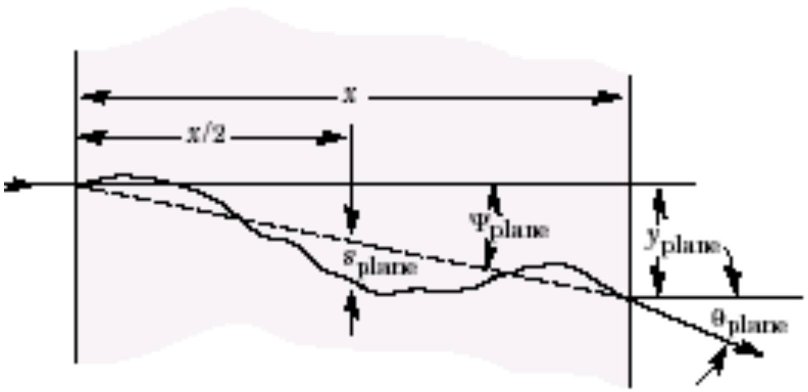
$$\left\langle y^2_{plane} \right\rangle \cong \frac{1}{6} \left\langle \vartheta^2_{ms} \right\rangle x^2$$

Scattering multiplo

Vediamo di ricavare $$\left\langle y_{plane}^2 \right\rangle \cong \frac{1}{6}\left\langle \vartheta_{ms}^2 \right\rangle x^2$$

A tale scopo consideriamo un elemento di spessore dx a profondità x e vediamo il contributo di dy^2 a $\langle y^2 \rangle$

$$y(x+dx) = y(x) + \vartheta_y(x)dx$$
$$\vartheta_y(x+dx) = \vartheta_y(x) + \delta\vartheta$$



$$\left\langle y^2(x+dx) \right\rangle = \left\langle y^2(x) \right\rangle + 2\left\langle y(x)\vartheta_y(x) \right\rangle dx$$
$$d\left\langle y^2 \right\rangle = 2\left\langle y(x)\vartheta_y(x) \right\rangle dx$$
$$d\left\langle \vartheta_y^2 \right\rangle = \left\langle \delta\vartheta^2 \right\rangle$$

$$d\left\langle y\vartheta \right\rangle = \left\langle \vartheta_y^2 \right\rangle dx \quad \text{infatti} \quad y(x+dx)\vartheta_y(x+dx) \cong y(x)\vartheta_y(x) + \vartheta_y^2(x)dx + y(x)\delta\vartheta$$
$$< y(x+dx)\vartheta_y(x+dx) > - < y(x)\vartheta_y(x) > = < \vartheta_y^2(x) > dx + < y(x)\delta\vartheta >$$

il < > dell'ultimo termine e' = 0 perche' $\langle d\theta \rangle = 0$ ed e' indipendente da y

Scattering multiplo

Ora:

$$\left\langle \vartheta_y^2 \right\rangle = \frac{\left\langle \vartheta_{ms}^2 \right\rangle}{2} = k \cdot s / 2 \qquad \text{con} \quad k = \left(21\left[MeV\right] \frac{1}{p\beta} \right)^2$$

$$\left\langle y\vartheta \right\rangle = \int_0^s kx\,dx = k\frac{s^2}{4} = \frac{1}{4} s \left\langle \vartheta_{ms}^2 \right\rangle$$

$$\left\langle y^2 \right\rangle = 2\int_0^s kx^2 \frac{1}{4} dx = \frac{1}{6} ks^3 = \frac{1}{6} \left\langle \vartheta_{ms}^2 \right\rangle s^2$$

Con s = X/Xo

$$\left\langle y_{plane}^2 \right\rangle \cong \frac{1}{6} \left\langle \vartheta_{ms}^2 \right\rangle s^2$$

In genere lo spostamento e' piccolo

Scattering multiplo

Notiamo: lo scattering multiplo è un fattore limitante per le misure.

- **Misure d' impulso** precisione della misura limitata dallo scattering multiplo.
- **Sciami elettromagnetici** dimensioni trasverse dello sciame dovute allo scattering multiplo.

Effect of Multiple Scattering on Resolution

- Approximate relation (PDG):

$$\theta_0 = \theta_{plane}^{RMS} \approx \frac{13.6MeV}{p\beta c} z \sqrt{\frac{L}{X_0}} \qquad \text{i.e.} \qquad \boxed{\theta_0 \propto \frac{1}{p}\sqrt{\frac{L}{X_0}}}$$

Charge of incident particle

Radiation length of absorbing material

- Apparent sagitta due to multiple scattering (from PDG):

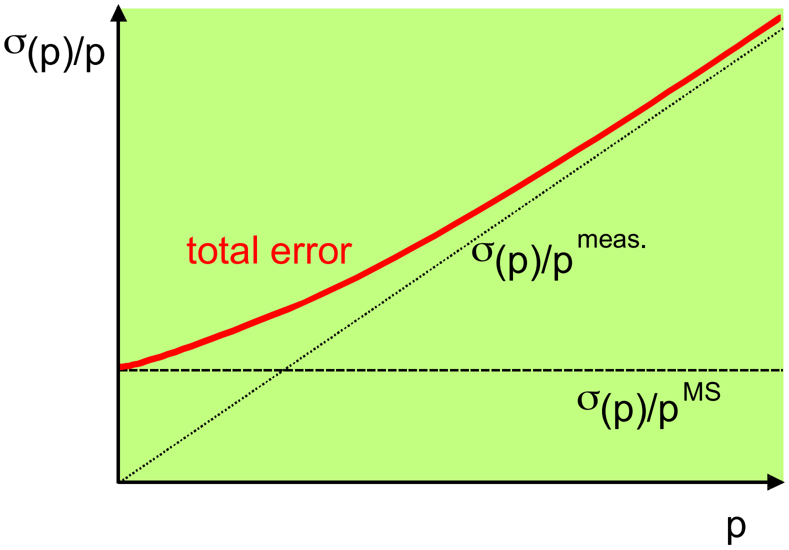
$$s_{plane} = \frac{L\theta_0}{4\sqrt{3}}$$

- Contribution to momentum resolution from multiple scattering:

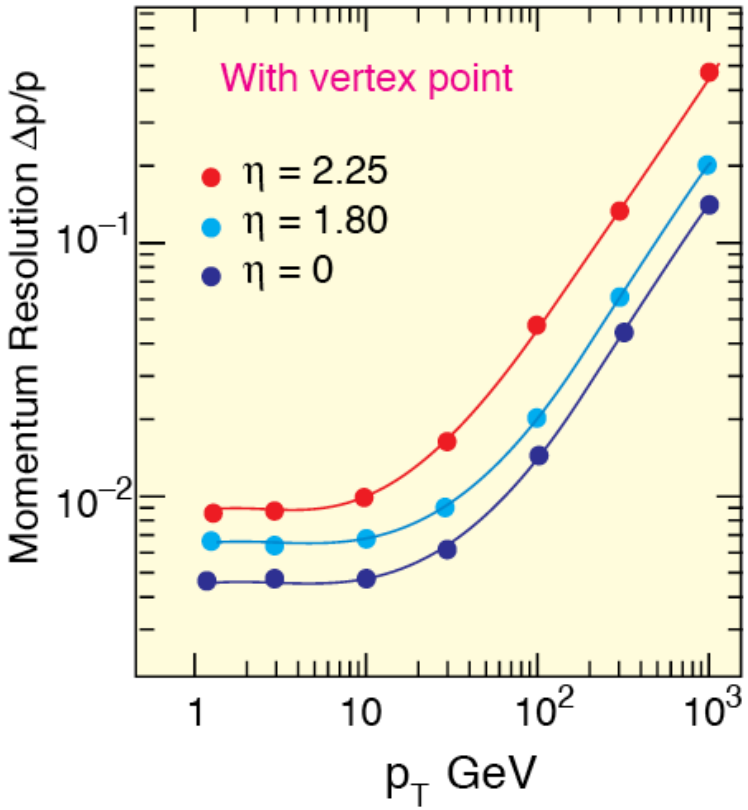
$$\left.\frac{\sigma_p}{p}\right|_{MS} = \frac{s_{plane}}{s} \approx \frac{0.05}{B\sqrt{LX_0}} \qquad \text{using } s \approx \frac{0.3BL^2}{8p_T} \qquad \text{i.e.} \qquad \boxed{\left.\frac{\sigma_p}{p}\right|_{MS} \propto \frac{1}{B\sqrt{LX_0}}}$$

Independent of p!

Effect of Multiple Scattering on Resolution



Estimated Momentum Resolution vs p_T in CMS



- Example:
  - p_T = 1 GeV/c, L = 1m, B = 1 T, N = 10, σ_x = 200μm:

$$\left.\frac{\sigma_p}{p}\right|_{meas} \approx 0.5\%$$

  - For detector filled with Ar, X_0 = 110m:

$$\left.\frac{\sigma_p}{p}\right|_{MS} \approx 0.5\%$$