Lecture 17 041219

- Il pdf delle lezioni puo’ essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1920/

Shock waves

$$[X] = X_2 - X_1$$

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0 \quad [B_n] = 0$$

$$[\rho v_n (v^2/2 + w) + \frac{1}{4\pi}(v_n B^2 - B_n(\vec{v} \cdot \vec{B}))] = 0 \quad [B_n v_t - B_t v_n] = 0$$

The analysis is simple in two limiting cases

The first is one in which the B field before the shock is normal to the shock surface, B_{t1}=0

This case is called "parallel" shock since the B field is parallel to shock normal direction

In such a case it is easy to show that B_{t2} =0 behind the shock

Since both B_n and B_t are continous with B_{n1} =B_{n2} and B_{t1}=B_{t2}=0, **it follows that the parallel shocks reduce to the pure hydrodynamic case, as if the B field is not present**

Shock waves

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0$$

$$[B_n] = 0$$

$$[\rho v_n (v^2/2 + w) + \frac{1}{4\pi}(v_n B^2 - B_n(\vec{v}\cdot\vec{B}))] = 0$$

$$[B_n v_t - B_t v_n] = 0$$

The analysis is simple in two limiting cases

The 2nd case is when the B field is parallel to the shock surface, ie perpendicular to the normal, B_{n1} =0

In such a case, from 3rd equation we see that v_t is continous → therefore we can choose a reference frame in which v_t=0 and the shock is a normal shock

From $[B_n v_t - B_t v_n] = 0$ And $[\rho v_n] = 0$ We get

$B_{t1}/B_{t2} = \rho_1/\rho_2$ While the others reduce to

$$[\rho v] = 0 \qquad [p + \rho v^2 + \frac{B_t^2}{8\pi}] = 0 \qquad [v^2/2 + w + \frac{B_t^2}{4\pi\rho}] = 0$$

Which are the hydrodynamics eqns with the additional magnetic terms is momentum and energy flux

Strong parallel shocks

For parallel strong shock the Rankine-Hugoniot read as

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)}{2/M_s^2 + (\gamma-1)} \quad \Rightarrow \quad \frac{\rho_2}{\rho_1} \approx \frac{(\gamma+1)}{(\gamma-1)}$$

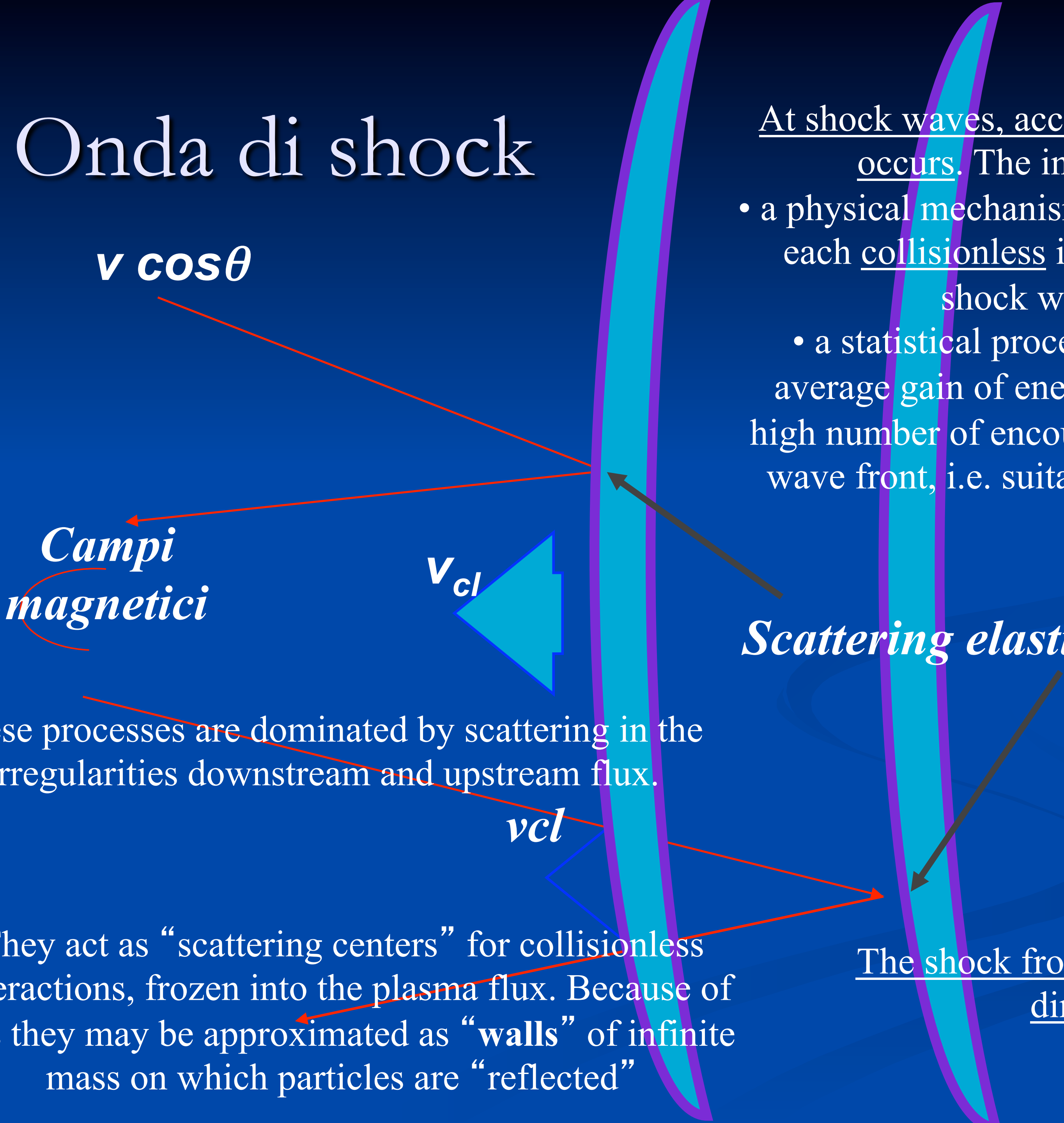
$$\frac{p_2}{p_1} = \frac{2\gamma M_s^2 - (\gamma-1)}{(\gamma+1)} \quad \Rightarrow \quad \frac{p_2}{p_1} \approx 2\gamma M_s^2$$

$$\frac{V_2}{V_1} = \frac{2+(\gamma-1)M_s^2}{(\gamma+1)M_s^2} \quad \Rightarrow \quad \frac{V_2}{V_1} \approx \frac{\gamma-1}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{[2+(\gamma-1)M_s^2][2\gamma M_s^2 - (\gamma-1)]}{(\gamma+1)^2 M_s^2} \quad \Rightarrow \quad \frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_s^2$$

$$M_{s2} = \frac{2+(\gamma-1)M_s^2}{2\gamma M_s^2 - \gamma + 1} \quad \Rightarrow \quad M_{s2} \approx \frac{\gamma-1}{2\gamma}$$

Onda di shock



<u>At shock waves, acceleration of particles occurs</u>. The ingredients are:

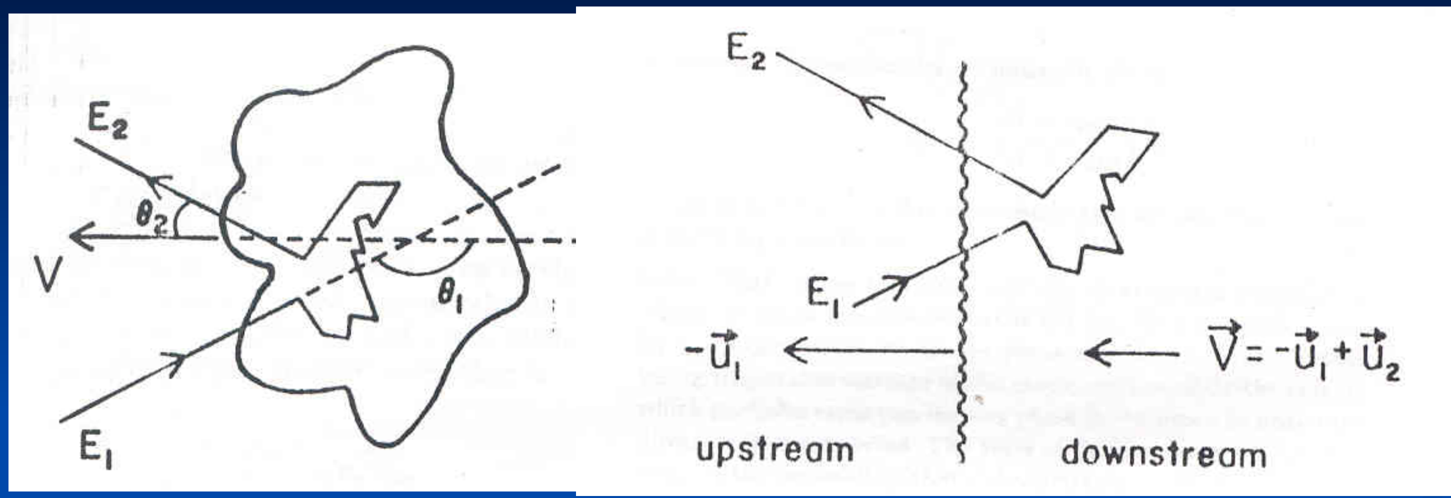
- a physical mechanism to change energy at each <u>collisionless</u> interaction with the shock wave and
- a statistical process which gives an average gain of energy with an enough high number of encounters with the shock wave front, i.e. suitable interaction time

These processes are dominated by scattering in the irregularities downstream and upstream flux.

They act as "scattering centers" for collisionless interactions, frozen into the plasma flux. Because of this they may be approximated as "**walls**" of infinite mass on which particles are "reflected"

<u>The shock front does not play any direct role.</u>

Isotropization



Particles in the cloud or around the shock front are scattered and isotropized by **diffusion** in the B field fluctuations, so that their average motion is the same of the cloud or of the fluid around the shock after few diffusion legnths. This means that the particles are in average at rest in the local fluid frame.

The process is called isotropization

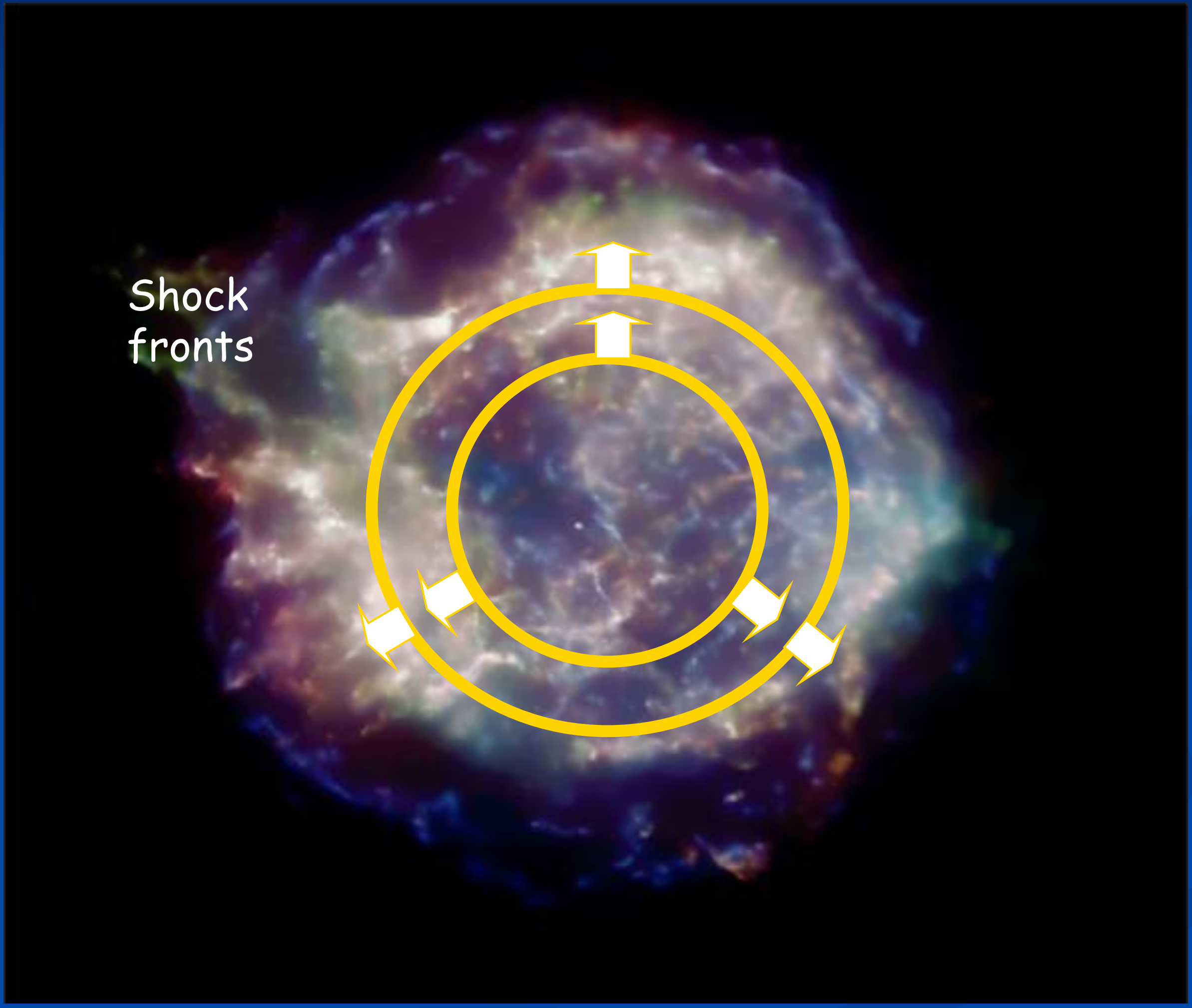
The process is collisionless therefore energy is not changed in the reference frame of the cloud or shock

Since the mass of particle is << M_{cloud} or M_{shock} we can, in first approximation, consider the cloud or the shock as "walls" of infinite mass on which the particle "bounce" forth and back

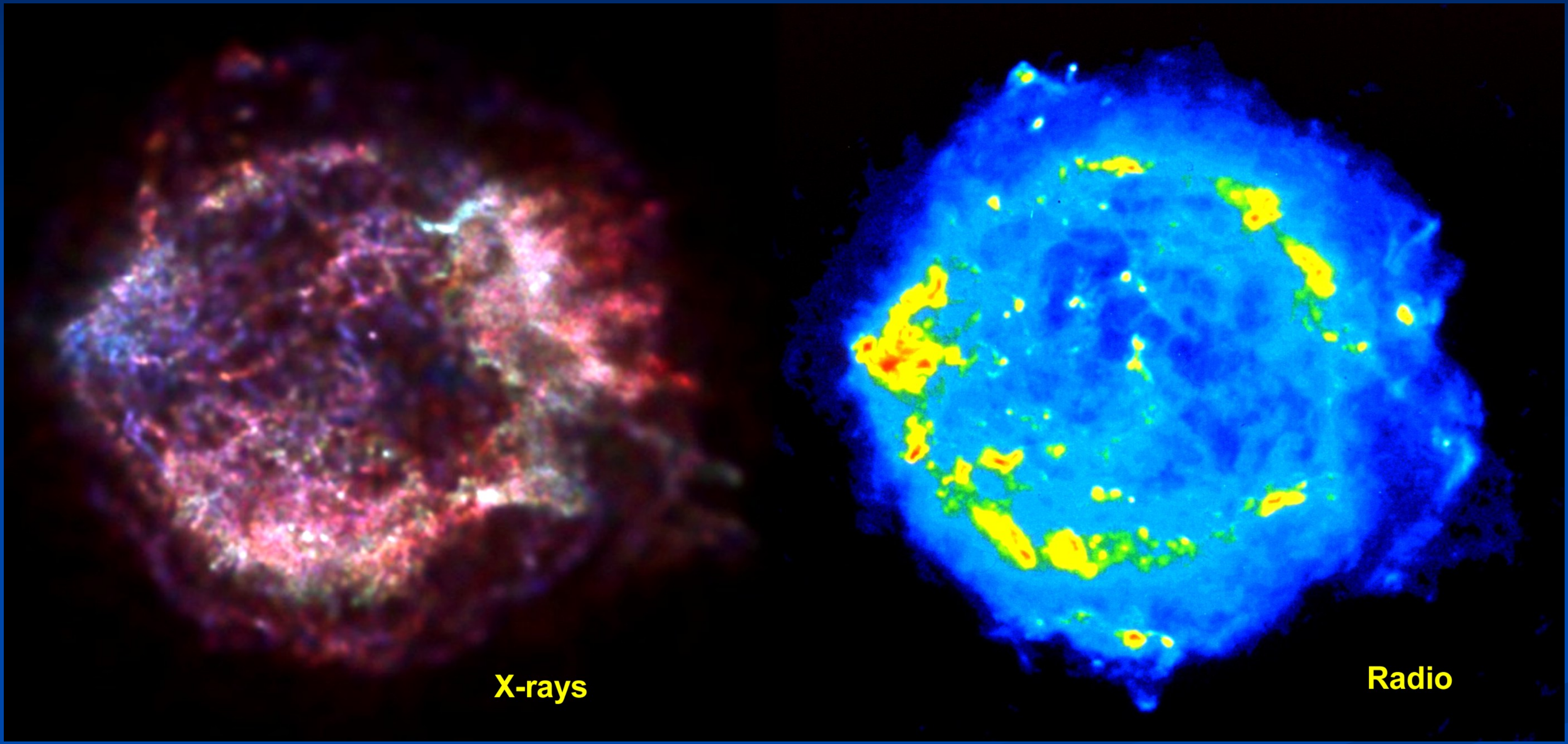
SuperNova Remnants (SNRs)

A supernova remnant (SNR) consists essentially of the stellar ejecta of the SN explosion embedded in a hot expanding bubble, preceded by swept-up interstellar material and an outer blast wave (strong shock) propagating into the interstellar medium

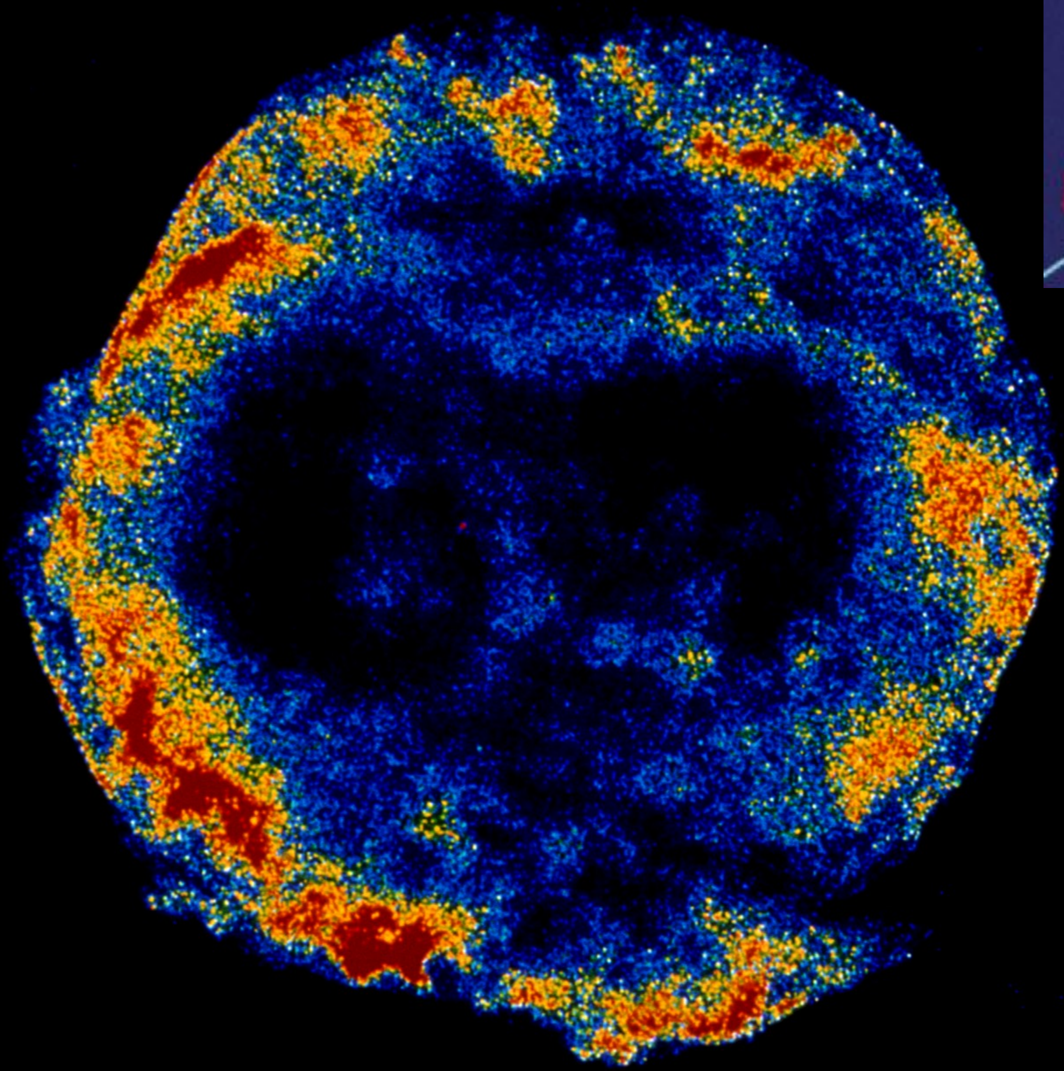
CasA Supernova Remnant in X-rays

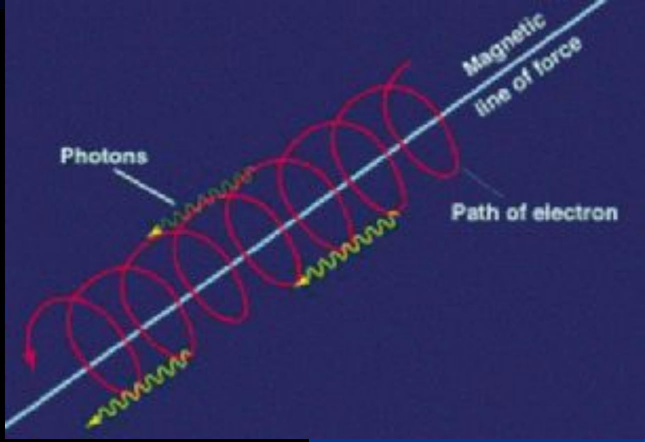


Supernova Remnant Cassiopeia A



2.9 pc, exploded 350 years ago





- Remnant of Tycho's supernova of 1572 AD

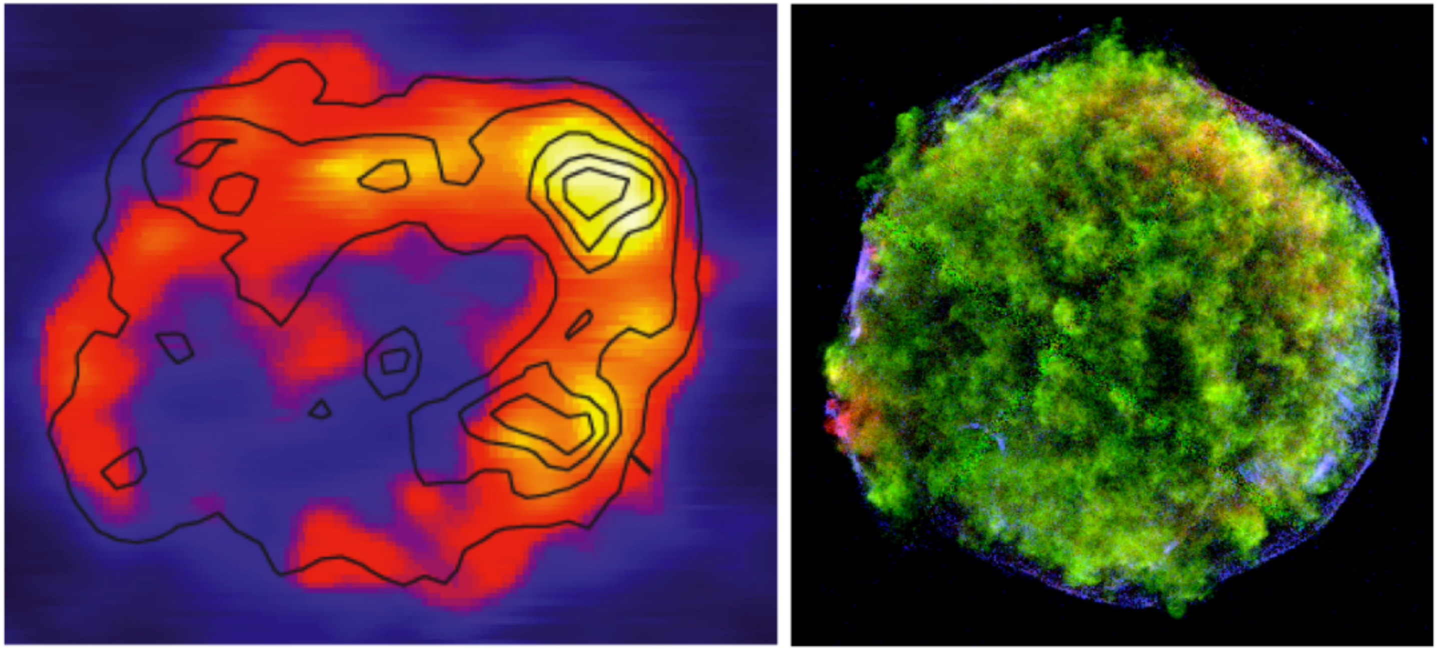
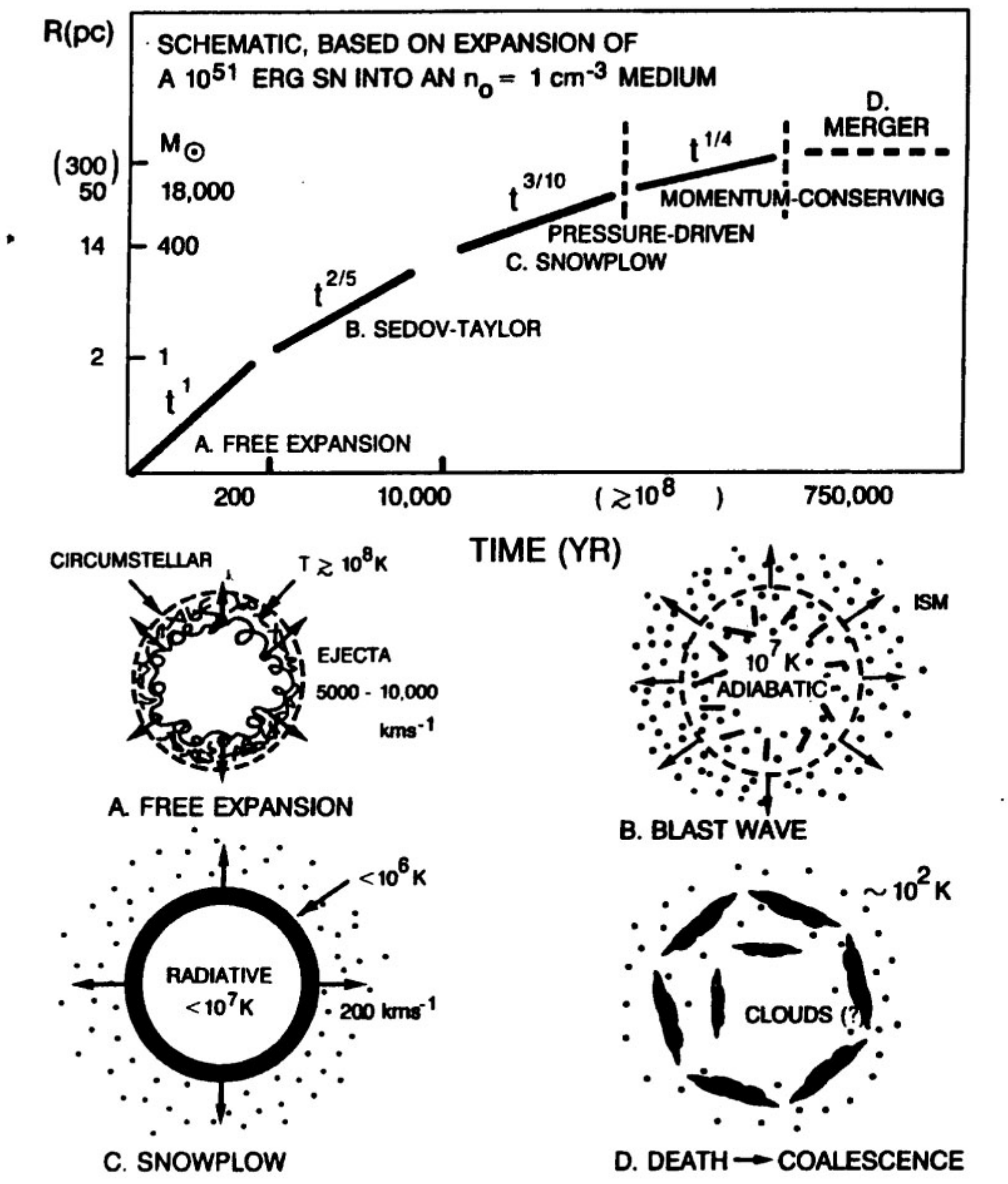


Fig. 4 *Left Panel:* Morphology of the RX J1713.7-3946. The colors illustrate the high energy gamma ray emission as measured by HESS (Aharonian et al, 2007), while the contours show the X-ray emission in the 1-3 keV band measured by ASCA (Uchiyama et al, 2002). *Right Panel:* Morphology of the Tycho SNR as measured with Chandra (Warren et al, 2005). The three colors refer to emission in the photon energy range 0.95 − 1.26 keV (red), 1.63 − 2.26 keV (green), and 4.1 − 6.1 keV (blue). The latter emission is very concentrated in a thin rim and is the result of synchrotron emission of very high energy electrons.

STANDARD SNR EVOLUTION



<u>Main properties:</u>

Different expansion stages:

- Free expansion stage (t<1000 yr) $R \propto t$
- Sedov-Taylor stage (1000 yr< t< 10,000 yr) $R \propto t^{2/5}$
- Pressure-driven snowplow (10,000 yr< t<250,000 yr) $R \propto t^{3/10}$
- Momentum-conserving (250,000<t<750,000 yr) $R \propto t^{1/4}$

DSA

If we concentrate only on the velocity discontinuity, we can easily understand the interest of shock waves for particles acceleration. In the shock frame, the upstream medium is coming toward the shock with a velocity v_1 (note of course that $v_1 = v_{sh}$ where v_{sh} is the shock velocity). Passing through the shock the gas slows down and the downstream medium is moving away from the shock with a velocity $v_2 = v_1/r = v_{sh}$ (r is the compression ratio)

Let us now consider an observer at rest in the upstream frame. He sees the shock approaching with a velocity $v_1 = v_{sh}$, he also sees the downstream medium approaching with a velocity[3] $\Delta v = v_1 - v_2 = \left(\frac{r-1}{r}\right) v_{sh}$ (see Fig. 3.8a).

Now for an observer at rest with respect to the downstream fluid, the shock is going away with the velocity v_2 we obtained before, but the upstream medium is coming toward the observer, again with a velocity $\Delta v = v_1 - v_2 = \left(\frac{r-1}{r}\right) v_{sh}$ (see Fig. 3.8b).

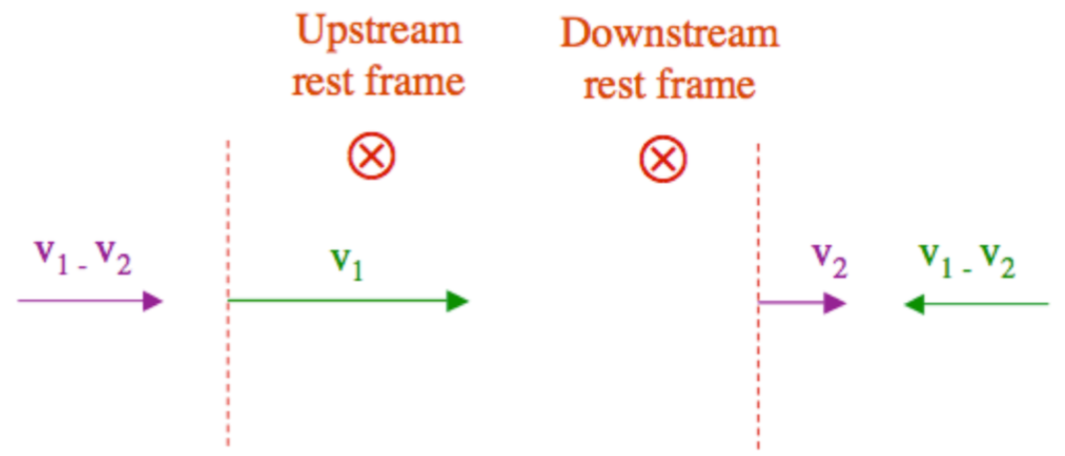
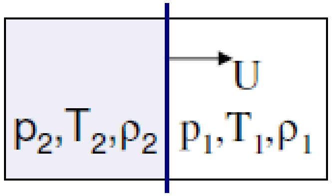


Fig. 3.8: Schematic view of the shock wave as represented in Fig. 3.7, this time seen in the upstream (left) and downstream (right) media rest frames.

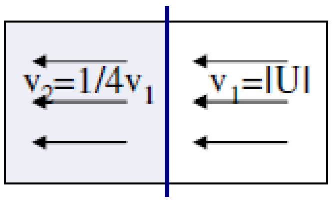
Infinite plane shock with a magnetic field perpendicular to it



a)

$U \gg c_s$ through stationary interstellar gas

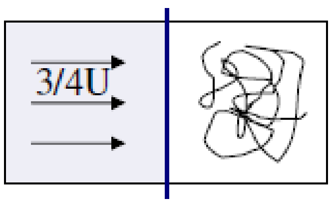
mass conservation: $\rho_1 v_1 = \rho_2 v_2$



b)

reference frame of the shock front at rest:

for monoatomic or fully ionised gas $\gamma=5/3$, $\rho 2/\rho 1=(\gamma+1)/(\gamma-1) = 4$

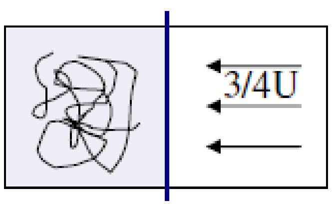


c)

upstream gas stationary:

isotropic distribution of velocities

the gas behind the shock moves with velocity $V=-v_1+v_2$



d)

downstream gas stationary:

isotropic distribution of velocities

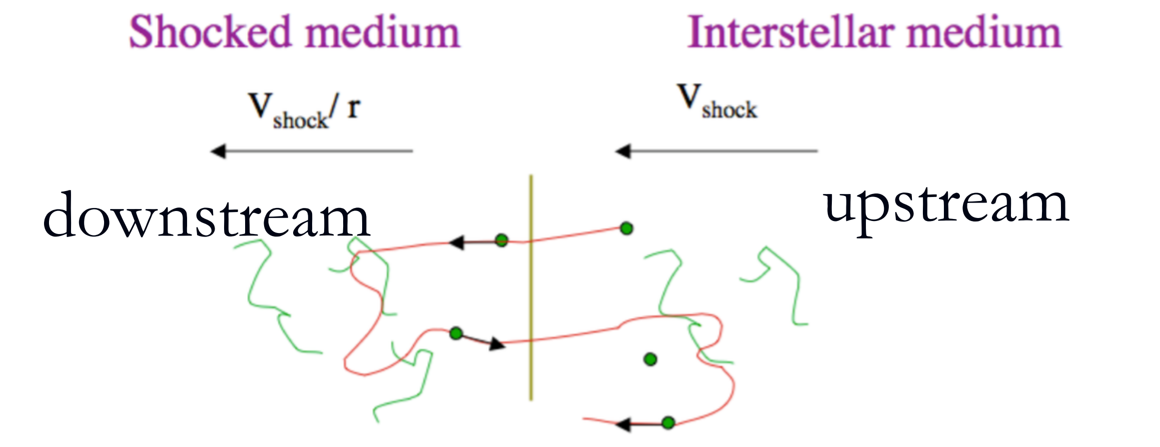
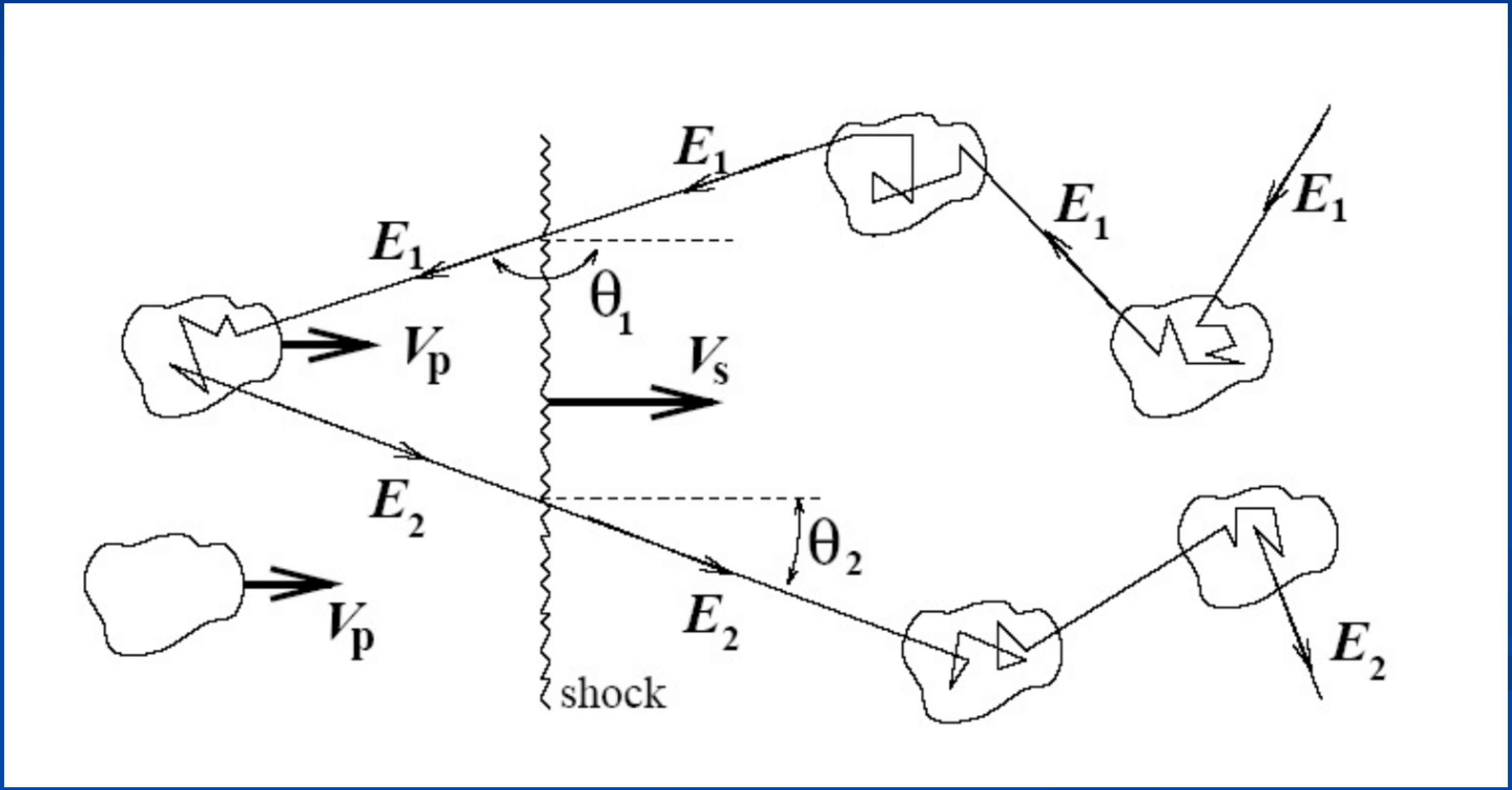


FIG. 3.9: Schematic view of a cycle as seen in the shock rest frame : the particle initially in the ISM (upstream medium) enters the shocked medium (downstream medium), it is then isotropized by the magnetic turbulence and reflected back to the upstream medium where the particle is isotropized again and eventually crossed the shock to start a new cycle.

Let us assume the both the upstream and downstream media are magnetized [4]. We are then in a situation where a particle coming from the upstream medium and passing through the shock would see the downstream medium as a "magnetic cloud" coming toward it (a cloud with a velocity Δv with respect to the upstream fluid rest frame). Likewise a particle coming from the downstream medium and passing through the shock would see the upstream medium as a "magnetic cloud" coming toward it (with a velocity Δv with respect to the downstream fluid rest frame). We can then understand that a particle which would cross several times the shock for instance from upstream to downstream then back upstream could gain energy by interacting with moving "magnetic clouds". The critical difference with the original mechanism proposed by Fermi is that with this configuration, all the collisions would now be head-on. It is then likely that this mechanism involving charged particles cycling across a shock front will turn out to be much more efficient than the original mechanism proposed by Fermi. To prove this statement we need to perform the calculation of the energy variation experienced by a charged particle during a cycle *upstream* $\rightarrow$ *downstream* $\rightarrow$ *upstream*.

Collisions on magnetic irregularities are always head-on



Energy gain in a cycle

Before calculating the mean energy gain experienced by a charged particles during a cycle, we need to set a series of physical hypotheses to define the framework of our calculation. The relevance of these hypothesis will be discussed later on, throughout this section :

- The shock is an infinite plane
- The media upstream and downstream of the shock have an infinite spatial extension

- There is no limitation in time, the relevant physical quantities are in a steady state
- There are magnetic field inhomogeneities, both upstream and downstream of the shock which "isotropize" energetic charged particles, the propagation of particles is diffusive in both media.
- The shock is non-relativistic $v_{sh} \ll c$, the charged particles are relativistic $v_{part} \simeq c \gg v_{sh}$.

For this calculation we use unprimed quantities for the upstream frame and primed quantities for the downstream frame. Let θ_{in} be the the angle between the particle velocity and the shock normal at the initial shock crossing in the upstream frame and θ'_{out} the angle of the particle with the shock normal in the downstream frame, when crossing the shock back to the upstream medium. On a cycle upstream → downstream → upstream, we have :

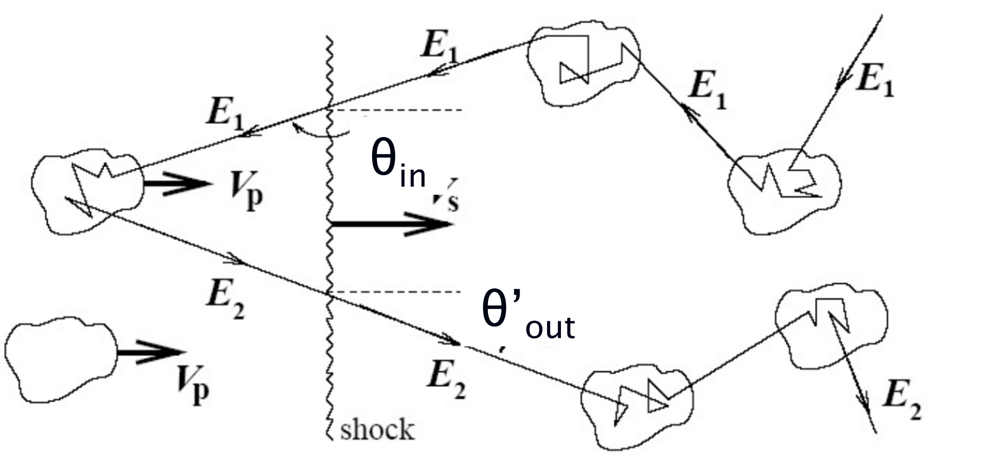
$$\begin{cases} E'_{in} = \gamma E_{in}(1 - \beta \cos\theta_{in}) \\ E_{out} = \gamma E'_{out}(1 + \beta \cos\theta'_{out}) \end{cases} \tag{3.29}$$

γ and β would correspond to γ_{cloud} and β_{cloud} in the original Fermi mechanism. In the case of a shock wave, they correspond to the velocity of the downstream medium in the upstream fluid frame (see above) and then we have $\beta = \Delta v/c$ and $\gamma = \frac{1}{\sqrt{1-\frac{\Delta v^2}{c^2}}}$. Using $E'_{in} = E'_{out}$, we get the already familiar expression :

$$\frac{\Delta E}{E} = \frac{\beta^2 - \beta\cos\theta_{in} + \beta\cos\theta'_{out} - \beta^2\cos\theta_{in}\cos\theta'_{out}}{1-\beta^2} \tag{3.30}$$

again, the mean fractional energy gain is obtained by averaging $\langle\cos\theta_{in}\rangle$ and $\langle\cos\theta'_{out}\rangle$, the result will however differ from the original case with magnetic clouds.

Energy gain in a cycle



To obtain $\langle\cos\theta_{in}\rangle$ and $\langle\cos\theta'_{out}\rangle$, we need to know the probability of crossing the shock with an angle between θ and $\theta+d\theta$. Let us neglect the shock velocity since $v_{sh} \ll v_{part}$. The particle crosses the shock with a velocity $v_{part}\cos\theta$ and then asuming a particle density n_0, the number of particles passing the shock with an angle between θ and $\theta+d\theta$, through a surface dS during a time dt is :

$$dN^4 = \frac{n_0}{4\pi} v_{part}\cos\theta d\Omega dS dt = \frac{n_0}{2} v_{part}\cos\theta\sin\theta d\theta dS dt \tag{3.31}$$

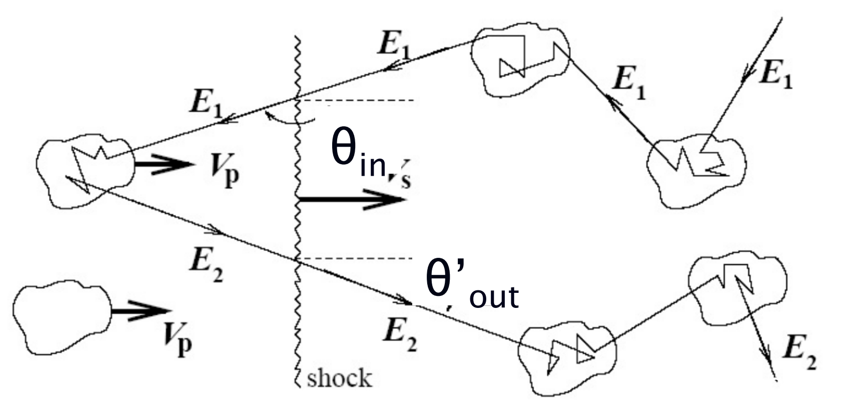
the probability of crossing the shock with an angle between θ and $\theta+d\theta$ is then proportional to $\cos\theta\sin\theta d\theta$. We then have :

$$\langle\cos\theta\rangle = \frac{\int_{\theta_{min}}^{\theta_{max}} \cos^2\theta\sin\theta d\theta}{\int_{\theta_{min}}^{\theta_{max}} \cos\theta\sin\theta d\theta} = \frac{\left[\frac{1}{3}\cos^3\theta\right]_{\theta_{min}}^{\theta_{max}}}{\left[\frac{1}{2}\cos^2\theta\right]_{\theta_{min}}^{\theta_{max}}} \tag{3.32}$$

for the crossing from upstream to downstream we have $\theta_{min} = \frac{\pi}{2}$ and $\theta_{max} = \pi$ (particle and shock velocities are antiparallel) which give :

$$\langle\cos\theta_{in}\rangle = -\frac{2}{3} \tag{3.33}$$

for the crossing back from downstream to upstream we have $\theta_{min} = 0$ and $\theta_{max} = \frac{\pi}{2}$ which give : $\langle\cos\theta'_{out}\rangle = \frac{2}{3}$



Energy gain in a cycle

Note that we implicitly got rid of some complications and of second order corrections by neglecting the shock velocity in this calculation. Since $v_{sh} \ll c$ we also have $\Delta v \ll c$ we can then neglect terms in β^2 in Eq. 3.30 calculating $\langle \Delta E/E \rangle$. We finally get :

$$\langle \frac{\Delta E}{E} \rangle = \frac{4}{3}\beta = \frac{4}{3}\beta_{sh}\left(\frac{r-1}{r}\right) \tag{3.35}$$

where $\beta_{sh} = v_{sh}/c$ and r is the above-mentioned compression ratio.

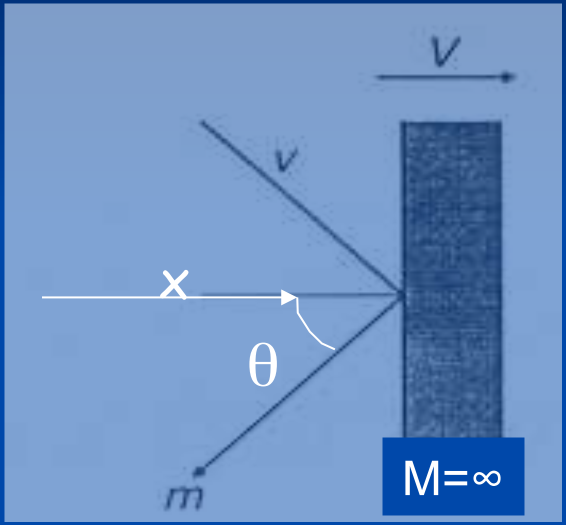
As anticipated, DSA is indeed an acceleration mechanism (since $\langle \frac{\Delta E}{E} \rangle$ is positive) with an energy gain proportional to β which is a very important step forward when comparing with the original Fermi mechanism. DSA is also called **first order Fermi mechanism** for obvious reasons.

Shock waves: Energy gain including the shock speed

The mechanism for energy gain is always the same

The gain is given by "elastic" collision in the wall rest frame and a relativistic boost to the lab.

- In the ref. frame of the walls, $E'_i = \gamma(E_i + Vp_{ix})$, $P'_{ix} = \gamma(p_{ix} + VE_i)$ (c = 1)



The collision is elastic (i.e. collisionless): $E'_f = E'_i$ and

$$p'_f = -p'_i$$

Going back to the lab frame

$$E_f = \gamma(E'_f - Vp'_{fx}) = \gamma(E'_i + Vp'_{ix})$$

$$p_{fx} = \gamma(p'_{fx} - VE'_f) = \gamma(-p'_{ix} - VE'_i)$$

Therefore $E_f = \gamma(E'_i + Vp'_{ix}) = \gamma^2[(E_i + Vp_{ix}) + V(p_{ix} + VE_i)] = \gamma^2[E_i + 2Vp_{ix} + V^2E_i]$

But $p_{ix} = v_{ix}E_i$ → $E_f = \gamma^2 [E_i + 2V v_{ix}E_i + V^2E_i] = \gamma^2 E_i[1 + 2V v_{ix} + V^2]$

→the relative gain per each scattering is (put back c)

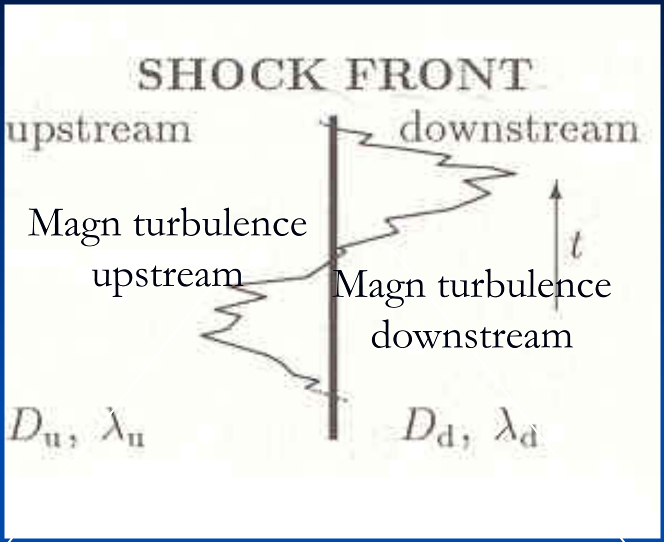
$$\Delta E/E_i = (E_f - E_i)/E_i = \gamma^2 [1 + 2V v_{ix} + V^2] - 1 = \gamma^2 [1 + 2V v\cos\theta/c^2 + V^2/c^2] - 1$$

$$\gamma^2 = [1-(V/c)^2]^{-1} \approx 1+(V/c)^2 \ if \ V \ll c \qquad \Delta E/E_i \approx [1+(V/c)^2][1+2Vv\cos\theta/c^2+(V/c)^2]-1$$

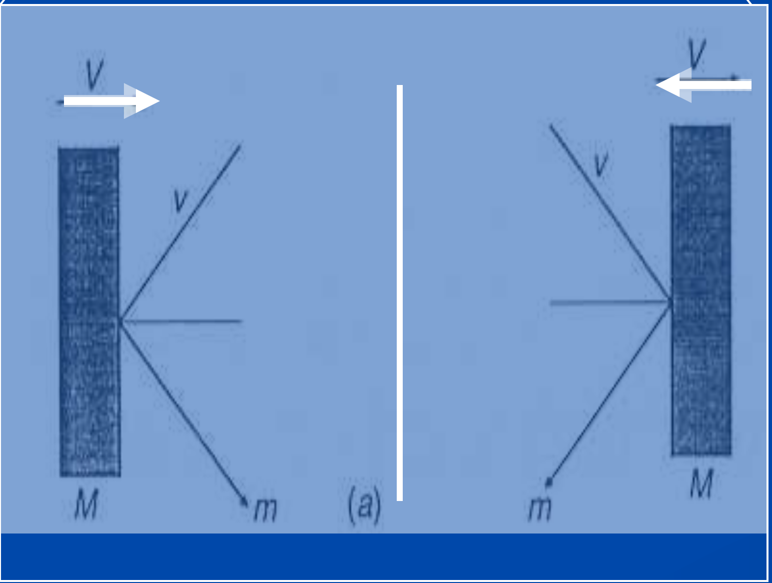
$$= 2Vv\cos\theta/c^2+(V/c)^2+(V/c)^2+(V/c)^2[2Vv\cos\theta/c^2+(V/c)^2] \qquad \approx 2Vv\cos\theta/c^2 + 2(V/c)^2$$

It is the gain for non-relativistic shock waves, i.e. V<< c

Diffusive Shock Acceleration



DSA is the dominant mech. for quasi-parallel shocks. In DSA particle scattering up- and down-stream is crucial→ high level of turbulence and irregularities are requested on both sides of the shock.
The magnetic fields on both sides are turbulent, so that the resulting scattering is quantified by the diffusion coefficient D or by the mean free paths λ



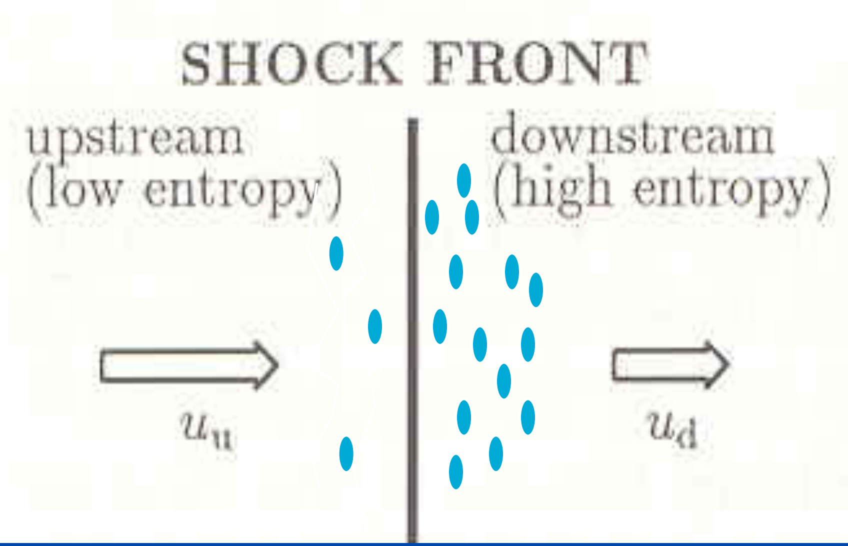
The crucial point is that in the ref frame in which one of the fluxes is at rest, a particle always sees the other side moving toward it (in a symmetric way for the 2 sides), so that the "collision" is always head-on.
Since the scattering centers are frozen into the plasma, particle scatt. back and forth through the shock front can be understood as repeated reflections between converging scattering centers (i.e. "walls" with infinite mass).

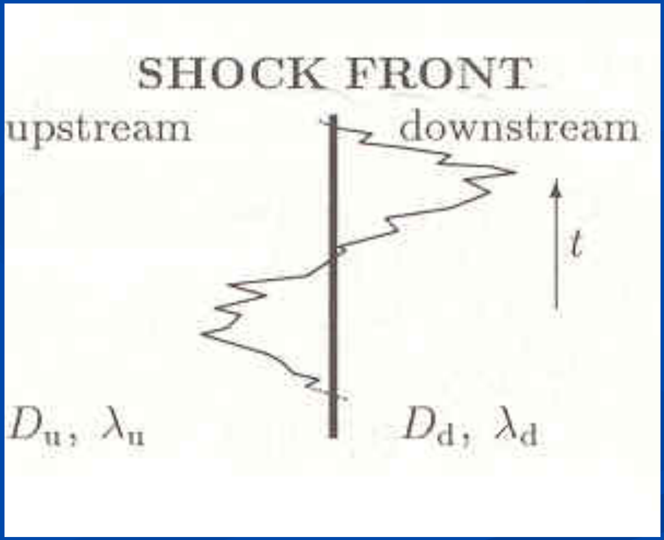
The B field fluctuations are frozen-in with the plasma and carried with the plasma flow

Shock waves: Energy gain

Because of the stochastic nature of the process we cannot follow a single particle, but we can work out the average energy gain.

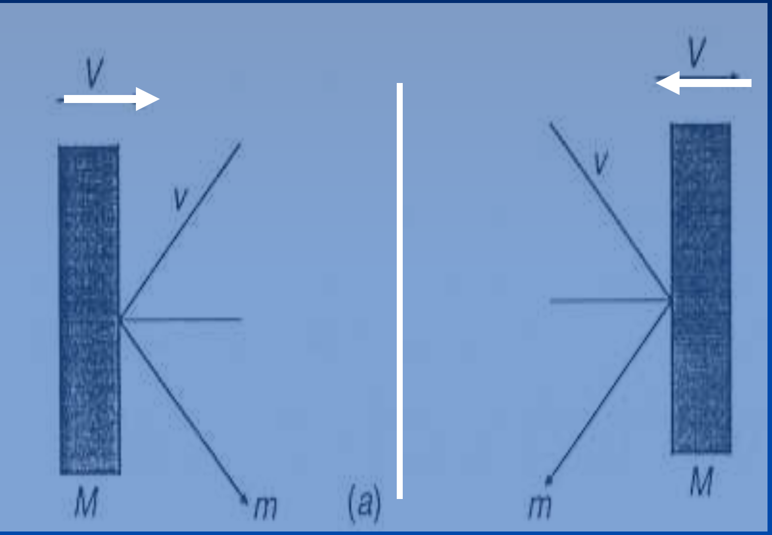
The basic property is that when a particle enters in the scattering region, because of the random distribution of the scattering centers, also its direction gets very rapidly randomized –key point- so that the average speed of the test particle is the same as for the flow and an isotropic distribution of directions can be assumed in a suitable ref. frame (the shock rest frame).





$$\langle \Delta E/E \rangle = \int_{\theta_1}^{\theta_2} P(\Omega) \frac{\Delta E}{E}(\cos\theta) \quad d\Omega$$

DSA: Energy gain (1st order Fermi mechanism)



P(θ) is prop. to the normal comp. of the particle velocity along the normal to the shock, vcosθ:
$P(\Omega)\, d\Omega = (\cos\theta/\pi)(2\pi \sin\theta d\theta) = 2\cos\theta d(\cos\theta)$ with $0<\theta<\pi/2$

$$< \Delta E/E > \approx \int p(\Omega) d\Omega [2 V v \cos\theta / c^2 + 2(V/c)^2]$$

$$\langle \Delta E/E \rangle = \frac{4}{3}\left(\frac{V}{c}\right)\left(\frac{v}{c}\right)$$

V<< c

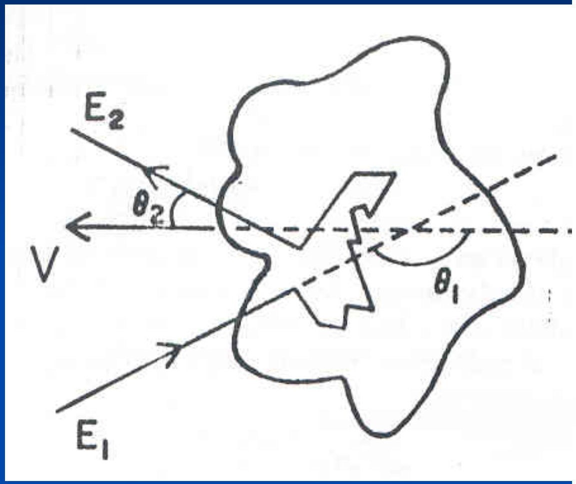
V is NOT the shock speed, but relative speed $|u_u - u_d|$ of the flow speed upstream and downstream in the SRF (i.e. Standing shocks).
In the lab, $u_u = U$, the speed of shock front and downstream con $u_d = (\rho_u/\rho_d)U \rightarrow V = (1 - \rho_u/\rho_d)U$

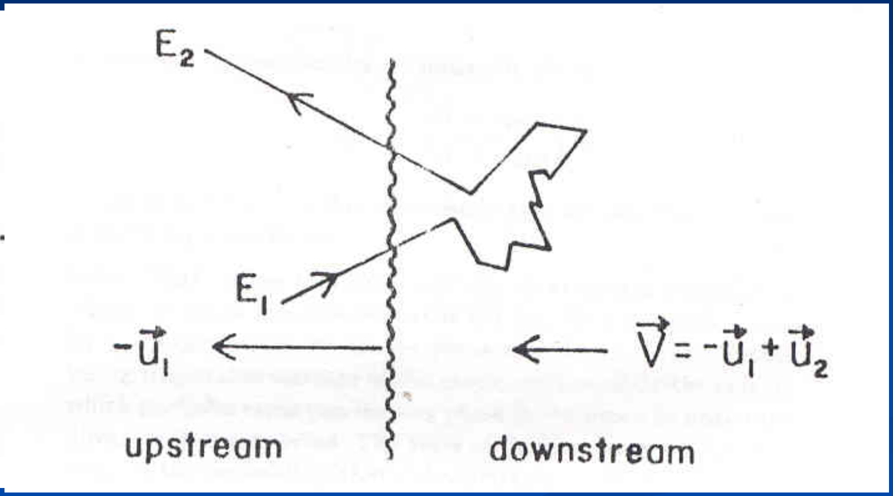
For V<<c, <ΔE/E> is of 1st order in V/c and is maximal for relativistic particles v ≈c

$$\langle \Delta E/E \rangle = \frac{4}{3}\left(\frac{V}{c}\right)$$

SW energy gain: another approach

Let consider another approach: the same physical mechanism is working both at a shock front both in a cloud of magnetized plasma with strong magnetic irregularities acting as collisionless scattering centers (e.g. this can be thought as the turbulent region ahead a shock front or even the whole galactic disk)





Downstream the shocked gas flows to the left with speed $\mathbf{u_2}$ relative to the shock front → in the lab it moves to the left with $\mathbf{V=-u_1 + u_2}$

In the rest frame of the moving cloud or in the shock rest frame, we have

$E_1'/c = \gamma(E_1/c - \beta_V p_{x1})$ $= \gamma(E_1/c)(1 - \beta_V \cos\theta_1)$ Because for ultra-relat. particles E/c=p

The collisions are elastic (because collision-less in the rest frame of the cloud/SW)

$E_1' = E_2'$ and going back to the labo

$E_2/c = \gamma(E_2'/c + \beta_V p_{x2}')$ $= \gamma(E_2'/c)(1 + \beta_V \cos\theta_2')$ $= \gamma(E_1'/c)(1 + \beta_V \cos\theta_2')$

$= \gamma^2(E_1/c)(1 - \beta_V \cos\theta_1)(1 + \beta_V \cos\theta_2')$ $= \gamma^2[(E_1/c)(1+\beta_V \cos\theta_2' - \beta_V \cos\theta_1 - \beta_V^2 \cos\theta_1 \cos\theta_2')$

therefore $\frac{\Delta E}{E} = \frac{E_2 - E_1}{E_1}$ $= \gamma^2(1+\beta_V \cos\theta_2' - \beta_V \cos\theta_1 - \beta_V^2 \cos\theta_1 \cos\theta_2') - 1$

SW energy gain: another approach

$$\frac{\Delta E}{E} = \frac{E_2 - E_1}{E_1} = \gamma^2(1+\beta_V \cos\theta'_2 - \beta_V \cos\theta_1 - \beta_V^2 \cos\theta_1 \cos\theta'_2) - 1$$

The crucial difference between the two cases comes when we take the angular averages to obtain the average fractional energy gain per encounter.

First, the average over θ_2' has be to done

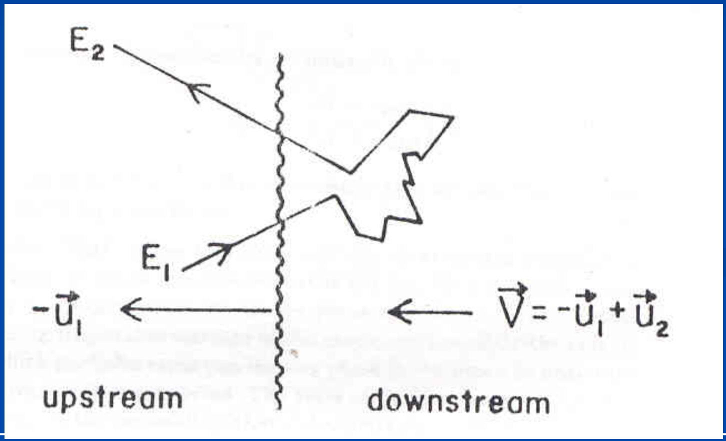
Cloud

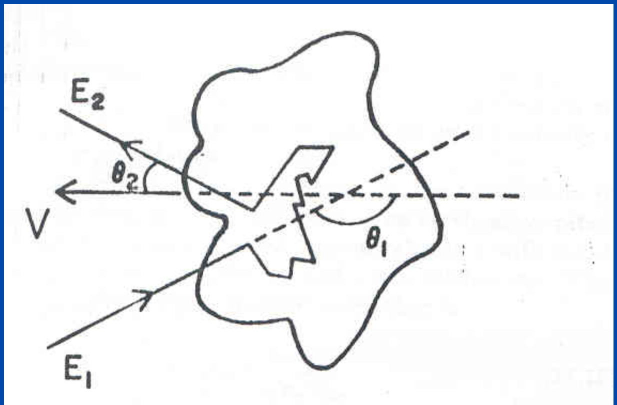
In S' (where the particle is at rest in average), the particle distr. is isotropic because of the random motion inside the cloud, therefore

$dn/d(\cos\theta_2') = \text{const.}, \quad -1 < \cos\theta_2' < 1$

Then $\langle \cos\theta_2' \rangle = 0$ and

$\Delta E/E_1 = \gamma^2 (1 - \beta_V \cos\theta_1) - 1$





Shock front

In S' we have again a isotropic distr. but we have to project on a plane (the shock front), therefore

$dn/d(\cos\theta_2') = 2\cos\theta_2', \quad 0 < \cos\theta_2' < 1$

Then $\langle \cos\theta_2' \rangle = 2/3$ and

$\Delta E/E_1 = \gamma^2 [1 + (2/3)\beta_V - \beta_V \cos\theta_1 - (2/3)\beta_V^2 \cos\theta_1] - 1$

SW energy gain: another approach

Shock front

In S' we have again a isotropic distr. but we have to project on a plane (the shock front), therefore

$dn/d(\cos\theta_2') = 2\cos\theta_2'$, $0 < \cos\theta_2' < 1$

Then $< \cos\theta_2' > = 2/3$ and

$\Delta E/E_1 = \gamma^2 [1 + (2/3)\beta_V - \beta_V \cos\theta_1 - (2/3)\beta_V^2 \cos\theta_1)] - 1$

The probability of the particles which cross the shock to arrive in a solid angle $d\Omega$ around θ dir in the time dt is:

$$dn \propto v\cos\theta d\Omega dt \rightarrow dp(\theta) \propto \cos\theta d(\cos\theta)$$

Normalizing so that prob is 1 for all the particles approaching the shock (ie those with $0 < \theta < \pi/2$) one gets

$$dp(\theta) = 2\cos\theta d(\cos\theta)$$

SW energy gain: another approach

Next we need to average over $\cos\theta_1$

2nd order

Cloud

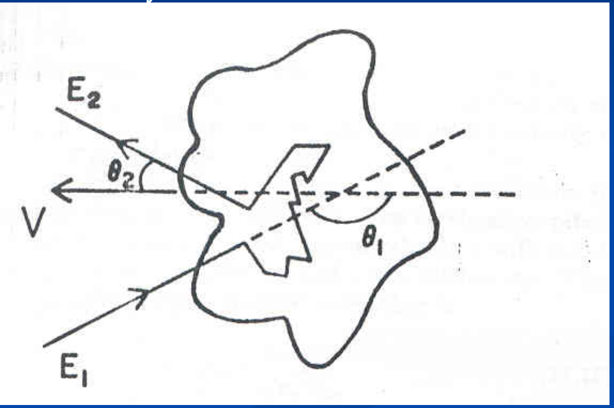
The probability of a collision is ~ to the relative speed between the cloud and the particle

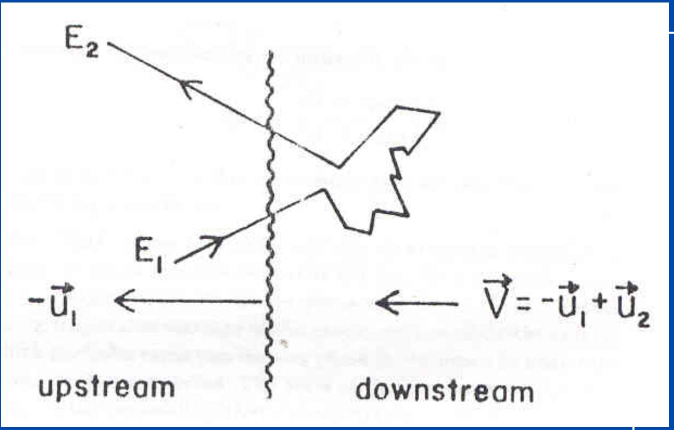
$dn/d(\cos\theta_1)=(c-V\cos\theta_1)/2c, \quad -1< \cos\theta_1<1$

Then $<\cos\theta_1>=-\beta_V/3$ and

$\Delta E/E_1=\gamma^2 (1-\beta_V\cos\theta_1) - 1= (1+\beta_V^2/3)/(1-\beta_V^2) - 1=$

$= (4/3)\gamma^2\beta_V^2 \approx$ **$(4/3)\beta_V^2$ if $\beta_V << 1$**





Shock front

We have again a isotropic distr. but we have to project on a plane (the shock front), therefore

$dn/d(\cos\theta_1)=2\cos\theta_1, \quad -1<\cos\theta_1<0$

Then $<\cos\theta_1>=-2/3$ and

$\Delta E/E_1=\gamma^2 [1+(4/3)V - (4/9)V^2] - 1=$

$= \gamma^2[(4/3)\beta + (5/9)\beta^2] \sim (4/3)\beta$ if $\beta <<1$

1st order

$$
\begin{aligned}
v_{\text{rel}} &= \left|\vec{v}_{\text{cloud}} - \vec{v}_{\text{particle}}\right| \\
&= \sqrt{(c - v\ \cos\theta_i)^2 + v^2\ \sin^2\theta_i} \\
&= c\sqrt{(1 - \beta\cos\theta_i)^2 + \beta^2\ \sin^2\theta_i} \\
&= c\sqrt{1 + \beta^2 - 2\beta\ \cos\theta_i} \\
&\simeq c\sqrt{(1 - 2\beta\ \cos\theta_i} \\
&\simeq c\,(1 - \beta\ \cos\theta_i)
\end{aligned}
$$

$\beta \ll 1$

Spectrum of accelerated particles

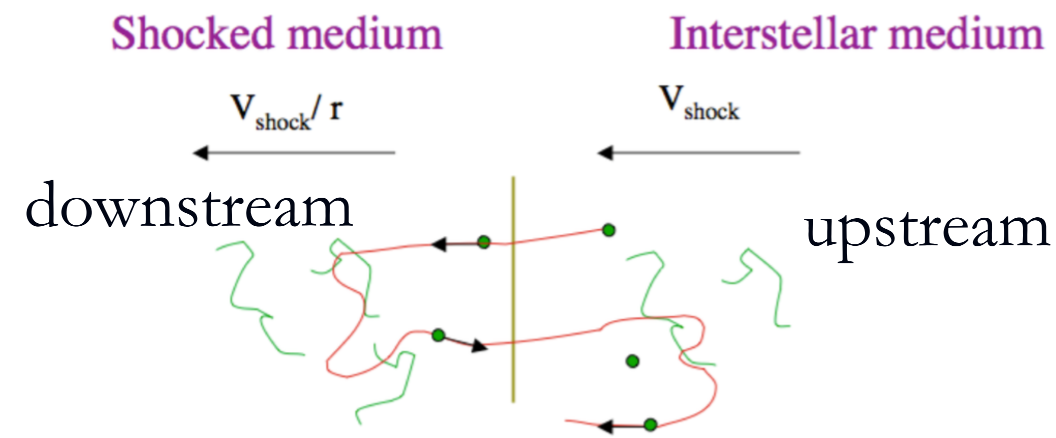
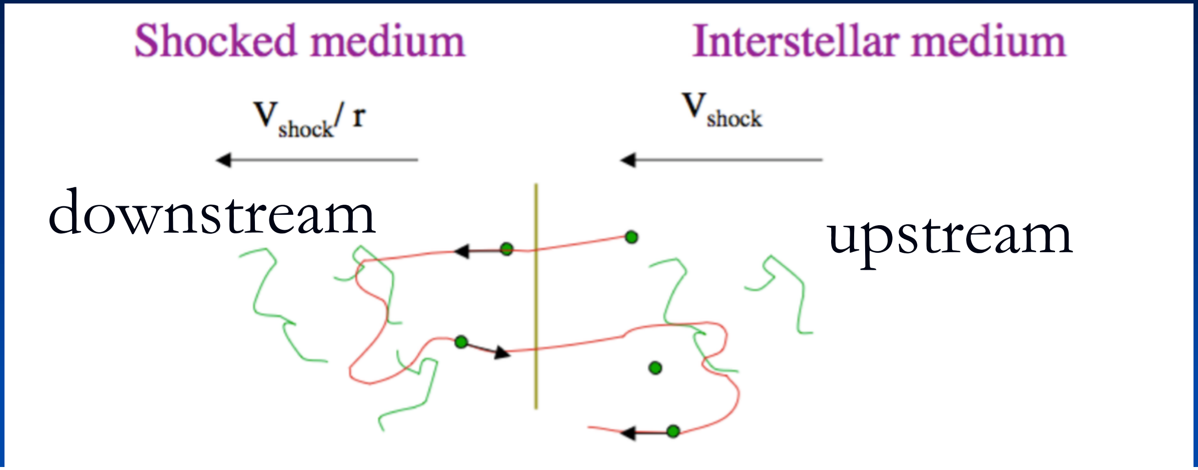


Fig. 3.9: Schematic view of a cycle as seen in the shock rest frame : the particle initially in the ISM (upstream medium) enters the shocked medium (downstream medium), it is then isotropized by the magnetic turbulence and reflected back to the upstream medium where the particle is isotropized again and eventually crossed the shock to start a new cycle.

After calculating the mean energy gain obtained after a single cycle upstream $\rightarrow$ downstream $\rightarrow$ upstream, one can try to estimate the number of cycles this particles are going to achieve before "leaving the system". At first place, it is important to understand what mechanism is going to limit the number of cycles a particle can perform. The critical point is that during each cycle a given has a probability not to come back to the shock, in other word a probability to escape from "the acceleration region".

Spectrum of accelerated particles



Within our hypothesis (infinite media upstream and downstream and steady state), the particles have no way of escaping upstream of the shock. Indeed since by hypothesis the accelerated particles are isotropized by the ambient magnetic fields the accelerated particles fluid has no net velocity with respect to the medium (either downstream or upstream) rest frame. As in the upstream medium frame the shock is "going after the particles", the return probability at the shock is 1, meaning that the escape probability is 0. In the downstream medium on the other hand, the cosmic-ray fluid has a zero velocity with respect to the ambient medium and the shock is going away with a velocity $v_2 = v_{sh}/r$. This means that the accelerated particles fluid is in average slowly advected away from the shock. The escape probability can be estimated by comparing the flux of particles being advected far away from the shock with the flux of particle entering the downstream medium by crossing the shock from the upstream medium. Before performing this trivial calculation, we can give an alternative reasoning to understand the impossibility of leaving the system in the upstream frame.

Spectrum of accelerated particles

Let us now consider individual particles rather than the "accelerated particles fluid" and let's think in term of diffusion as we did in the chapter dedicated to Galactic cosmic-rays. Let us assume a particle enters the upstream medium (crossing the shock from upstream) at $t = 0$. Since we have an infinite amount of time available and since the upstream medium is infinite (at least in the framework of our hypotheses) diffusion theory tells us that the probability for the particle to come back to the plane where it crossed the shock within the infinite amount of time available is 1. Then the probability to cross the shock again would be 1 even if the shock was not moving. The fact that the shock is going after the particle in the upstream medium makes it even easier for the particle to cross the shock again.

Downstream, the probability for the particle to come back to the plane where it crossed the shock (from upstream) is still one, but the problem is that during the time it took for the particle to come back, the shock has moved away. If one calculated, using diffusion theory, the probability for a particle, not only to come back to where it last crossed the shock, but where the shock actually is at a given time t between 0 and ∞ (which is actually the condition for the particle to come back to the shock and be able to cross it again and which is, unlike in the upstream medium, a more stringent condition than just coming where it last crossed the shock) then one would find a probability smaller than 1, as a result of the shock going away with respect to the downstream fluid frame.

Spectrum of accelerated particles

Let us now calculate the escape probability using our first reasoning in terms of accelerated particles fluid. Let n_0 be the accelerated particle density. Due to the global advection of the downstream fluid away from the shock front with a velocity v_2, the flux of accelerated particles passing through a unit surface very far away from the shock is $\phi_{esc} = n_0 v_2$. On the other hand the flux of particles crossing the shock from upstream to downstream is given by :

$$\phi_{ud} = \frac{n_0}{2} v_{part} \int_{\pi/2}^{\pi} \cos\theta \sin\theta d\theta = \frac{n_0}{4} v_{part} \simeq \frac{n_0}{4} c \qquad (3.36)$$

The escape probability is then simply the ratio of the two fluxes :

$$P_{esc} = \frac{\phi_{esc}}{\phi_{ud}} \simeq \frac{4}{r} \beta_{sh} \qquad (3.37)$$

$r = \rho_2/\rho_1 = u_1/u_2$ is the shock compression ratio

We then have obtained so far $\langle \Delta E \rangle = \frac{4}{3} \left(\frac{r-1}{r} \right) \beta_{sh} E = kE$ and $P_{esc} = \frac{4}{r} \beta_{sh}$. We have everything we need to predict the slope of the accelerated particle spectrum :

Spectrum of accelerated particles

After 1 cycle, the mean energy of particles injected with the energy E_0 is :

$$E_1 = (1 + k)E_0 \tag{3.38}$$

thus after n cycles :

$$E_n = (1 + k)^n E_0 \Leftrightarrow n = \frac{\ln(E/E_0)}{\ln(1 + k)} \tag{3.39}$$

on the other hand, if we initially injected N_0 particles then after n cycles we have :

$$N_n = N_0(1 - P_{esc})^n = N_0(1 - P_{esc})^{\frac{\ln(E/E_0)}{\ln(1+k)}} = N(\geq E_n) \tag{3.40}$$

the right hand side just meaning that particles which have achieved n end up with energies greater or equal to E_n (since they will either escape at the cycle or continue for more cycles). Using $a^{\ln b} = e^{\ln a \ln b} = b^{\ln a}$ and dropping the subscript n we get :

With a = (1-P_{esc})

$$N(\geq E) = N_0 \left(\frac{E}{E_0}\right)^{\frac{\ln(1-P_{esc})}{\ln(1+k)}} \tag{3.41}$$

Spectrum of accelerated particles

and since the shock is non-relativistic, $\beta_{sh} \ll 1 \Leftrightarrow P_{esc} \ll 1$ and $k \ll 1$:

Taylor's expansion of exponent

$\ln(1 - P_{esc}) \approx -P_{esc}$

$\ln(1 + k) \approx k$

$$P_{esc} = \frac{\phi_{esc}}{\phi_{ud}} \simeq \frac{4}{r}\beta_{sh}$$

$$N(\geq E) = N_0 \left(\frac{E}{E_0}\right)^{\frac{-P_{esc}}{k}} \tag{3.42}$$

moreover

$$N(\geq E) = \int_E^{+\infty} n(E)dE \tag{3.43}$$

where $n(E)dE$ is the number of particles with energy between E and $E + dE$ and is the quantity we are looking for. Then,

$r = \rho_2/\rho_1 = u_1/u_2$

$$n(E) = \left|\frac{dN(\geq E)}{dE}\right| = (x-1)\frac{N_0}{E_0}\left(\frac{E}{E_0}\right)^{-x} \tag{3.44}$$

with (after developing P_{esc}/k),

$$x = \frac{r+2}{r-1} \tag{3.45}$$

We obtain a power law spectrum with a *spectral index* x depending only on the shock compression ratio r! For a monoatomic gas and a strong shock ($\gamma_a = 5/3$, $M_1 \gg 1$), the slope of the power law is $x = 2$ and is "universal"[5].

The Diffusion-Convection Equation: A more formal approach

Obviously the above derivation is rather basic and more advanced techniques exist. While it is not the goal to explain all of these in detail, it is instructive to go over the main ideas. Apart from being relevant results themselves, these theoretical efforts are often the inspiration for simulating the acceleration process. Essentially there exist two approaches to finding the spectrum. In the first one, by Bell[72], one considers the behaviour of one individual particle and then take averages, like was done above. The second approach, which follows the papers by Krymskii[70], Axford, Leer and Skadron[71] and Blandford and Ostriker[74], uses a macro-approach where one considers the distribution function of the particles. Many good reviews on the theoretical efforts in the last decades exist[75][76][59][61].

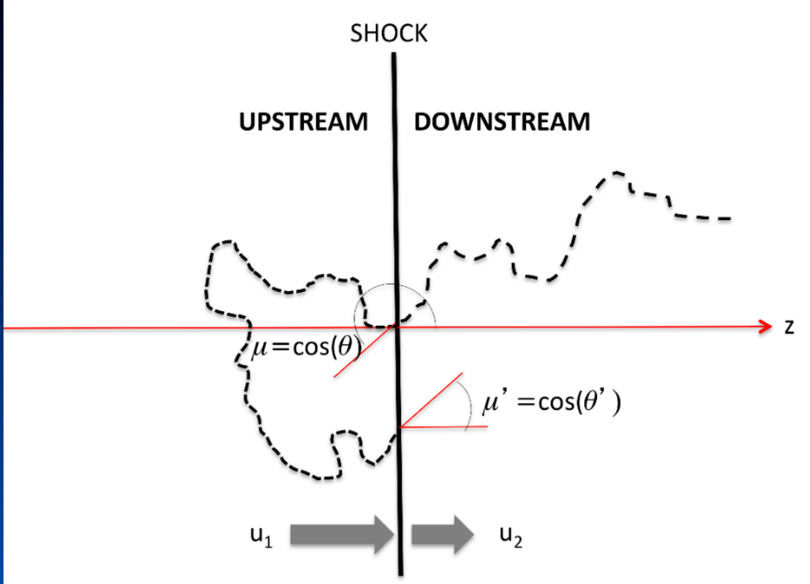
The Diffusion-Convection Equation: A more formal approach

The main idea of the first order mechanism described above is that the accelerated particles diffuse on each side of the shock, allowing for head-on collisions required on shock crossing, leading to the alternative name 'diffusive shock acceleration'. Therefore, to theoretically model the acceleration process, one can try to solve the distribution function f(x,p,t) on each side of the shock from the transport equation for diffusive transport (or actually more appropriately called the Fokker-Planck equation, since we are concerned with individual particles and not flows) and then connect the distributions at the shock front. Most generally the equation is

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla f = \nabla(\boldsymbol{\kappa} \nabla f) + \frac{1}{3} \nabla \cdot \mathbf{V} p \frac{\partial f}{\partial p} + \mathbf{Q}$$

with **V** the plasma flow speed. The second term on the left hand side represents advection by the scattering centres which are moved by the background motion. The first term on the right describes ordinary diffusion, with generally κ the anisotropic diffusion tensor. The second term on the right represents adiabatic momentum changes as the large scale motion of the medium is diverging or converging at the shock. This last term is part of the continuous energy losses term of the transport equation $\partial/\partial E(b(E)N)$ with $b(E)$ = adiabatic energy loss, Q is the source term.

The diffusion approach



For a stationary parallel shock, namely a shock for which the normal to the shock is parallel to the orientation of the background magnetic field (see Fig. 6) the transport of particles is described by the diffusion-convection equation (Skilling, 1975a) (see (Blandford and Eichler, 1987) for a detailed derivation), which in the shock frame reads: Stationary shock means steady state solutions, $\partial f/\partial t = 0$

$$u\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] + \frac{1}{3}\frac{du}{dz}p\frac{\partial f}{\partial p} + Q, \qquad (34)$$

where $f(z,p)$ is the distribution function of accelerated particles, normalized in a way that the number of particles with momentum p at location z is $\int dp 4\pi p^2 f(p,z)$. In Eq. 34 the LHS is the convection term, the first term of the RHS is the spatial diffusion term. The second term on the RHS describes the effect of fluid compression on the accelerated particles, while $Q(x,p)$ is the injection term.

A few comments on Eq. 34 are in order: 1) the shock will appear in this equation only in terms of a boundary condition at $z = 0$, and the shock is assumed to have infinitely small size along z. This implies that this equation cannot properly describe the thermal particles in the fluid. The distribution function of accelerated particles is continuous across the shock. 2) In a self-consistent treatment in which the acceleration process is an integral part of the processes that lead to the formation of the shock one would not need to specify an injection term. Injection would result from the microphysics of the particle motions at the shock. This ambiguity is usually faced in a phenomenological way, by adopting recipes such as the thermal leakage one (Malkov, 1998; Gieseler et al, 2000) that allow one to relate the injection to some property of the plasma behind the shock. This aspect becomes relevant only in the case of non-linear theories of DSA, while for the test particle theory the injection term only determines the arbitrary normalization of the spectrum. However it is worth recalling that while these recipes may apply to the case of protons as injected particles, the injection of heavier nuclei may be much more complex. In fact, it has been argued that nuclei are injected at the shock following the process of sputtering of dust grains (Meyer et al, 1997; Ellison et al, 1997).

For the purpose of the present discussion I will assume that injection only takes place at the shock surface, immediately downstream of the shock, and that it only consists of particles with given momentum p_{inj}:

$$Q(p, x) = \frac{\eta n_1 u_1}{4\pi p_{inj}^2} \delta(p - p_{inj}) \delta(z) = q_0 \delta(z), \quad (35)$$

where n_1 and u_1 are the fluid density and fluid velocity upstream of the shock and η is the acceleration efficiency, defined here as the fraction of the incoming number flux across the shock surface that takes part in the acceleration process. Hereafter I will use the indexes 1 and 2 to refer to quantities upstream and downstream respectively.

The key here is that a parallel shock does nothing locally to energetic particles; there is no discontinuity in the magnetic field and the particles simply continue their helical paths across the front. Thus the complete momentum space distribution function, as measured in the shock frame, must be continuous across the front.

The spatial distribution of the source term is arbitrary (or determined by the observations, if feasible)

The diffusion approach

The compression term vanishes everywhere but at the shock since $du/dz = (u_2 - u_1)\delta(z)$. Integration of Eq. 34 around the shock surface (between $z = 0^-$ and $z = 0^+$) leads to: **

$$\left[D\frac{\partial f}{\partial z}\right]_2 - \left[D\frac{\partial f}{\partial z}\right]_1 + \frac{1}{3}(u_2 - u_1)p\frac{df_0}{dp} + q_0(p) = 0, \tag{36}$$

where $f_0(p)$ is now the distribution function of accelerated particles at the shock surface. Particle scattering downstream leads to a homogeneous distribution of particles, at least for the case of a parallel shock, so that $[\partial f/\partial z]_2 = 0$. In the upstream region, where $du/dz = 0$ the transport equation reduces to:

$$\frac{\partial}{\partial z}\left[uf - D\frac{\partial f}{\partial z}\right] = 0,$$

Quantity in parenthesis is constant. Since must go to 0 when z → -∞, it must be zero everywhere

and since the quantity in parenthesis vanishes at upstream infinity, it follows that

$$\left[D\frac{\partial f}{\partial z}\right]_1 = u_1 f_0. \tag{38}$$

At the shock front upstream, $[D(\partial f/\partial z)]_1 - [uf]_1 = 0$. But $u = u_1$ and $f \rightarrow f_0$ at 0- and therefore (38) holds

** Integration at the eshock

$$\int_{-\varepsilon}^{+\varepsilon} dz\left[\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right]-u\frac{\partial f}{\partial z}+\frac{1}{3}\frac{du}{dz}p\frac{\partial f}{\partial p}+Q(z,p,t)\right]$$

f is continous at shock (in [-ε, +ε])

du/dz = $(u_2 - u_1)\,\delta(z)$

$Q(z,p,t) = q_0(p)\delta(z)$

$$\int_{-\varepsilon}^{+\varepsilon} dz\left[\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right]-u\frac{\partial f}{\partial z}+\frac{1}{3}(u_2-u_1)\delta(z)p\frac{\partial f}{\partial p}+q_0(p)\delta(z)\right]$$

The contribution from ∂f/∂z is = 0, as can be seen integrating by parts due to the continuity of f

$$\left[D\frac{\partial f}{\partial z}\right]_2-\left[D\frac{\partial f}{\partial z}\right]_1+\frac{1}{3}(u_2-u_1)p\frac{\partial f_0}{\partial p}+q_0(p)$$

f_0 is the distribution at the shock (due to the δ(z)) and represents the distribution function of the accelerated particles

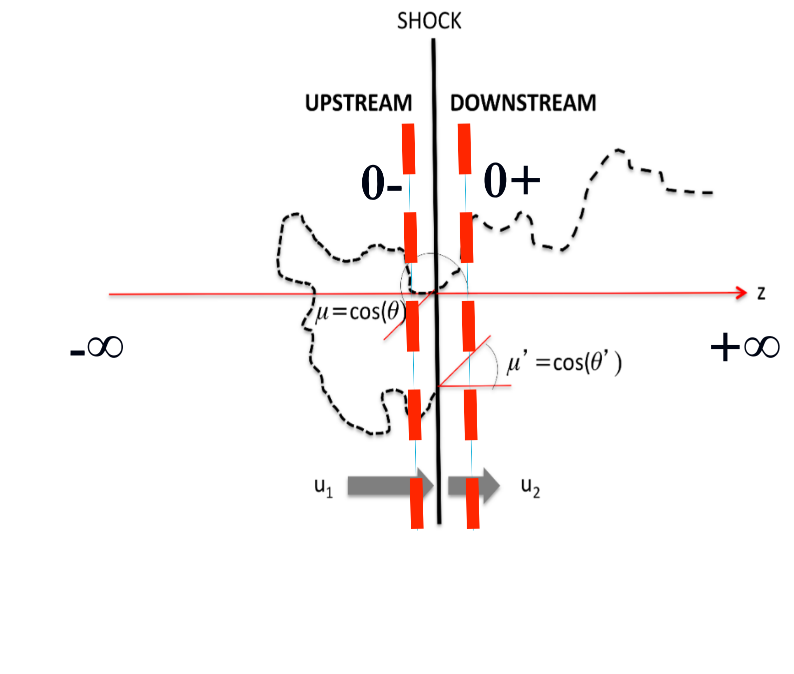


Fig. 6 Illustration of test-particle acceleration at a collisionless shock. In the shock frame the plasma enters from the left with velocity u_1 and exits to the right with velocity $u_2 < u_1$. Here the test particle is shown to enter downstream with cosine of the pitch angle μ (as measured in the upstream plasma frame) and exit with a cosine of the pitch angle μ' (as measured in the downstream plasma frame).

The diffusion approach

Using this result in Eq. 36 we obtain an equation for $f_0(p)$

$$u_1 f_0 = \frac{1}{3}(u_2 - u_1)p\frac{df_0}{dp} + \frac{\eta n_1 u_1}{4\pi p_{inj}^2}\delta(p - p_{inj}), \quad *** \tag{39}$$

which is easily solved to give: The shock compression ratio $r = \rho_2/\rho_1 = u_1/u_2$

$$f_0(p) = \frac{3r}{r-1}\frac{\eta n_1}{4\pi p_{inj}^2}\left(\frac{p}{p_{inj}}\right)^{-\frac{3r}{r-1}}. \tag{40}$$

The spectrum of accelerated particles is a power law in momentum (and not in energy as is often assumed in the literature) with a slope α that only depends on the compression ratio r:

$$\alpha = \frac{3r}{r-1}. \tag{41}$$

The slope tends asymptotically to $\alpha = 4$ in the limit $M_s \to \infty$ of an infinitely strong shock front. The number of particles with energy ϵ is $n(\epsilon)d\epsilon = 4\pi p^2 f_0(p)(dp/d\epsilon)d\epsilon$, therefore $n(\epsilon) \propto \epsilon^{-\alpha}$ for relativistic particles and $n(\epsilon) \propto \epsilon^{(1-\alpha)/2}$ for non-relativistic particles. In the limit of strong shocks, $n(\epsilon) \propto \epsilon^{-2}$ ($n(\epsilon) \propto \epsilon^{-3/2}$) in the relativistic (non-relativistic) regime.

*** solution

$$u_1 f_0 = \frac{1}{3}(u_2 - u_1) p \frac{df_0}{dp} + \frac{\eta n_1 u_1}{4\pi p_{inj}^2}\delta(p - p_{inj})$$

Dividing by u_1 both sides and using the definition of r

$$f_o = \frac{1}{3}(\frac{1}{r} - 1) p \frac{df_o}{dp} + \frac{\eta n_1}{4\pi p_{inj}^2}\delta(p - p_{inj})$$

And rearranging

$$-\frac{3r}{r-1} f_o = p \frac{df_o}{dp} + \frac{3r}{r-1}\frac{\eta n_1}{4\pi p_{inj}^2}\delta(p - p_{inj})$$

q_o

when $p \neq p_{inj}$, the equation is

$$-\frac{3r}{r-1} f_o = p \frac{df_o}{dp}$$

the solution is

$$f_o(p) = k\left(\frac{p}{p_{inj}}\right)^{-\frac{3r}{r-1}}$$

Imposing the continuity of f at p_{inj}, $f(p_{inj}) = q_0$, we get

$$k = \frac{3r}{r-1}\frac{\eta n_1}{4\pi p_{inj}^2}$$

The diffusion approach

$$f_o(p) = \frac{3r}{r-1}\frac{\eta n_1}{4\pi p_{inj}^2}\left(\frac{p}{p_{inj}}\right)^{-\frac{3r}{r-1}}$$

The slope tends asimptotically to α = 4 in the limit M →∞ of an infinitely strong shock front.

The number density of particles with energy between E and E + dE is

$$n(E)dE = 4\pi p^2 f_o(p(E))\frac{dp}{dE}dE$$

For ultrarelativistic particles (UR) p = E and $n(E) \propto E^{\alpha-2}$

For non relativistic particles (NR) p = $(2mE)^{1/2}$ and $n(E) \propto E^{(1-\alpha)/2}$

For strong shocks α = 4 and

$n(E) \propto E^{-2}$ For UR particles

$n(E) \propto E^{-3/2}$ For NR particles

The diffusion approach

$$f_o(p) = \frac{3r}{r-1}\frac{\eta n_1}{4\pi p_{inj}^2}\left(\frac{p}{p_{inj}}\right)^{-\frac{3r}{r-1}}$$

Some points are worth being mentioned: the shape of the spectrum of the accelerated particles does not depend upon the diffusion coefficient. On one hand this is good news, in that the knowledge of the diffusion properties of the particles represent the greatest challenge for any theory of particle acceleration. On the other hand, this implies that the concept of maximum energy of accelerated particles is not naturally embedded in the test particle theory of DSA. In fact, the power law distribution derived above does extend (in prin-

contains a divergent energy, thereby implying a failure of the test particle assumption. Clearly the absence of a maximum energy mainly derives from the assumption of stationarity of the acceleration process, which can be achieved only in the presence of effective escape of particles from the accelerator, a point which is directly connected to the issue of maximum energy, as discussed in

SOME IMPORTANT COMMENTS

- THE STATIONARY PROBLEM DOES NOT ALLOW TO HAVE A MAX MOMENTUM!
- THE NORMALIZATION IS ARBITRARY THEREFORE THERE IS NO CONTROL ON THE AMOUNT OF ENERGY IN CR
- AND YET IT HAS BEEN OBTAINED IN THE TEST PARTICLE APPROXIMATION
- THE SOLUTION DOES NOT DEPEND ON WHAT IS THE MECHANISM THAT CAUSES PARTICLES TO BOUNCE BACK AND FORTH
- FOR STRONG SHOCKS THE SPECTRUM IS UNIVERSAL AND CLOSE TO E^{-2}
- IT HAS BEEN IMPLICITELY ASSUMED THAT WHATEVER SCATTERS THE PARTICLES IS AT REST (OR SLOW) IN THE FLUID FRAME

Acceleration time

* see eqn. 3.36

In the assumption of isotropy, the flux of particles that cross the shock from downstream to upstream is $n_s c/4$,* which means that the upstream section is filled through a surface Σ of the shock in one diffusion time upstream with a number of particles $n_s(c/4)\tau_{diff,1}\Sigma$ (n_s is the density of accelerated particles at the shock). This number must equal the total number of particles within a diffusion length upstream $L_1 = D_1/u_1$, namely:

$$n_s \frac{c}{4} \Sigma \tau_{diff,1} = n_s \Sigma \frac{D_1}{u_1}, \tag{45}$$

which implies for the diffusion time upstream $\tau_{diff,1} = \frac{4D_1}{cu_1}$. A similar estimate downstream leads to $\tau_{diff,2} = \frac{4D_2}{cu_2}$, so that the duration of a full cycle across the shock is $\tau_{diff} = \tau_{diff,1} + \tau_{diff,2}$. The acceleration time is now:

$$\tau_{acc} = \frac{E}{\Delta E/\tau_{diff}} = \frac{3}{u_1 - u_2}\left[\frac{D_1}{u_1} + \frac{D_2}{u_2}\right]. \tag{46}$$

Acceleration time

Define a cycle $\equiv U \to$ shock $\to D \to$ shock $\to U$, so $\tau_{\rm acc} = E/(dE/dt) = \tau_{\rm cycle}/\xi$. But, $\tau_{\rm cycle} = ?$

Let t_2 be the average time spent in D before returning to the shock, i.e., for $(S \to D \to S)$.

Average distance in D through which particle diffuses in time t is $\approx \sqrt{k_2 t}$, with $k_2 =$ diffusion coefficient in D.

Average distance through which the particle is advected in D in time t is $= u_2 t$.

The particle is lost downstream (i.e., low probability of returning to shock) if $u_2 t > \sqrt{k_2 t}$. Thus, 'returning zone boundary' is given by the condition $u_2 t \approx \sqrt{k_2 t}$. Thus only the particles within a distance $d_2 \approx \frac{k_2}{u_2}$ from the shock return to the shock.

Number of particles per unit area within the 'returning zone' is $n_{\rm CR} d_2 = n_{\rm CR} \frac{k_2}{u_2}$. Thus,

$t_2 \approx n_{\rm CR} d_2 / r_{\rm cross}$, i.e., $t_2 = \frac{4}{c} \frac{k_2}{u_2}$.

Similarly, for $S \to U \to S$), we have $t_1 = \frac{4}{c} \frac{k_1}{u_1}$. Thus, $\tau_{\rm cycle} = \frac{4}{c} \left(\frac{k_1}{u_1} + \frac{k_2}{u_2} \right)$.

Thus, $r_{\rm acc} = \frac{R-1}{3R} u_1 \left(\frac{k_1}{u_1} + \frac{k_2}{u_2} \right)^{-1}$.

Note that the higher is the diffusion, the longer is the acceleration time

A CRUCIAL ISSUE: the maximum energy of accelerated particles

1. The maximum energy is determined in general by the balance between the Acceleration time and the shortest between the lifetime of the shock and the loss time of particles

2. For the ISM, the diffusion coefficient derived from propagation is roughly

$$D(E) = 3\times 10^{29}\, E_{GeV}^{\alpha} \qquad \alpha \approx 0.3 - 0.6$$

$$\tau_{acc}(E) = \frac{3}{u_1 - u_2}\left[\frac{D_1(E)}{u_1} + \frac{D_2(E)}{u_2}\right]$$

For a typical SNR the maximum energy comes out as FRACTIONS OF GeV !!!

Particle Acceleration at Parallel Collisionless Shocks works ONLY if there is additional magnetic scattering close to the shock surface that makes the diffusion slow

Acceleration time scale

- Energy gain at each crossing : $\Delta E = \frac{v_1}{c} E$ if $r = 4$.
- What is the duration of cycle ?
- It depends on particle isotropization timescale
- We introduce the diffusion coefficient D in $m^2.s^{-1}$
- During time t, particle diffuse in its medium on a typical length :

$$l_d = \sqrt{Dt}$$

- In the meantime, the shock has moved on a distance : $l_s = vt$
- Equating the two length give the residence timescale :

$$t = \frac{D}{v^2}$$

Acceleration time

$$\langle \Delta t_{cycle}(E) \rangle = 4 \left(\frac{D_1(E)}{v_1 c} + \frac{D_2(E)}{v_2 c} \right) \quad (3.46)$$

where D_1 and D_2 are the diffusion coefficients for the upstream and downstream media respectively. Then for the acceleration time we have :

$$\langle t_{acc}(E) \rangle = \frac{\langle \Delta t_{cycle}(E) \rangle}{\langle \frac{\Delta E}{E} \rangle} \propto \frac{D(E)}{v_{sh}^2} \quad (3.47)$$

The energy dependence of the acceleration time is then given by the energy dependence of the diffusion coefficient which is related to the type of magnetic turbulence. As we saw in Chapt. 2 that for a Kolmogorov turbulence, n the limit $r_L(E) \ll \lambda_{max}$ we have $D(E) \propto E^{1/3}$. We will make a different assumption in the following mostly for the sake of simplicity.

$D(R)=\lambda(R)c\beta/3$

Indeed, in most calculation, the *Bohm scaling* (or Bohm approximation) is usually assumed for the energy evolution of the diffusion coefficient. In the general case : $D(E) = \frac{1}{3} l_{scat} v$ where l_{scat} is the scattering length (or mean free path), *i.e* the length on which the particle get "randomized" and losses the memory of its initial direction and v is the velocity of the particle. The Bohm diffusion coefficient is obtained by assuming $l_{scat}(R) = r_L(E)$, where $r_L = \frac{P}{ZeB} = \frac{R}{Bc}$ is the Larmor radius of a particle of momentum P and rigidity R[7] (see Chapt. 2). We then have,

$$D_{Bohm}(R) = \frac{1}{3} r_L(R) c \quad (3.48)$$

Acceleration time

With the Bohm hypothesis, we then get :

$$\langle t_{acc}(R) \rangle \propto \frac{R}{B} \times \frac{1}{v_{sh}^2} \tag{3.49}$$

(a dependence in $R^{1/3}$ would be expected for a Kolmogorov turbulence).

Before going further we can obtain some useful scaling laws (the reader is invited to recalculate them) :

$$r_L(R) \simeq 1.1 \times \left(\frac{R}{10^{15}\mathrm{V}}\right) \times \left(\frac{B}{\mu\mathrm{G}}\right)^{-1} \mathrm{pc} \tag{3.50}$$

$$D_{Bohm}(R) \simeq 3.4 \times 10^{28} \times \left(\frac{R}{10^{15}\mathrm{V}}\right) \times \left(\frac{B}{\mu\mathrm{G}}\right)^{-1} \mathrm{cm}^2\,\mathrm{s}^{-1} \tag{3.51}$$

Let us now quantify a little our finding using typical parameters probably at play in a supernova remnant. We remind that *supernova remnants* are what remains of a supernova hundreds to tens of thousands of years after a supernova explosion. The phenomenon is explained by the propagation of a supersonic plasma, emitted at the time of the supernova event, through the interstellar medium. The shock wave is known to heat up the shocked interstellar medium and to accelerate electrons and nuclei which emit non thermal radiation up to very high energies. Supernova remnants are considered as one of the best source candidate for the acceleration of Galactic cosmic-rays[8]. The

The physical parameters which are important to discuss CR acceleration by supernova remnant evolve with time[9], we will limit ourselves for the type being to typical values thought to be at play in the vicinity of the shock wave a few hundred years after the supernova explosion.

The typical value of the magnetic field and the shock velocity is thought to vary from remnants to remnants $B \sim [100, 500]\,\mu$G, $v_{sh} \sim [1000, 10000]\,\mathrm{km\,s^{-1}}$. Let us use $B = 100\ \mu$G and $v_{sh} = 3000$ $\mathrm{km\,s^{-1}}$ as typical values. For a proton at 100 GeV we have $D_{Bohm}(100GeV) \simeq 3.4 \times 10^{22}\,\mathrm{cm^2\,s^{-1}}$. Thus,

$$\langle \Delta t_{cycle} \rangle_{100\,\mathrm{GeV}} \simeq \frac{4D(E)}{cv_{sh}} \simeq 1.5\,10^4\,\mathrm{s} \quad (3.52)$$

$$\langle t_{acc} \rangle_{100\,\mathrm{GeV}} = \frac{\langle \Delta t_{cycle} \rangle_{100\mathrm{GeV}}}{\langle \frac{\Delta E}{E} \rangle} \simeq \frac{4D(E)}{v_{sh}^2} \simeq 1.5\,10^6\,\mathrm{s} \simeq 17.5\,\mathrm{days} \quad (3.53)$$

Using $t_{acc}(E) \propto E$ (assuming Bohm scaling) we get :

$$\langle t_{acc} \rangle_{10^{15}\,\mathrm{eV}} \simeq 480\,\mathrm{years} \quad (3.54)$$

Although the discussion of cosmic-ray acceleration in supernova remnants goes of course far beyond the order of magnitude calculation we just made, we can use our results to compare with what we obtained for the second order Fermi mechanism we discussed in the previous section. The comparison speaks for itself. With DSA we now get much more reasonable acceleration times which can be achieve within the source lifetime (see next chapter for a more complete (although oversimplified) discussion) and which can also compete with energy loss mechanisms.

Acceleration time

- The diffusion distance upstream of the shock is then :

$$l_d = \frac{D_1}{v_1}$$

- It is possible to show that the duration of a cycle is :

$$\tau_{acc} = \frac{4}{c}\left(\frac{D_1}{v_1} + \frac{D_2}{v_2}\right)$$

- We use in general the Bohm diffusion coefficient

$$D = \frac{pc}{3ZeB}$$

- Then assuming $D_1 = D_2$, we have

$$\tau_{acc} = \frac{4(r+1)E}{3ZeBv_1c} \propto E$$

- The energy gain rate is then :

$$\frac{dE}{dt} = \frac{\Delta E}{\tau_{acc}} = \frac{r-1}{r(r+1)} ZeBv_1^2$$

Acceleration time

- For a typical young supernova remnant shock : $v_1 \sim 10^4 km/s$ and $B = 2nT$:

$$\frac{dE}{dt} \sim 30keV/s \left(\frac{v}{10^4 km/s}\right)^2 \left(\frac{B}{2nT}\right)$$

- To reach 100 TeV, only $\sim$ 100 yrs are necessary.
- **Efficient acceleration mechanism !**
- E_{max} is limited by several factors :
  - Energy losses (Coulomb, radiative, inelastic collisions)
  - Time : age of the system limits the energy
  - Particle escape (geometry effect)
- Maximal energy depends of system physical conditions

Particle confinement and maximum achievable energy

At this stage, it is useful to discuss the some of the assumptions we made to calculate the accelerated particle spectrum, *i.e* infinite media (upstream and downstream) and steady conditions (shock lasting forever).

In any astrophysical source, the acceleration region has in principle a finite spatial extension (and of course a finite age and activity duration) and as a consequence, the Larmor radius of the accelerated particles cannot exceed the size of the acceleration site.

Taking for instance $B = 100\ \mu G$ and $L_{source} = 1$ pc, one can estimate a maximum energy E_{max}^{size} such that $r_L(E_{max}^{size}) = L_{source}$. With these parameter, one gets $E_{max}^{size} \simeq Z \times 10^{17}$ eV or in terms of the maximum rigidity $R_{max}^{size} \simeq 10^{17}$ V.

Particle confinement and maximum achievable energy

An argument very close to that one was used by M. Hillas (in 1984) to build the famous *Hillas diagram* estimating the maximum energy achievable for cosmic-rays in different types of sources as a function of their size and their ambient magnetic field. Hillas constructed his famous diagram in order to select viable sources for cosmic-ray acceleration above 10^{20} eV, a version of it is visible in Fig. 3.10. We must note that the "Hillas condition" is a necessary but not sufficient condition to estimate whether or not a given energy can be achieve in a given type of source. In most case as we will discuss later the maximum energy estimated with this criterion is overoptimistic. Moreover energy losses, which may limit the acceleration process (see next chapter) and which are specific to a given type of source, are not accounted for in this argument.

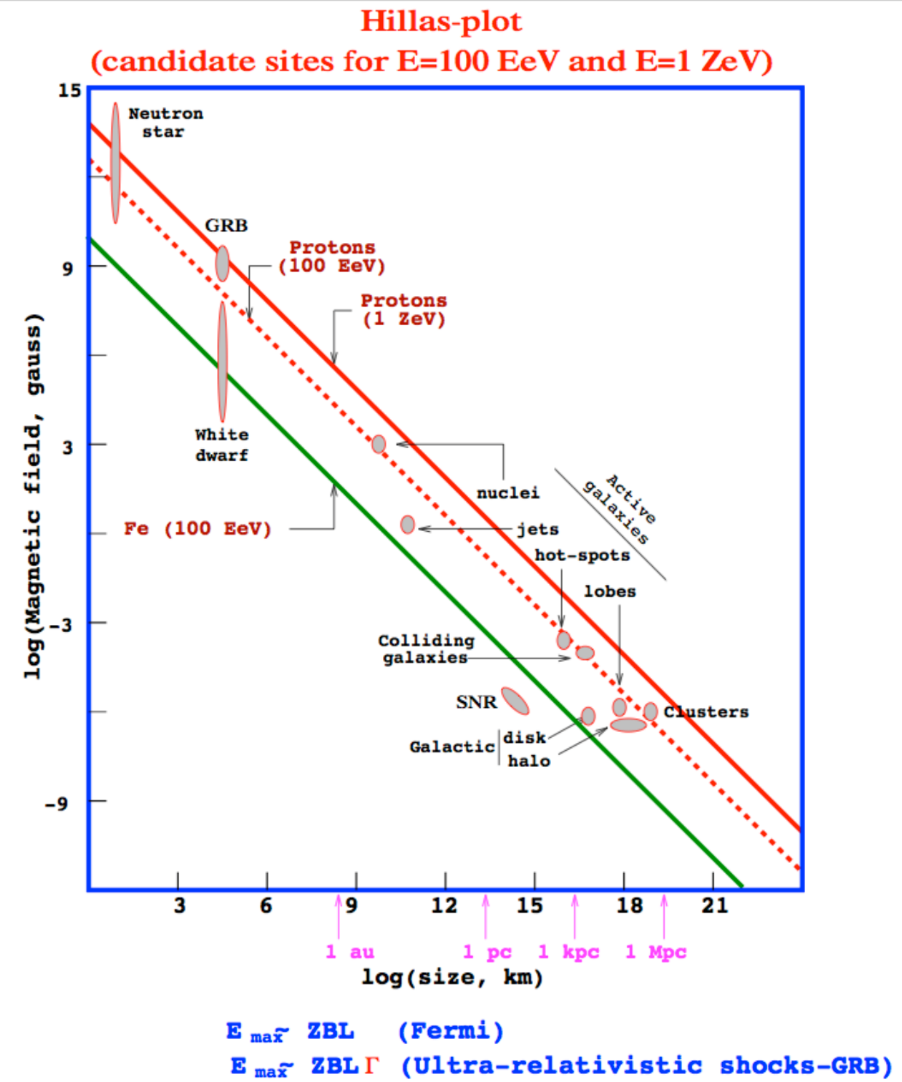


Fig. 3.10: Hillas diagram constructed by plotting different sources characteristics in the (size, B) plane. Sources above the green line may be able to accelerate Fe nuclei above 10^{20} eV, while those above the dashed and full red lines may be able to accelerate protons above 10^{20} and 10^{21} eV respectively. Note that the Hillas must be interpreted with the greatest caution : *(i)* The Hillas criterion is a necessary but not sufficiently condition to reach a given maximum energy. *(ii)* The maximum energies estimated using the Hillas criterion are overoptimistic (see text). *(iii)* Energy losses during the acceleration process (which may limit the maximum reachable energy) are not taken into account.

Particle confinement and maximum achievable energy

A slightly higher level argument can be used to derive a more quantitative estimate. In diffusion theory, one can show that the typical distance from the shock a particle manages to reach upstream or downstream and which is called the diffusion length is given by :

$$L_{diff} \simeq \frac{D(E)}{v_{sh}} \tag{3.55}$$

when the value of L_{diff} reaches a value of the order of the spatial extension of the magnetized region upstream of the shock then particle escape in the upstream region becomes efficient (and the approximation we made for our calculation is broken). We can then define a maximum energy due to particle confinement E_{max}^{conf} such that $L_{diff}(E_{max}^{conf}) = L_{up}$, where L_{up} is the size of the magnetized region upstream of the shock. Since particles with energy of the order of E_{max}^{conf} will escape efficiently upstream rather than re-crossing the shock, this escape mechanism is going to limit the maximum energy achievable by cosmic-rays which was not the case in our idealized calculation. This is however a good thing since the particles escaping downstream of the shock were not accelerated anymore but were still trapped within the source without any possibility to escape through the interstellar medium until the phenomenon at the origin of the acceleration dissipates. These particles would be likely to experience energy losses while being confined in the source. Particles escaping upstream by de-confinement eventually escape through the interstellar medium and can propagate throughout the Galaxy.

Particle confinement and maximum achievable energy

Particles with energies close to E_{max}^{conf} can escape efficiently upstream while at much lower energy the probability for reaching the boundary of the magnetized region upstream of the shock is very low[10]. For these low energy particles the above approximation of an infinite spatial extension for the upstream medium is relatively accurate.

$$L_{diff}(R) \simeq 110 \times \left(\frac{R}{10^{15}\mathrm{V}}\right) \times \left(\frac{B}{\mu\mathrm{G}}\right)^{-1} \times \left(\frac{v_{sh}}{1000\,\mathrm{km\,s^{-1}}}\right)^{-1} \mathrm{pc} \qquad (3.56)$$

Using the same supernova remnant parameters as before and assuming $L_{up} = 0.1$ pc (the size of the upstream magnetised region is often assumed to be of the order of 1/10th of the size of remnant for which we use $L_{source} = 1$ pc). We get $E_{max}^{conf} \simeq Z \times 3\,10^{14}$ eV (or $R_{max}^{conf} \simeq 3\,10^{14}$ V) which is indeed much smaller than the above calculated E_{max}^{size}.

The important thing to understand here is that particles escaping upstream are directly released in the ISM and will eventually propagate through the Galaxy. They are the cosmic-rays we expect to detect on Earth. Since only particles with rigidities close to R_{max}^{conf} have a significant probability to escape upstream by reaching the boundary of the magnetized region, the spectrum of particles escaping upstream is expected to be much harder than the spectrum of accelerated cosmic-rays (one might call this phenomenon a "high pass filter effect"). This very important fact is illustrated in Fig. 3.11.

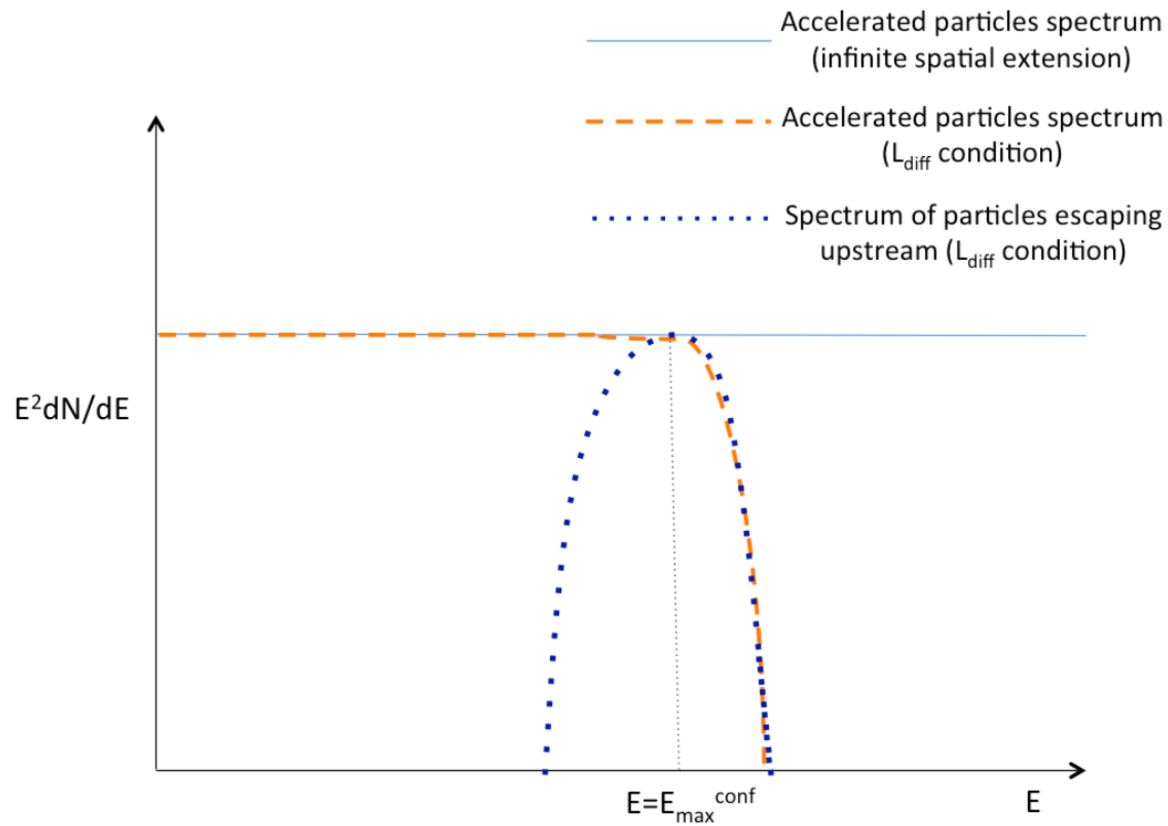


Fig. 3.11: Schematic view explaining the impact of the existence of the diffusion length and the limited confinement capabilities of a source : When the spatial extension of the source is assumed to be infinite, the spectrum of the accelerated cosmic-rays is proportional to E^{-2} and can reach arbitrarily large energies (blue line). When the spatial extension of the source is considered (and the condition on $L_{diff}(R)$ applied), the spectrum of the accelerated cosmic-rays is still proportional to E^{-2} but the acceleration mechanism becomes inefficient above E_{max}^{conf} since particles start to escape upstream of the shock rather than coming back to the shock and starting a new cycle (orange dashed line). Finally the dark blue dotted line shows the spectrum of particles which manage to escape upstream of the shock. The spectrum is very hard (almost a Dirac distribution) since the escape probability only becomes significant for energies close to E_{max}^{conf}. Above this energy the efficient escape upstream prevent the particles from performing extra cycles and reaching higher energies.

Maximum energy : size limit

- If the shock is not an infinite plane, the particle diffusing far upstream won't be catched up
- E.g. : a SNR has a spherical shape of radius R
  - Particle distribution upstream : $n(x) = n_0 e^{-V_c/D\ x}$
  - Characteristic length $l = \frac{D}{u}$
  - Escape if $l \geq \eta R_c$ that is using the Bohm limit ($\lambda_{diff} = r_g$) :

NB: typo $R_c = R_s$, Vsc = V_s

$$D = \frac{p\ c}{3\ Z\ e\ B} = \eta VscR_s \Rightarrow E_{max,size} = 3ZeB\eta V_s R_s$$

with $\eta \leq 1$ a factor controlling the escape probability

  - In the adiabatic evolution phase : $R_c = R_{ST} t^{2/5}$ and $V_c = V_{ST} t^{-3/5}$

$$E_{max,size} = 3Ze\eta B\ V_{ST} R_{ST} t^{-1/5}$$

$$E_{max,taille} = 125\,TeV\ Z \left(\frac{\eta}{0.2}\right) \left(\frac{E}{10^{44} J}\right)^{\frac{2}{5}} \left(\frac{n_0}{10^6 m^{-3}}\right)^{-\frac{2}{5}} \left(\frac{t}{10^3 ans}\right)^{-1}$$

Age limit

Another source of limitation for the acceleration of cosmic-rays might come from the "age" of the source (in other word the amount of time during which the source has been active). This criterion is relatively straightforward to understand : if a source has an age t_{source} then only the rigidities for with $t_{acc}(R) \leq t_{source}$ can be potentially reached.

For instance, we saw earlier that $\langle t_{acc}(10^{15}\ \mathrm{V})\rangle \simeq 480$ yr. Thus for an SNR with $B = 100\ \mu$G and $v_{sh} = 3000\ \mathrm{km\,s^{-1}}$ the maximum rigidity due to the age of the source after 480 yr will be $R_{max}^{age} \simeq 10^{15}$ V. Since in the Bohm hypothesis, $\langle t_{acc}(R) \propto R\rangle$ then if v_{sh} and B were constant with time, we would expect $R_{max}^{age} \propto t$.

Age Limit

Let us know discuss the comparison between R_{max}^{age} and R_{max}^{conf}, assuming for the time being v_{sh} and B are constant with time. After 480 yr, we have $L_{source} = v_{sh} \times t \simeq 1.45$ pc. Let us take $L_{up} \simeq 0.1\, L_{source} \simeq 0.145$ pc. Then we would get with the same calculation as above $R_{max}^{conf}(480\text{ yr}) \simeq 5\,10^{14}$ V. Then at $t = 480$ yr we have $R_{max}^{conf} \simeq R_{max}^{age}$ (within a factor of 2) and it should remain so at least while v_{sh} and B are constant (in which case as we saw $R_{max}^{age} \propto t$ and moreover $L_{up} \propto L_{source} \propto t \Rightarrow R_{max}^{conf} \propto t$.

In this situation the escape of particles becomes efficient at a rigidity which is practically the same as the maximum rigidity the cosmic accelerator can achieve (due to its age). This is actually a quite optimum situation and we can understand this points by considering two extreme case :

(i) if $R_{max}^{conf} \gg R_{max}^{age}$, in this case, the escape of particles at the maximum energy achievable is inefficient and then the accelerated particles remain trapped in the source.

(ii) if $R_{max}^{conf} \ll R_{max}^{age}$, in this case, particles escape becomes efficient at rigidities much lower than the maximum energy the source could achieve. This efficient escape would "break" the acceleration mechanism (see above) before reaching the rigidity R_{max}^{age}.

The case $R_{max}^{conf} \simeq R_{max}^{age}$ is then optimum since the criteria for an efficient particle escape upstream and for the acceleration of the particles to the maximum possible energy due to the age of the source are met at the same time.

Maximum energy : age limit

- Maximal energy of a particle in an accelerator for time t
- Using the acceleration rate :

$$E_{max} = \int_0^t \frac{dE}{dt} dt = \int_0^t \frac{r-1}{r(r+1)} Z\ e\ B\ V_s^2 dt$$

- If we assume the shock velocity is constant

$$E_{max,age} = \frac{r-1}{r(r+1)} Z\ e\ B\ V_s^2 t$$

- Or for $r = 4$ and parameters typical of a supernova remnant :

$$E_{max,age} \sim 12 TeV \left(\frac{B}{1nT}\right) \left(\frac{V_s}{5\ 10^3 km/s}\right)^2 \left(\frac{t}{100yrs}\right)$$

SNR : maximum energies

- Maximum energy grows with time until t so that

$$E_{max} = E_{max,size}(t) = E_{max,age}(t)$$

- This limit is typically of the order of

$$E_{max} \approx 120\,TeV\;Z\left(\frac{B}{1nT}\right)$$

- **With typical B fields, Supernova Remnants can't accelerate up to the knee ($3\;10^{15}$ eV) !**
- Once this limit is reached, the most energetic particles escape slowly ($E_{max} \propto t^{-1/5}$)
- Electrons are very quickly limited by radiative losses
- The higher B, the lower $E_{max,e}$
- **The highest energy particles escape slowly the SNR shock : efficient acceleration for a limited period of time**

Toy model of the time evolution of a SNR

As mentioned in the previous chapter, the SNR phenomenon is caused by the ejection of a supersonic plasma (or wind) at the time of the supernova event. The supersonic flow propagates in the surrounding ISM, the gas swept by the shock is both compressed and heated (see Eqs. 3.21 to 3.28) and particles are expected to be accelerated by the DSA mechanism. The flow propagates until it becomes subsonic and eventually mixes with the ISM, the whole process lasts in total up to 10^5 years. From the point of view of particles acceleration a SNR probably remain efficient during $\sim 10^4$ years or so. During this period of time there are mainly two phases in the shock propagation

Toy model of the time evolution of a SNR

(i) The free expansion phase which lasts a few hundreds of years in general (depending on the characteristics of the wind and of the surrounding ISM). During the phase, the shock velocity v_{sh} is in good approximation constant which means that the shock radius R_{sh} is proportional to the elapsed time, $R_{sh} \propto t$. The time evolution of the magnetic field in the vicinity of the shock is not precisely known while it is of course a quantity of major importance for particle acceleration. We will assume in the following that the shock magnetic field B remains constant during the free expansion phase (ı.e, proportional to v_{sh}). It is however important to note that scenarios involving very strong magnetic fields at the very early times of the shock propagation are currently extensively studied[1].

---

[1] These scenarios are relatively attractive since (as we will discuss later) there is no signature of the acceleration of cosmic-rays up to the knee energy in the photon spectra of the SNRs observed in γ-rays. Since those SNRs are all more than 100 years old, it can be invoked that those SNRs are not able anymore to accelerate cosmic-rays up to the knee energy but that they were earlier in their evolution. The scenario we will discuss in the following is different since we will assume B is constant during the free expansion phase.

Explosion and free expansion phase

- Kinetic energy released during Supernova explosion $E_k \sim 10^{44}$J
- Sphere of ejected matter (total mass M_{ej}) expanding in a uniform density medium.
- At the shock itself, V_s is constant (no energy loss). The shock radius is then $R_s = V_s t$.
- The total kinetic energy is :

$$E_k \sim \frac{1}{2} M_{ej} V_s^2$$

- For an explosion of $E_k = 10^{44}$ J with $M_{ej} = 1 M_\odot$

$$V_s = \sqrt{2\frac{E_c}{M_{ej}}} = 10000 km/s \left(\frac{E_c}{10^{44} J}\right)^{1/2} \left(\frac{M_{ej}}{1 M_\odot}\right)^{-1/2}$$

- **In the early phases after the explosion the shock speed can be has large as 0.03c !**

(ii) The Sedov-Taylor phase : after a few hundreds of years, once the shock has swept a mass of ISM greater or equivalent to its own, the shock starts to decelerate. During the Sedov-Taylor phase the velocity of the shock radius is expected to scale as $R_{sh} \propto t^{2/5}$ and then $v_{sh} \propto t^{-3/5}$. We will not discuss in detail the implications of the Sedov-Taylor phase for the SNR gas dynamics but just the aspects of the evolution which are relevant for particle acceleration within our simplified model, the interested reader can find many more details in the chapter 16 of M. Longair's book, "*High Energy Astrophysics*". The Sedov-Taylor scaling is a standard result of fluid dynamics obtained by dimensional analysis (see M. Longair's book "*Theoretical Concepts in Physics*") and experimentally observed in the behavior of shock waves produced by nuclear explosions[2]. The evolution of the magnetic field during this phase is not well known, we will assume, as in the free expansion phase that the magnetic field is proportional to the shock velocity[3], $B \propto v_{sh}$.

Transition to the Sedov-Taylor phase

- When the ejected mass is equal to the mass of the swept up interstellar matter, hydrodynamic evolution changes. SNR is now an adiabatic phase where the evolution is guided by the inital kinetic energy and the mass of the swept up matter
- Temperatures are very large (so $T > 10^8$ K) so that radiative cooling of the thermal energy is inefficient and evolution is adiabatic
- Transition when :

$$\frac{4}{3}\pi R_s^3 \rho_0 = \frac{4}{3}\pi v_s^3 t^3 \rho_0 = M_{ej}$$

- That is :

$$t_{ST} = 120yrs \left(\frac{E}{10^{44}J}\right)^{-\frac{1}{2}} \left(\frac{n_0}{10^6 m^{-3}}\right)^{-\frac{1}{3}} \left(\frac{M_{ej}}{1M_\odot}\right)^{\frac{5}{6}}$$

$$R_{ST} = 2.2pc \left(\frac{n_0}{10^6 m^{-3}}\right)^{-\frac{1}{3}} \left(\frac{M_{ej}}{1M_\odot}\right)^{\frac{1}{3}}$$

Sedov-Taylor phase

An autosimilar solution allows to describe the adiabatic evolution of an expanding shock wave.
Simply, if we assume that a constant fraction of the energy is in the form of kinetic energy

$$\rho R^3 V^2 = const$$

$$R^{3/2} \frac{dR}{dt} = const$$

Therefore

$$R^{5/2} \propto t \quad \text{and} \quad R \propto t^{2/5} \quad V \propto t^{-3/5}$$

Let us assume, for our toy model, that the initial velocity of the shock is $v_{sh} = 3000\ \mathrm{km\,s^{-1}}$, the magnetic field $B = 100\ \mu\mathrm{G}$ and that the free expansion phase lasts for 480 years before entering in the Sedov-Taylor phase[4]. Under these assumptions, the time evolution of R_{sh} and v_{sh} are represented in Fig. 4.1.

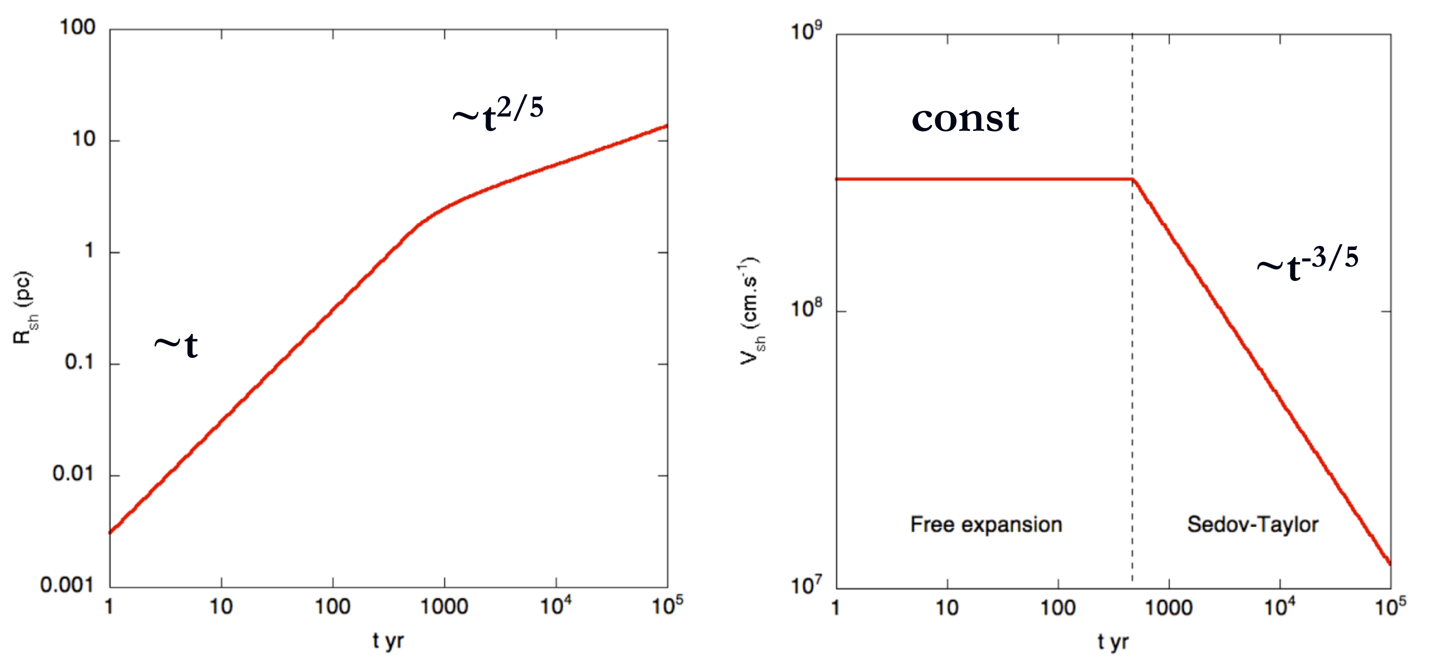


Fig. 4.1: Time evolution of r_{sh} (left) and v_{sh} for our SNR toy model parameters (see text).

Free expansion phase

From our discussions in the last paragraph of Chapt. 3, we already know the evolution of R_{max}^{conf} and R_{max}^{age} in the free expansion phase. We know that $R_{max}^{age}(480 \text{ yr}) \simeq 10^{15}$ V and $R_{max}^{conf}(480 \text{ yr}) \simeq 5\,10^{14}$ V. Since we are assuming the Bohm scaling for the rigidity evolution of the diffusion coefficient and as a result that $t_{acc}(R) \propto R$ then we get $R_{max}^{age} \propto t$. On the other hand since v_{sh} =cst and then
constant
$R_{sh} \propto t$, with our assumption of $L_{up} \simeq 0.1 R_{sh}$ (see the last paragraph of the previous chapter) then we have $L_{up} \propto t$ which gives (still due to the Bohm scaling assumption with yields $L_{diff}(R) \propto R$) $R_{max}^{conf} \propto t$. We can deduce the time evolution of R_{max}^{age} and R_{max}^{conf} during the free expansion phase, for our choice of physical parameters :

$$R_{max}^{age} \simeq 10^{15} \left(\frac{t}{480 \text{ yr}} \right) \text{ V for t} \leq 480 \text{ yr} \tag{4.1}$$

and

$$R_{max}^{conf} \simeq 5\,10^{14} \left(\frac{t}{480 \text{ yr}} \right) \text{ V for t} \leq 480 \text{ yr} \tag{4.2}$$

Sedov-Taylor phase

The time evolution of the different maximum rigidities can be calculated very simply from the definitions of R_{max}^{conf} and R_{max}^{age}.

R_{max}^{age} is such that $t_{acc}(R_{max}^{age}) = t_{source}$. Since,

$$t_{acc}(R) \simeq \frac{4D(R)}{v_{sh}^2} \propto \frac{R}{Bv_{sh}^2} \text{ then } R_{max}^{age} \propto tBv_{sh}^2 \propto tt^{-3/5}t^{-6/5} \propto t^{-4/5} \quad (4.3)$$

using the assumed Bohm scaling of the diffusion coefficient, the scaling with time of v_{sh} during the Sedov-Taylor phase and our assumption on the magnetic field evolution.

On the other hand, R_{max}^{conf} is such that $L_{diff}(R_{max}^{conf}) = L_{up} \simeq 0.1R_{sh}$. Since,

$$L_{diff}(R) \simeq \frac{D(R)}{v_{sh}} \propto \frac{R}{Bv_{sh}} \text{ then } R_{max}^{conf} \propto R_{sh}Bv_{sh} \propto t^{2/5}t^{-3/5}t^{-3/5} \propto t^{-4/5} \quad (4.4)$$

(try to redo this reasoning assuming for instance that the magnetic field is constant, or using $D(R) \propto R^{1/3}$ rather than the Bohm scaling)

Sedov-Taylor phase

We then get during the Sedov-Taylor phase we then have

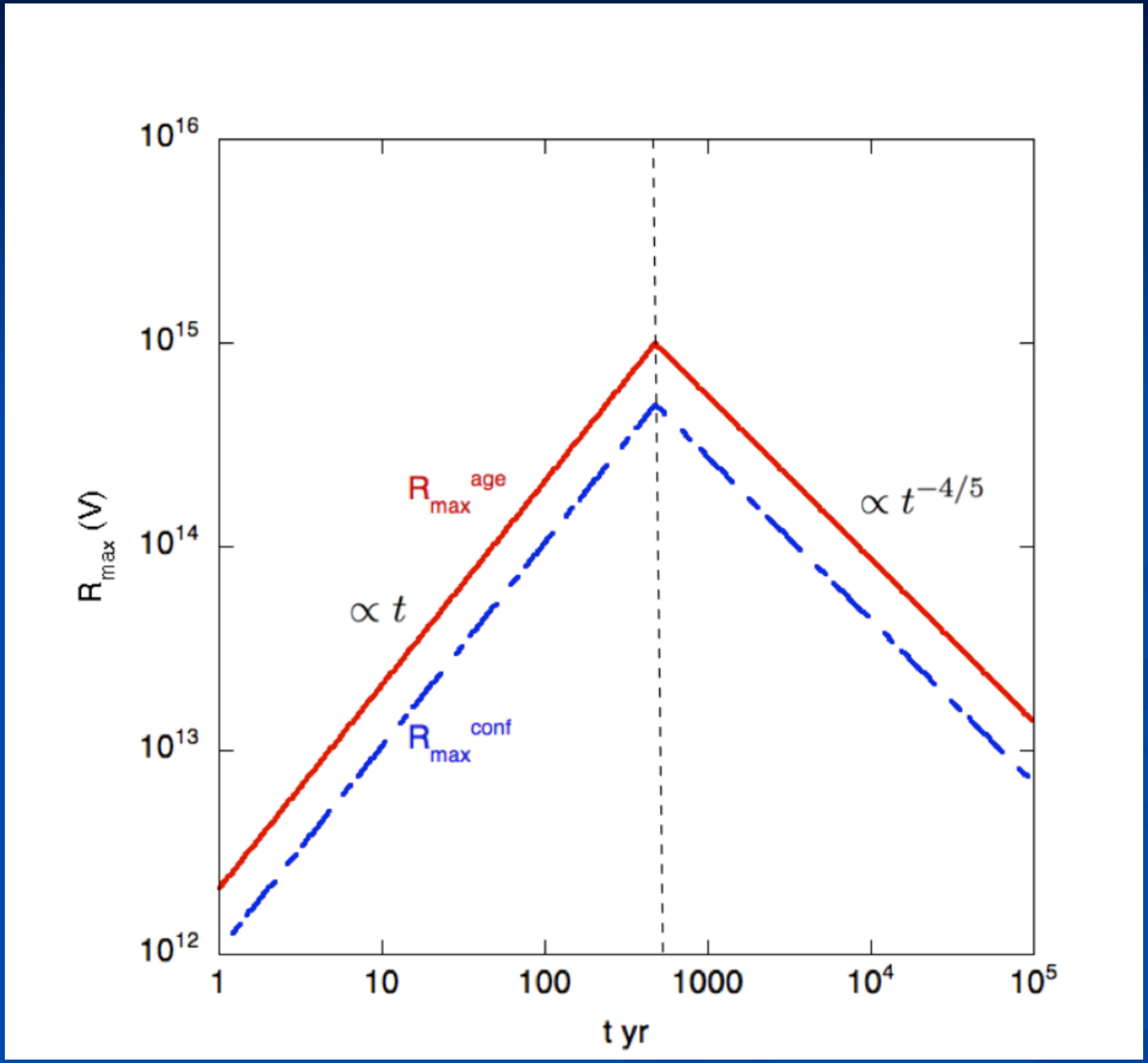
$$R_{max}^{age} \simeq 10^{15} \left(\frac{t}{480 \text{ yr}} \right)^{-4/5} \text{ V for t} > 480 \text{ yr} \tag{4.5}$$

and

$$R_{max}^{conf} \simeq 5\,10^{14} \left(\frac{t}{480 \text{ yr}} \right)^{-4/5} \text{ V for t} > 480 \text{ yr} \tag{4.6}$$

We see that, within the assumptions of our toy model[5], R_{max}^{age} and R_{max}^{conf} evolve the same way time (see Fig. 4.2). As a result, as far the escape of cosmic-rays into the ISM is concerned the discussion we had at the of the previous chapter remains true during the whole shock propagation which means that during the whole shock propagation in the ISM, cosmic-rays with rigidities close to $R_{max}^{age}(t)$ escape efficiently from the source. The value of the maximum achievable rigidity increases during the free expansion phase reaching its maximum value before decreasing during the Sedov-Taylor phase as illustrated in Fig. 4.2.

[5] Let us stress again that the evolution we found is only true for the set of simplifying assumptions we made : Bohm scaling, $B \propto v_{sh}$. The scaling with time would be different had we assumed B constant during the whole evolution for instance.


10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
R_{max} (V)
R_{max}^{age}
R_{max}^{conf}
$\propto t$
$\propto t^{-4/5}$
1
10
100
1000
10^4
10^5
t yr