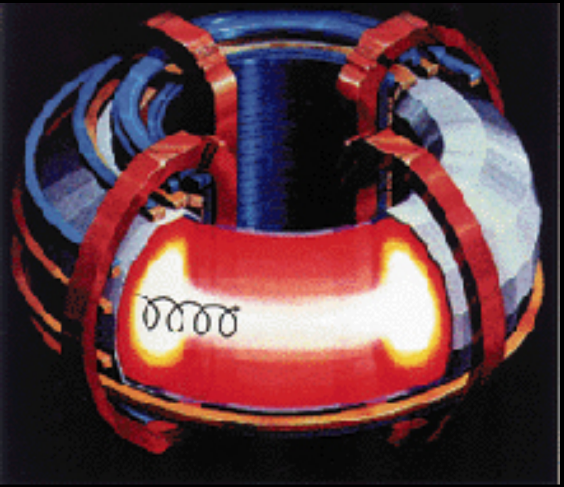
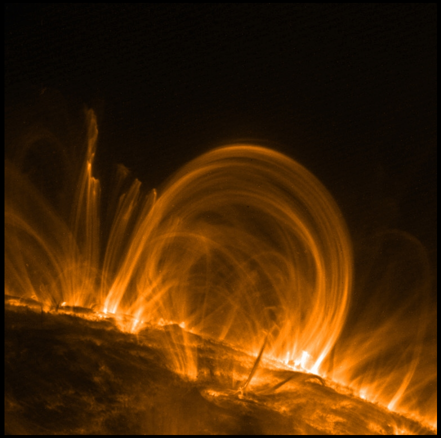
Lecture 15 271119

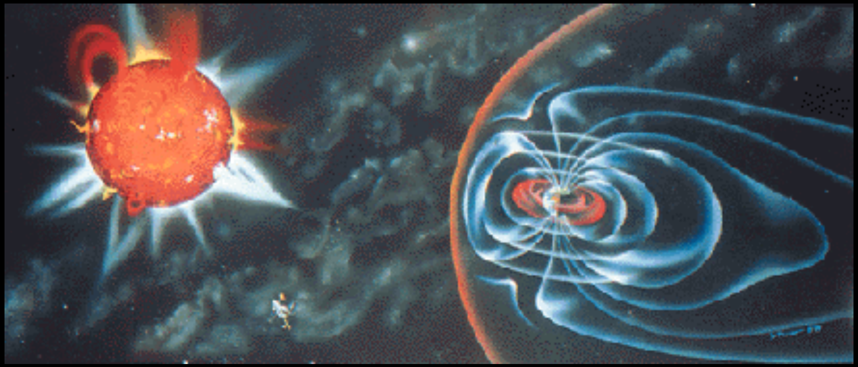
- Il pdf delle lezioni puo’ essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1920/

The slides are taken from http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1920/bibliography/hydrodynamics_achterberg.pdf

The Wide Range of Plasmas







Plasmas

Plasma

Density low to neglect particles collisions…
…but high enough to allow elm interactions.
The overall charge is nearly zero ➔ quasi-neutrality
Free charged particles→ currents→self magnetic fields→ auto-interactions
➔ Magnetohydrodynamics

Plasma is the more common state - the 4th state - of matter in universe:

99% of "normal" matter is plasma

MHD

The MHD studies the interactions of ensembles of charged particles and electromagnetic fields, i.e. plasmas

Limiting cases:

1) external fields assigned→ determine the particle motion from Lorentz force, $ma=(q/c)(E+v \times B)$ or $\rho dv/dt = \rho_c E+ jxB$, $\rho_c = qn$, $\rho =mn$

2) charges and current distributions assigned→ determine the fields from maxwell equations

The MHD is different for two reasons:

(a) it deals with many particles systems (plasmas) which exhibit collective behavior (bulk motion, oscillations, instabilities,...)

(b) fields E and B are not prescribed but determined by the positions and motions of the particles

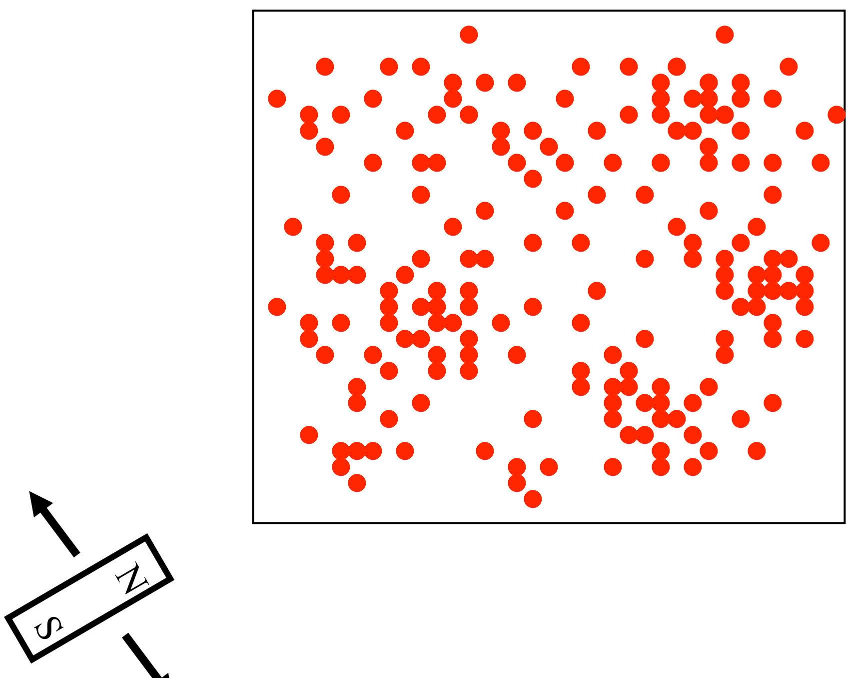
➔ field and motion equations must be solved simultaneously and self-consistently: we are looking for a set of particle trajectories and fields patterns such that the particles *generate* the field patterns as they move along their orbits and the fields patterns force the particles to move in exactly these orbits

In a plasma the generated self-fields act as coupling device between the individual particles

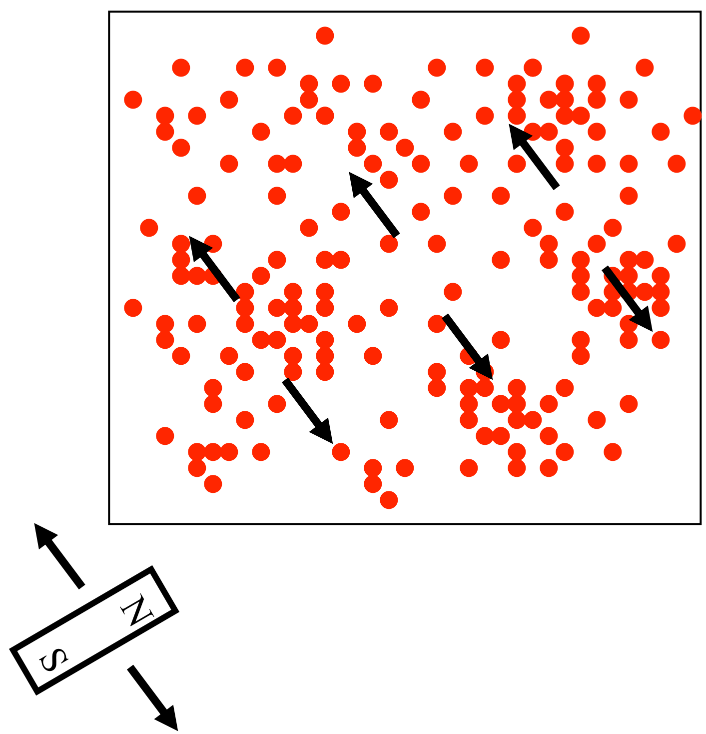
Plasmas Respond to B-Fields

• ***Magneto-HydroDynamics (MHD) = hydrodynamics flow+electro-magnetic phenomena***

Regular Gas



Plasma



MHD

The problem is mechanical, thermodynamical and electromagnetic

We need to find the matter distribution ρ, charge distr. ρ_c, current density j, speed v and the fields E and B, with given boundary conditions

Assumptions:
(a) the medium cannot be magnetized nor polarized $\varepsilon = \mu = 0$
(b) small velocities compared to c
(c) highly conductive medium, ie $E/B \gg 1$
(d) displacement current small compared to induction current: $\partial E/\partial t \ll \mu_o j$

Motion equations

To get the motion equations for the fluid we have to modify the hydrodynamic equations

The mass conservation law remains unchanged $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

In the momentum equation a new term appears due to the Lotentz force acting on the moving charges

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}$$

The energy equation must be modified too, since the presence of currents implies electrical resistance and this, in turn, implies dissipation and therefore heating

From elementary physics we know that the rate of heating (ie power dissipated) is given by Joule's law P = dE/dt= j^2/σ, being σ the fluid conductivity, therefore

While in absence of dissipation, we got for a fixed mass element $T\frac{Ds}{Dt} = 0$

In presence of resistive dissipation it becomes $\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$

$$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$$

Motion equations

Why should we have currents in the fluid?

Because elm fields could be present: from elementary physics, we get Ohm's law

In a stationary medium, an electric field generates a current density $j=\sigma E'$

In our case, the fluid is in motion wrt the laboratory frame (or, for astronomers, of the observer!) and the electric and magnetic fields are measured in this frame

From this it follows that the electric and magnetic fields felt by the fluid in its reference (rest) frame are different from those measured in laboratory

Remember that the Lorentz transformations for the fields are

$\vec{E}' = \gamma[\vec{E} + (\vec{v} \times \vec{B})/c]$ being E and B the fields in the lab

We assumed that v<<c, therefore we can approximate as $\vec{E}' \approx [\vec{E} + (\vec{v} \times \vec{B})/c]$

This is the field felt by the fluid in its rest frame

Motion equations

$$\vec{E}' \approx [\vec{E} + (\vec{v} \times \vec{B})/c]$$

Therefore the Ohm's law becomes $\vec{j}' \approx \sigma[\vec{E} + (\vec{v} \times \vec{B})/c]$

Generally speaking, the conductivity is a tensor and it's easy to argue that, in presence of strong B fields, σ (as diffusion coefficients, like heat transport or viscosity) is different if we consider the directions parallel or normal to the field, but it can be shown that the conduction coefficients along the two directions differ by 3π/32=0.295, so that we will omit the distinction and treat σ as a scalar

Of course, j is the spatial component of a 4-vector (cρ,**j**) and is expected to transform when we go from rest frame to the lab frame

But the Lorentz transf mixes charge density and current, and we have seen that we can safely assume ρ=0 because of charge-neutrality, so in the limit v<< c→

$$\vec{j}' \approx \gamma \vec{j} \approx \vec{j}$$

In these hypothesis, then, current density does not change when passing from rest frame to the lab frame and the Ohm's law is

$$\vec{j} \approx \sigma[\vec{E} + (\vec{v} \times \vec{B})/c]$$

Motion equations

Together with the fluid equations, we have to take into account also the Maxwell equations to describe the elm fields

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \approx 0$$

Due to charge neutrality ρ (remember that it is the algebric sum of positive and negative charges)~0

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

It follows that the only way to have E fields in MHD is by induction!

And that the induced field strength is, to order of magnitude, E~vB/c, being V a typical speed

$$\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} + \frac{1}{c}\frac{\partial E}{\partial t} \approx (4\pi/c)\vec{j}$$

Since the displacement current is negligible

Since the E field in MHD is due exclusively to induction, from rotor equation we find that E~ LB/cT, where L and T are typical lengths and time scale over which the elm varies and L/T is clearly a typical speed

From this it follows that the displacement current is, to order of magnitude, ~ LB/c^2T^2

This must be compared with rotor of B, which, to order of magnitude is B/L

The ratio of the two terms is $L^2/(cT)^2 \sim (v/c)^2$, so that for v<< c can be neglected

$$\vec{\nabla} \cdot \vec{B} = 0$$

Remains unchanged

MHD eqns summary

In the simplest case of a single charge specie the equations are

Maxwell field equations

$$\vec{\nabla}\cdot\vec{E}=0 \quad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=(4\pi/c)\vec{j} \quad \vec{\nabla}\times\vec{E}=-\frac{\partial B}{\partial t}$$

Ohm's law

$$\vec{j}=\sigma(\vec{E}+\vec{v}\times\vec{B}/c)$$

Equations of motion (Euler eqn for ideal fluids)

$$\rho\frac{\partial\vec{v}}{\partial t}+\rho(\vec{v}\cdot\vec{\nabla})\vec{v}=-\vec{\nabla}p+\vec{j}\times\vec{B}/c$$

Mass conservation

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}(\rho\vec{v})=0$$

Energy equation

$$\rho T\frac{Ds}{Dt}=\frac{j^2}{\sigma}$$

Equation of state

$$\frac{d(\frac{p}{\rho^{\gamma a}})}{dt}=0$$

MHD induction equation

Let assume that σ = const. in time and space.

Combining Faraday and Ohm law we eliminate E and J from equations

$$\vec{E} = \vec{j}/\sigma - \vec{v} \times \vec{B}/c$$

$$\vec{\nabla} \times (\vec{j}/\sigma - \vec{v} \times \vec{B}/c) = -\frac{1}{c}\frac{\partial B}{\partial t} \quad \Longrightarrow \quad \nabla \times \vec{j} = \sigma(\nabla \times (\mathbf{v} \times \mathbf{B}) / \mathbf{c} - \frac{\partial \mathbf{B}}{c \partial t})$$

From Ampere law $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} \quad \Longrightarrow \quad \frac{c}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{j}$

$$\frac{c}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c}\frac{\partial B}{\partial t})$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\nabla^2 \vec{B} + \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) = -\nabla^2 \vec{B} \quad \Longrightarrow \quad -\frac{c^2}{4\pi}\nabla^2 \mathbf{B} = \sigma(\nabla \times (\mathbf{v} \times \mathbf{B} / \mathbf{c}) - \frac{\partial \mathbf{B}}{c \partial t})$$

$$\boxed{\frac{c^2}{4\pi\sigma}\nabla^2 \vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t}}$$

The coefficient of laplacian $\lambda = \frac{4\pi\sigma}{c^2}$ is called magnetic diffusivity

Or $\eta = c^2/4\pi\sigma$ is the magnetic diffusion coefficient

This equation, called induction equation, is very useful because only the B field appears in it

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

N.B.:

(i) → B once v is known

(ii) In MHD, v and B are primary variables: induction eqn + eqn of motion → basic physics

(iii) $\mathbf{j} = c\nabla \times \mathbf{B} / 4\pi \quad and \quad \mathbf{E} = -\mathbf{v} \times \mathbf{B} + \mathbf{j} / \sigma$ are secondary variables

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \underset{\mathbf{A}}{\nabla \times (\mathbf{v} \times \mathbf{B})} + \underset{\mathbf{B}}{\eta \nabla^2 \mathbf{B}}$$

(iv) B changes due to transport + diffusion

(v) $\boxed{\frac{A}{B} = \frac{L_0 v_0}{\eta} = R_m}$ -- * magnetic Reynold number *

eg, $\eta = 1\ \mathrm{m}^2/s$, $L_0 = 10^5$ m, $v_0 = 10^3$ m/s --> $\mathbf{R_m = 10^8}$

(vi) A >> B in most of Universe -->

B frozen to plasma -- keeps its energy

Except SINGULARITIES -- j & ∇ B large

Form at NULL POINTS, B = 0

(a) If $R_m << 1$

- The induction equation reduces to

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}}$$

- B is governed by a diffusion equation
  → field variations on a scale L_0
  diffuse away on time *

$$\boxed{t_d = \frac{L_0^2}{\eta}}$$ *

with speed $v_d = L_0 / t_d \boxed{= \frac{\eta}{L_0}}$

- E.g.: sunspot (η= 1 m^2/s, L_0 = 10^6 m), $t_d = 10^{12}$ **sec**;
  for whole Sun (L_0 = $7\text{x}10^8$ m), $t_d = 5\text{x}10^{17}$ **sec**

MHD: ideal limit

$$\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} + \vec{\nabla}\times\vec{v}\times\vec{B} = \frac{\partial B}{\partial t}$$

To understand what happens when we are not in ideal limit, let consider a fluid in which the second term in the LHS is negligible

$$\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} = \frac{\partial B}{\partial t}$$

This is perfectly analogous to the heat diffusion equation, which by definition represents the dissipation: even if there is some heat concentrated in a small region, after a while it diffuses into all the space

→ the B field diffuses in space on a time scale

$$T_d = \frac{4\pi\sigma L^2}{c^2}$$

MHD: ideal limit

$$T_d = \frac{4\pi\sigma L^2}{c^2}$$

To understand whether the ideal MHD approximation fits the astrophysical situations, let calculate T_d for some concrete situation

Assume a conductivity for a hydrogen gas completely ionized of $\sigma = 6.98 \times 10^7 \frac{T^{3/2}}{ln\Lambda}\ s^{-1}$

Where T is the temperature in K and lnΛ~ 30 is an approximate factor called Coulomb logarithm (it arises from semi-classical calculation)

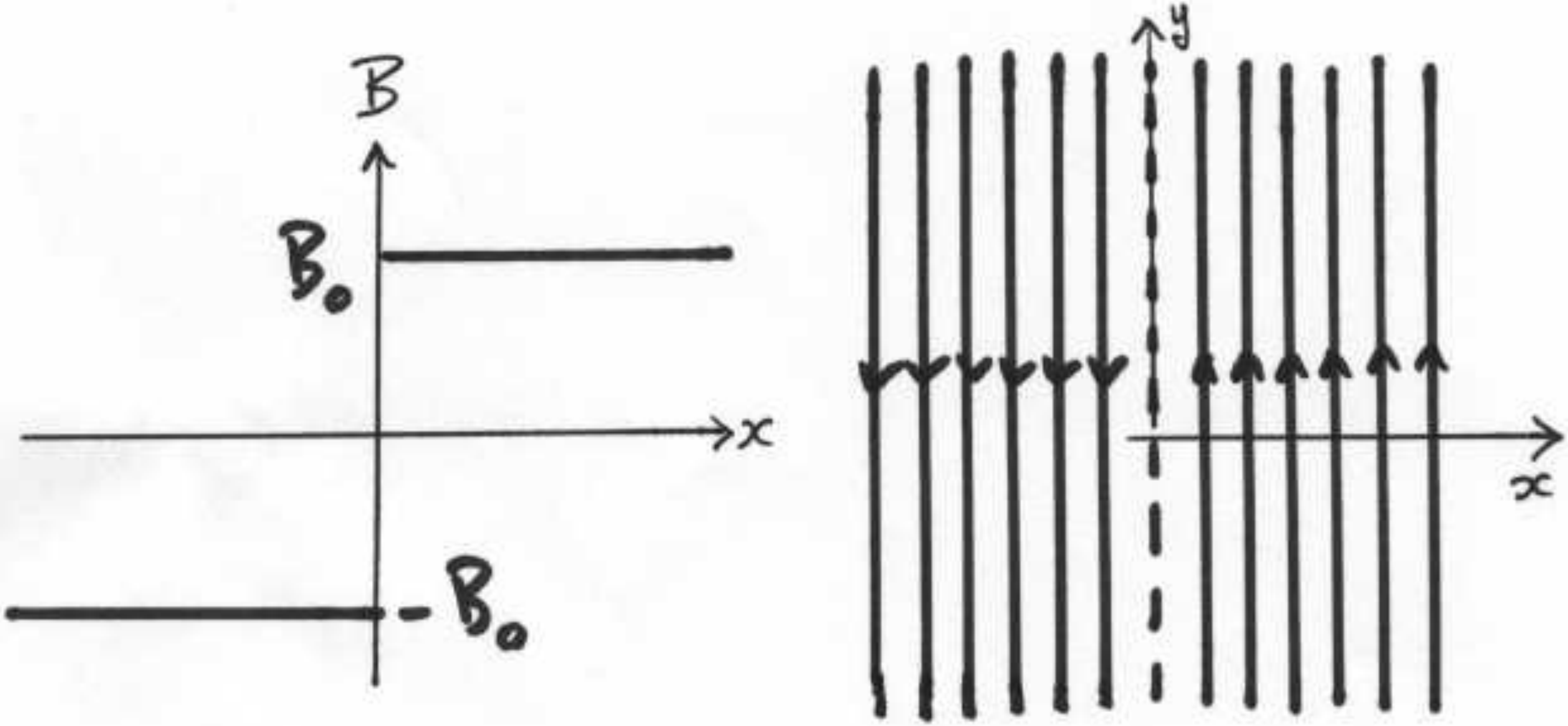
For a stellar interior for which $T \sim 10^7$ K and $L \sim 10^{11}$ cm, we find $T_d \sim 3\text{x } 10^{11}$ years, greater than universe lifetime, T_U

For a galaxy, L is very huge and $T_d > T_U$

But even for the smallest objects this time is long: in the inner regions of accretion disks around a black hole, with dimensions of ~ 100 Schwarzschild radii (~10^{15} cm), we find $T_d \sim 10^{20}$ years

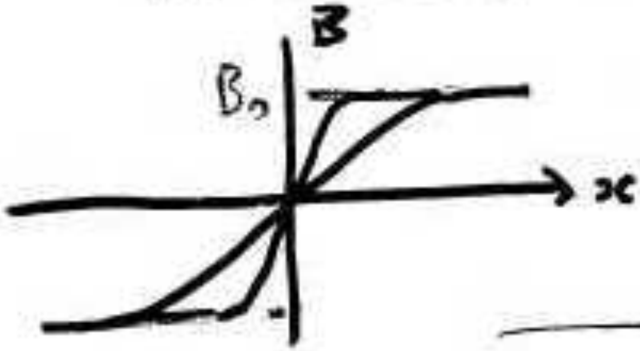
In other words, the astrophysical fluids are (almost) all such that the dissipation time of the B field is greater than Universe lifetime: the B field is never dissipated in typical astrophysical conditions

Example: given B(x,0) as in figure, determine B(x,t) in the hypothesis that diffusion dominates



To Solve $\quad \dfrac{\partial B}{\partial t} = \eta \dfrac{\partial^2 B}{\partial x^2} \quad - (1)$

for $B(x,t)$



where $B(x,0) = \begin{cases} B_0, & x > 0 \\ -B_0, & x < 0 \end{cases} \quad - (2)$

Since initial conditions have no natural scale, seek a self-similar solution of the form

$$B(x,t) = f(V), \quad \text{where } V = \frac{x}{t^{1/2}}$$

Then $\quad \dfrac{\partial B}{\partial t} = \dfrac{df}{dV}\left(-\dfrac{x}{2t^{3/2}}\right) = -\dfrac{V}{2t}\dfrac{df}{dV}$

$$\frac{\partial B}{\partial x} = \frac{df}{dV}\frac{1}{t^{1/2}}, \quad \frac{\partial^2 B}{\partial x^2} = \frac{d^2 f}{dV^2}\frac{1}{t}$$

Thus (1) becomes $-\frac{V}{2}\frac{df}{dV} = \eta \frac{d^2f}{dV^2}$

Put $\frac{df}{dV} = F$ and $V = U\sqrt{4\eta}$ $\Rightarrow$ $\frac{dF}{dU} = -2UF$

Integrate $F = \text{const}\ e^{-U^2}$

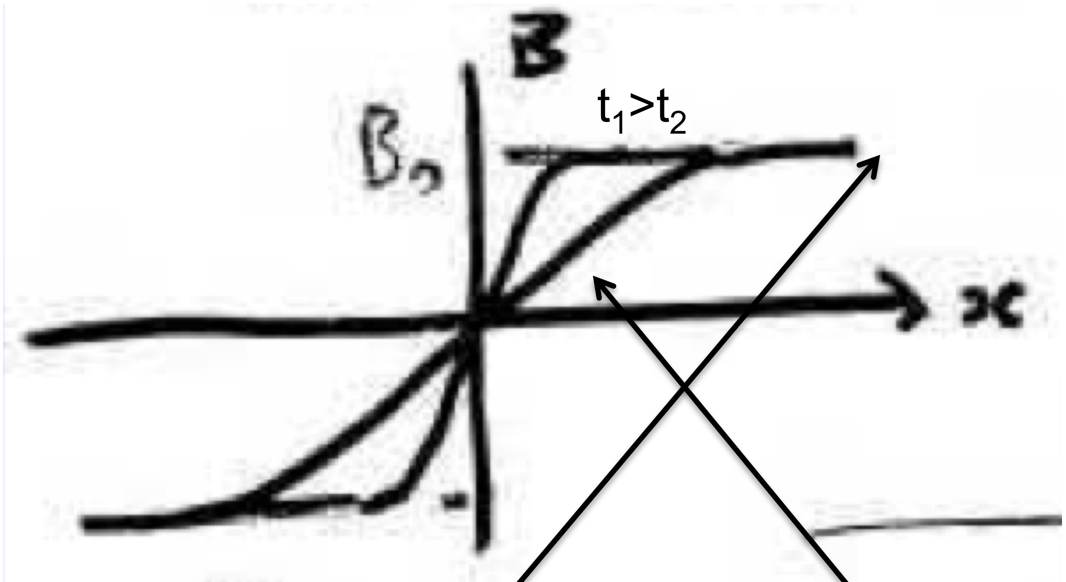
Integrate $f(U) = c\int_0^U e^{-u^2}du$

where $t \to 0, U \to \infty \Rightarrow c = \frac{2B_0}{\sqrt{\pi}}$ from (2)

i.e. $B = \frac{2B_0}{\sqrt{\pi}}\int_0^{x/\sqrt{4\eta t}} e^{-u^2}du$

$$For\ x >> \sqrt{4\eta t}, \quad B \approx B_0$$ Not dependent on x

$$For\ x << \sqrt{4\eta t}, \quad B \approx B_0\, x / \sqrt{\pi\eta t}, \quad so\ slope \downarrow with\ t$$



$$For\ x >> \sqrt{4\eta t}, \quad B \approx B_0, \quad \textcolor{red}{indep\, x}$$

$$For\ x << \sqrt{4\eta t}, \quad B \approx B_0\, x / \sqrt{\pi\eta t}, \quad \textcolor{red}{so\, slope \downarrow with\, t}$$

MHD: ideal limit

The equations simplify in the so called "ideal MHD" when R>>1 or $\sigma \rightarrow \infty$

In such a case the energy equation turns back to the form of pure entropy conservation

$$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma} \quad \Longrightarrow \quad \rho T \frac{Ds}{Dt} = 0 \quad \Longrightarrow \quad \frac{Ds}{Dt} = 0$$

Euler eqn remains the same $\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}/c$

With the aid of $(c/4\pi)\vec{\nabla} \times \vec{B} = \vec{j}$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Mass conservation law is the same $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

The field equation simplifies

$$\frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t} \quad \Longrightarrow \quad \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t}$$

These are the MHD equations in the ideal limit

(b) If $R_m >> 1$

The induction equation reduces to

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})}$$

and Ohm's law -->

$$\boxed{\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}}$$

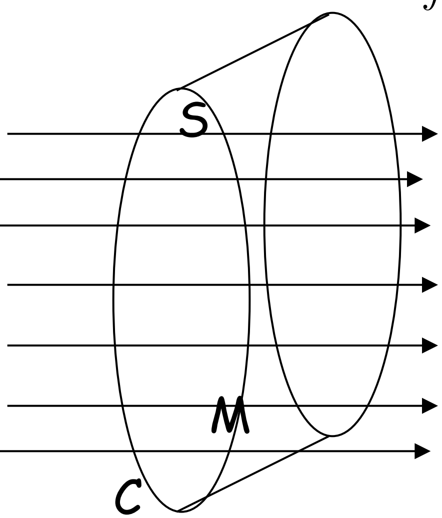
Magnetic field is "* frozen to the plasma *"

MHD: Frozen-in flux

Let assume that the B field is $B(r,t_o)$ at time t_o

The magnetic flux through a surface S enclosed by a curve C is

$$\Phi = \int_S \vec{B} \cdot d\vec{S}$$



if the curve moves, Φ changes because:
(a) the B field changes in time and (b) the field lines move into or out of S

As C moves, it creates a cylinder (a "flux tube") with a mantle surface M

Every field line leaving or entering C is associated with a flux of the SAME line through M

$$d\Phi = dt \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_M \vec{B} \cdot d\vec{S_M}$$

MHD

$$d\Phi = dt\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_M \vec{B}\cdot d\vec{S_M}$$

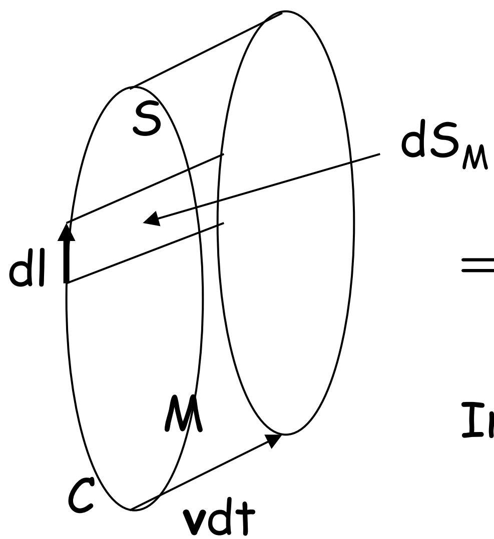
The surface element on M can be written as $d\vec{S_M} = \vec{v}\times d\vec{l}dt$

$$d\Phi = dt\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_C \vec{B}\cdot \vec{v}\times d\vec{l}dt$$

$$d\Phi/dt = \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_C \vec{B}\cdot \vec{v}\times d\vec{l}$$

$$\vec{B}\cdot\vec{v}\times d\vec{l} = -\vec{v}\times\vec{B}\cdot d\vec{l}$$

$$d\Phi/dt = \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} - \int_C \vec{v}\times\vec{B}\cdot d\vec{l}$$



$$= \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} - \int_S \vec{\nabla}\times\vec{v}\times\vec{B}\cdot d\vec{S}$$

With the Stokes' theorem for the last term

$$= \int_S \left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times\vec{v}\times\vec{B}\right]\cdot d\vec{S} = -\int_S \vec{\nabla}\times\vec{j}/\sigma\cdot d\vec{S} = -\int_C \vec{j}/\sigma\cdot d\vec{l}$$

In the convective limit, $R_M >> 1$ or $S << 1$, ie $\sigma\rightarrow\infty$

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times\vec{v}\times\vec{B} = 0 \quad\Rightarrow\quad d\Phi/dt = 0$$

The magnetic flux cannot change→ therefore field lines must be swept away with the plasma motion, ie B is tied to the particles in the element: FROZEN FIELD→ <u>flux tube does not break when plasma shuffles around!</u>

Frozen-in Condition

In astrophysical systems with high R_m, we can imagine the magnetic flux to be frozen in the plasma and to move with the plasma flows

Suppose we have straight magnetic field liens going through a plasma column

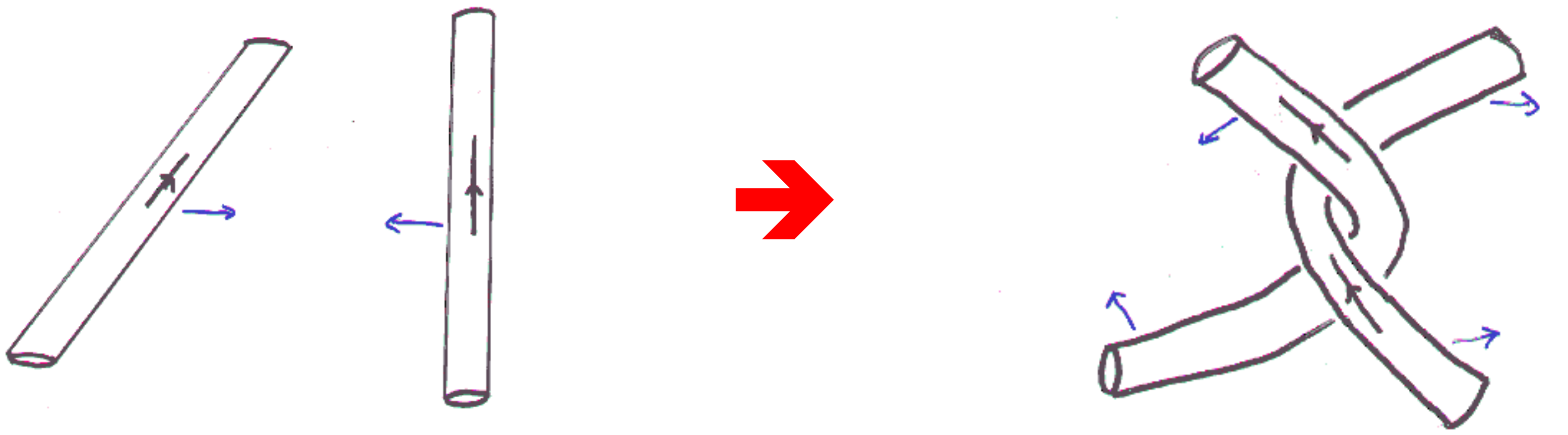
If the plasma column is bent, the magnetic field lines are bent too

On the other hand, if one end of the plasma column is twisted (because, for istance, is in rotation), then the magnetic field lines are twisted too

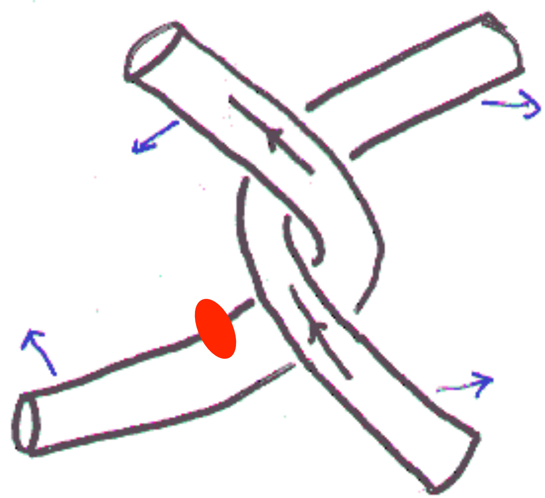
As a result of frozen-in theorem, the B field in a astrophysical system can be almost regarded as a plastic material which can be bent, twisted or distorted by making the plasma move appropriately

This view of a magnetic field is radically different from that we encounter in laboratory situations, where it appears as something rather passive which we can switch on or off by sending a current through a coil. In the astrophysical setting, the magnetic field appears to acquire a life of its own

Magnetic Field Lines Can’t Break

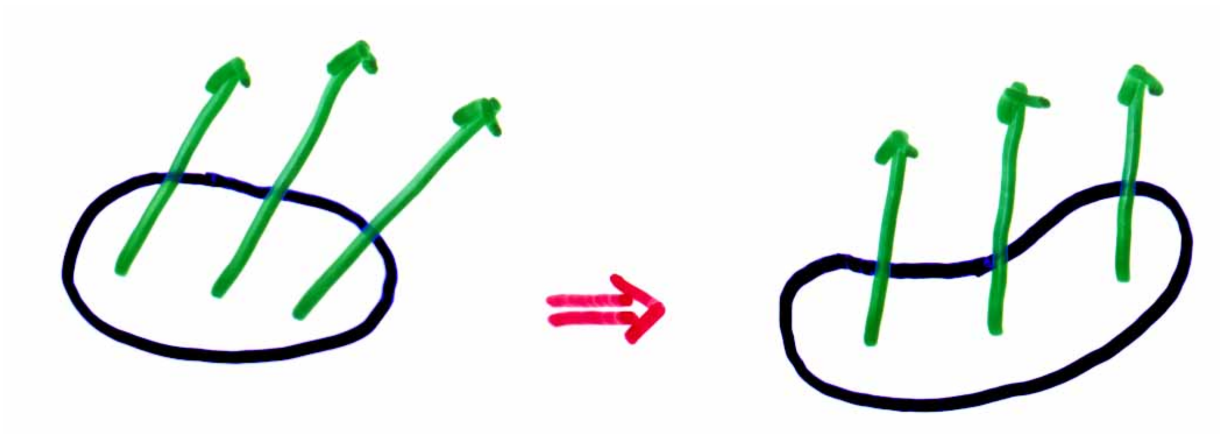


Everything Breaks Eventually



Frozen-in Condition

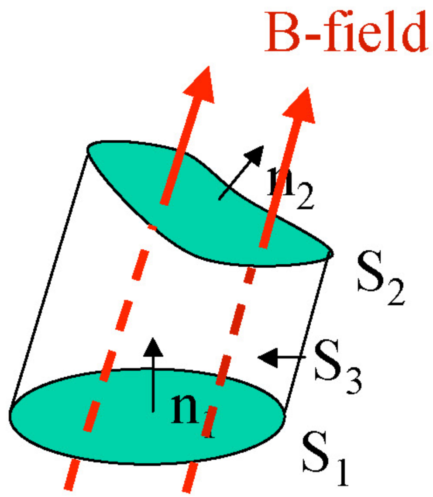
- In a simple form of plasma, the plasma moves so that the magnetic flux through any surface is preserved.



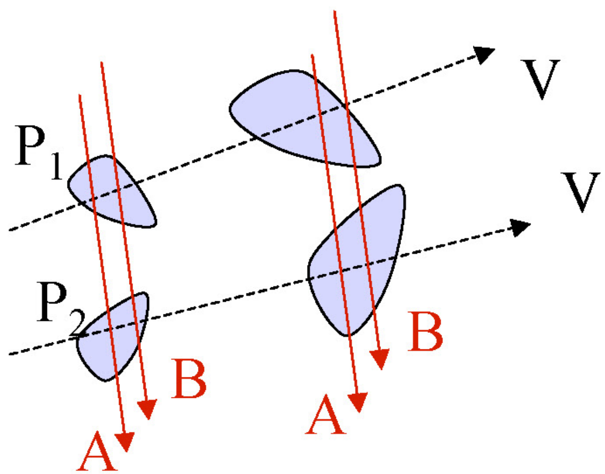
The frozen-in field condition is exactly the opposite of dissipation (ie magnetic diffusion), which tends to decrease the flux through the spire

MHD

We can define a magnetic flux tube: by taking the closed loop and moving it parallel to the field it encloses.
The surface, or tube S_3, thus created has zero flux through it and consequently the fluid elements that form the flux tube at one moment, form the flux tube at all instants.



Also: if two fluid elements P_1 and P_2 are originally linked by the field lines A and B, they will remain connected by field lines A and B whatever the individual motions V_1 and V_2 of the individual volumes.



Frozen-in field: an example

When an astronomical object shrinks due to gravitational attraction, its magnetic field is expected to become stronger

If r is the radius of the equatorial cross-section of the body through which the the magnetic field of the order of B is passing, then the flux is of the order of Br^2

If the field is perfectly frozen, then this flux should remain an invariant during the contraction of the object

Some neutron stars are believed to have B fields of the order of 10^{12} G

Let us see if we can explain this magnetic field by assuming that the neutron star formed due to the collapse of an ordinary star of which the B field got compressed

A star like the Sun has a radius $r=10^{11}$ cm and B~ 10 G at poles

Since the radius of a typical NS is 10^6 cm, the equatorial area would decrease by a factor 10^{10} if an ordinary star were to collapse to become the NS

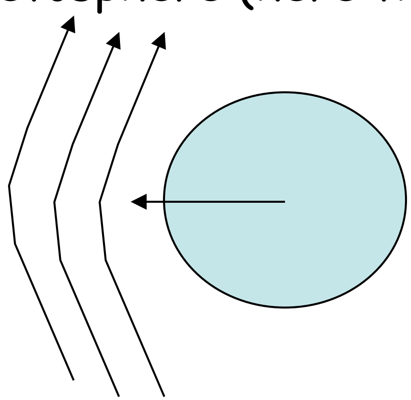
If the magnetic flux remained frozen during the collapse, the the initial field of 10 G would finally become 10^{11} G

MHD

$$d\Phi/dt = \int_S [\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{v} \times \vec{B}] \cdot d\vec{S} - \int_S \vec{\nabla} \times \vec{j}/\sigma \cdot d\vec{S} = - \int_C \vec{j}/\sigma \cdot d\vec{l}$$

A reversal of the concept, the frozen-out field, exists

If a field free plasma bubble moves toward a region filled with a magnetic field, then it pushes the field away until the pressures equals. The field cannot enter into the bubble because the magnetic flux inside the bubble would change. This happens with the solar wind frozen-out of the magnetosphere (here the field inside in the convective limit...$(R_M)_{mgtsph} >> 1$



Magnetic force:

$\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j}$ Ampere-Maxwell Law

Multiply vectorially with B both sides

The Lorentz force can be decomposed in two forces:

$$\begin{aligned} \mathbf{j}\times\mathbf{B} &= (\nabla\times\mathbf{B})\times\frac{\mathbf{B}}{\mu} \\ &= (\mathbf{B}\cdot\nabla)\frac{\mathbf{B}}{\mu} - \nabla\left(\frac{B^2}{2\mu}\right) \end{aligned}$$

Magnetic field lines have a

Tension $B^2/\mu \rightarrow$ force when lines curved

Pressure $B^2/2\mu \rightarrow$ force from high to low B^2

Magnetic force(s)

The internal forces acting on a charged fluid are the mechanical pressure, already seen and the Lorentz force, which is new because due to the particle charge and the B field

$$\vec{f} = (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Using again the identity $(\vec{\nabla} \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2$

$$\vec{f} = \frac{1}{4\pi}[(\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2]$$

The first term can be rewritten

We note that it represents B times the derivative along the field line

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B\frac{\partial \vec{B}}{\partial s}$$

Where *s* is a curvilinear coordinate along the field line

Magnetic force(s) $\vec{f} = \frac{1}{4\pi}[(\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2]$

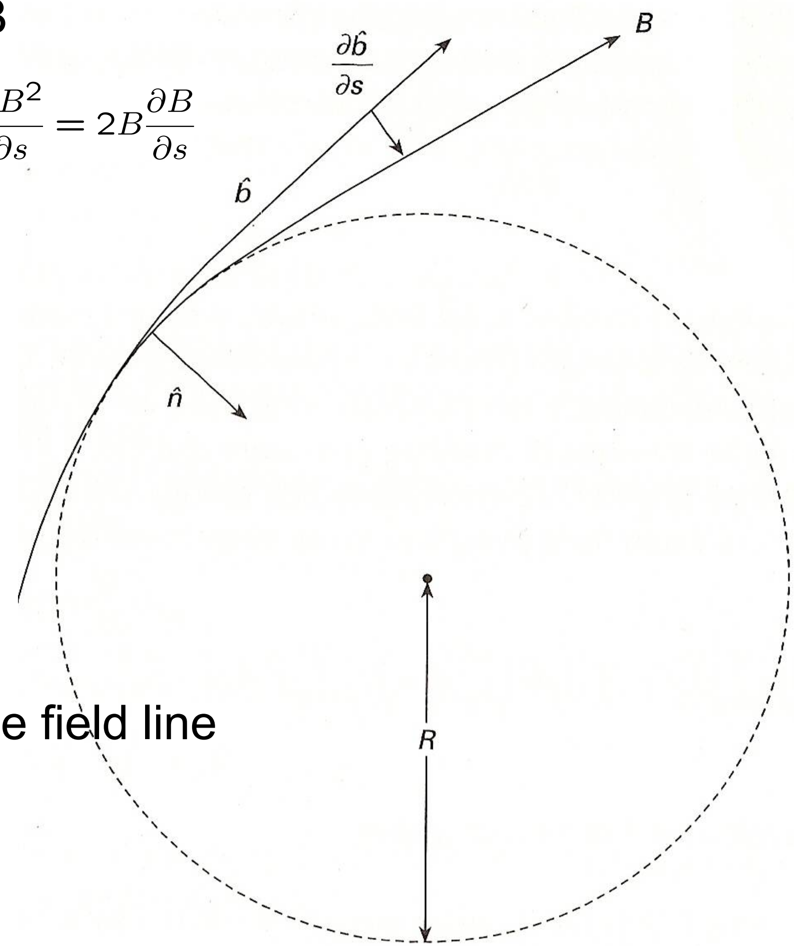
$(\vec{B}\cdot\vec{\nabla})\vec{B} = B\frac{\partial \vec{B}}{\partial s}$ Where *s* is a curvilinear coordinate along the field line

In general, B varies both in intensity and direction $\vec{B} = B\hat{b}$

With ***b*** is the local tangent versor to B

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = B\frac{\partial (B\hat{b})}{\partial s} = B^2\frac{\partial \hat{b}}{\partial s} + B\hat{b}\frac{\partial B}{\partial s}$$

but $\frac{\partial B^2}{\partial s} = 2B\frac{\partial B}{\partial s}$



$$\Rightarrow \quad (\vec{B}\cdot\vec{\nabla})\vec{B} = B^2\frac{\partial \hat{b}}{\partial s} + \hat{b}\frac{\partial B^2}{2\partial s}$$

The first term is due to the change of the field direction. The versor derivative is, by definition, perp to the versor (ie field line) and measures the curvature of the field $\frac{\partial \hat{b}}{\partial s} = \frac{\hat{n}}{R}$

The 2nd term may rewritten as $\hat{b}\frac{\partial B^2}{2\partial s} = \hat{b}\vec{\nabla}_{||}(\frac{B^2}{2})$

Where $\text{grad}_{||}$ is the gradient component along the field line

$$\Rightarrow \quad (\vec{B}\cdot\vec{\nabla})\vec{B} = B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{||}\frac{B^2}{2}$$

Magnetic force(s)

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2}$$

Lorentz force is therefore $\vec{f} = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2} - \frac{1}{2}\vec{\nabla}B^2]$

Writing grad as $\vec{\nabla} = \hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel}$ We get

$$\vec{f} = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2} - (\hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel})\frac{B^2}{2}] = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} - \hat{n}\nabla_{\perp}\frac{B^2}{2}]$$

Parallel component of the gradient cancels with the corresponding pressure term, leaving only the perp component: both terms give forces normal to the field line (as it is reasonable to expect since the Lorentz force is ~ vxB

Magnetic force(s)

$$\vec{f} = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2} - (\hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel})\frac{B^2}{2}] = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} - \hat{n}\nabla_{\perp}\frac{B^2}{2}]$$

The first term represents a force directed along the istantaneous center of curvature: it tends therefore to shorten the field line

This term is very similar to the elastic force exerted on a guitar string when is tweaked: it exerts a force that tends to bring the string back to its minimum length

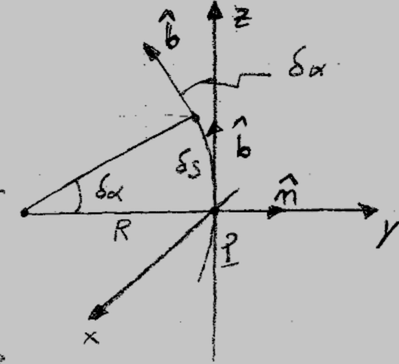
That's why it's called magnetic tension

The 2nd term is the gradient of a scalar quantity→ $B^2/2$ can be interpreted as a magnetic pressure, opposite to the gradient: the more the field is increasing along perp dir, the more is the force that tends to decrease the field...in terms of field lines this means when you try to compress field lines, there is a force which opposes to the compression

Curvature can be eliminated since

$$\vec\nabla_\perp B = -\frac{B}{R}\hat m$$

In P, $\vec B(0,0,B)$ e $\boxed{\frac{\partial B_x}{\partial z}=0}$ Due to osculator circle in P



$$\vec\nabla\times\vec B=0$$

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=0 \qquad \boxed{-\frac{\partial B_z}{\partial x}+\frac{\partial B_x}{\partial z}=0} \qquad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=0$$

$$\boxed{\frac{\partial B_z}{\partial x}=0}$$

$$\to \vec\nabla B = \frac{\partial}{\partial x}\left[\left(B_x^2+B_y^2+B_z^2\right)^{1/2}\right] = \frac{\cancel{2}B_x\partial_x B_x+\cancel{2}B_y\partial_x B_y+\cancel{2}B_z\partial_x B_z}{\cancel{2}B} = \overbrace{\left(\frac{B_z}{B}\right)}^{=1}\frac{\partial B_z}{\partial x}=0$$

$$\frac{\partial}{\partial y}\left[(\cdots)^{1/2}\right] = \frac{B_x\partial_y B_x+B_y\partial_y B_y+B_z\partial_y B_z}{B} = \cancel{\frac{B_z}{B}}\frac{\partial B_z}{\partial y}=\frac{\partial B}{\partial y}$$

$$\frac{\partial}{\partial z}\left[(\cdots)^{1/2}\right] = \frac{\partial B}{\partial s} = (\hat b\cdot\vec\nabla)B \equiv \nabla_\parallel B$$

Gradient lies in the yz-plane

But $\vec\nabla=\vec\nabla_\parallel+\vec\nabla_\perp \Rightarrow \nabla_\perp B=\frac{\partial B}{\partial y} \Rightarrow \vec\nabla B=\nabla_\parallel B\,\hat b+\nabla_\perp B\,\hat n$

The trasverse comp (B_y) changes when moving along field line

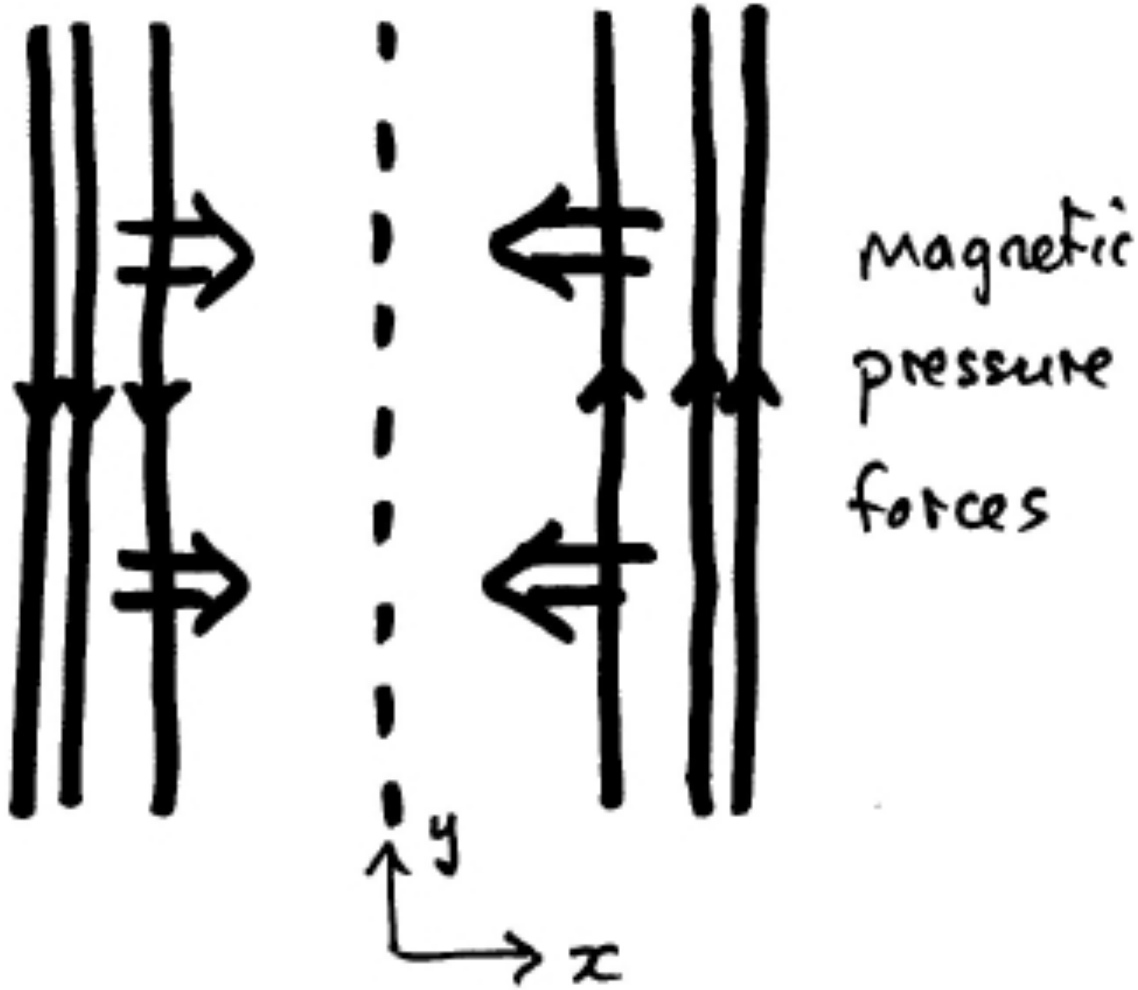
$$dB_y \approx -B\sin\delta\alpha \approx -B\,\delta\alpha \approx -B\frac{\delta s}{R} \Rightarrow \frac{\partial B_y}{\partial s}=\frac{\partial B_y}{\partial z}\approx -\frac{B}{R}$$

But $\frac{\partial B_y}{\partial z}=\frac{\partial B_z}{\partial y}$ from $\vec\nabla\times\vec B=0 \Rightarrow \frac{\partial B_z}{\partial y}\approx -\frac{B}{R}$

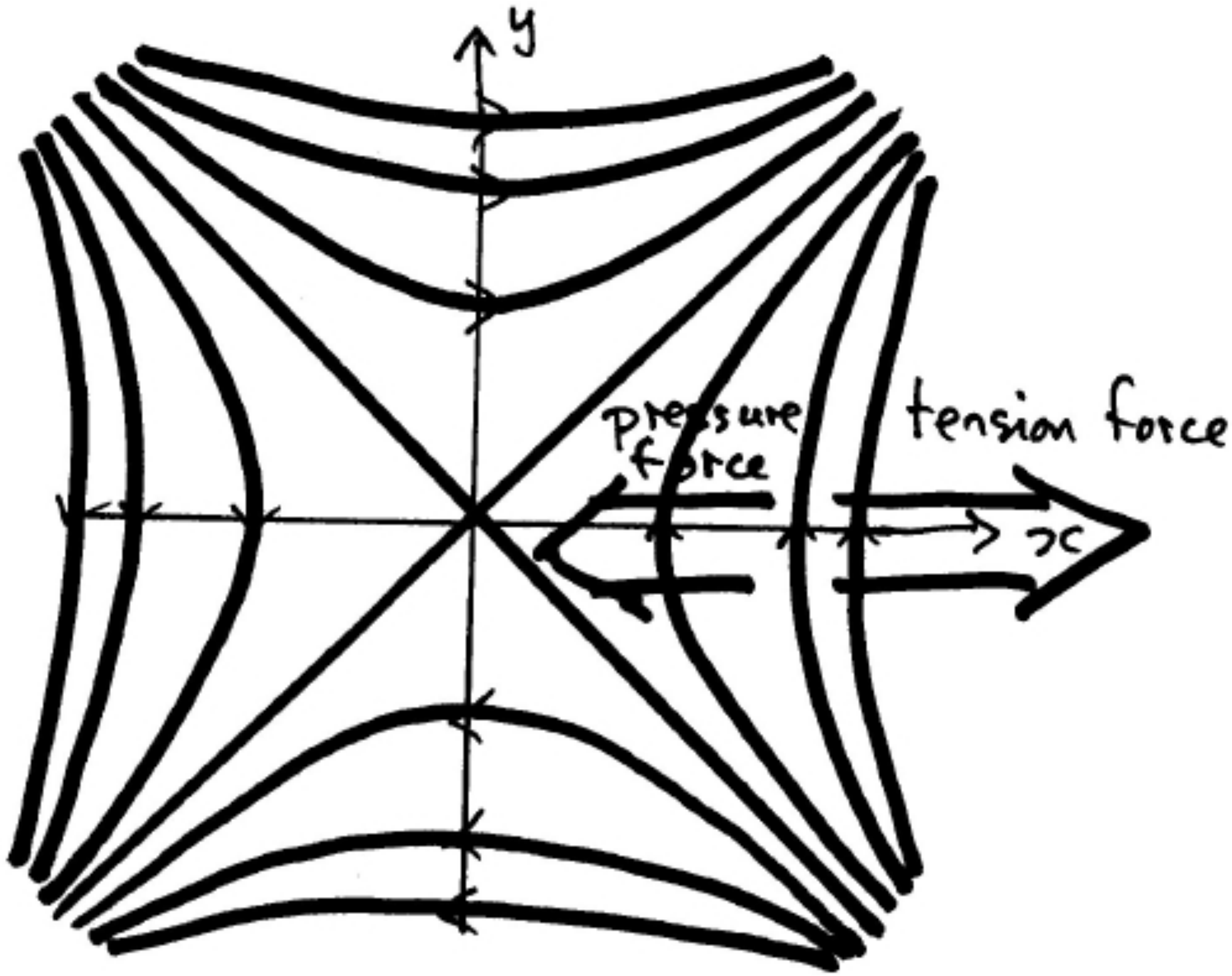
$\Rightarrow \nabla_\perp B\approx -\frac{B}{R}$ or in vector form $\frac{\vec\nabla_\perp B}{B}=-\frac{\hat n}{R}$ (with $\vec\nabla_\perp=\hat m\nabla_\perp$)

Ex.

$$\mathbf{B} = x\,\hat{\mathbf{y}}$$



Ex. $$\mathbf{B} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$$



Conservative form

We have already seen that the equation of hydrodynamics can be written in a manifestly conservative form

Einstein convention adopted

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\,\boldsymbol{I}) = 0$$

or in components

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i(\rho v_i v_j + p\delta ij) = 0$$

The tensor $\mathbf{R_{ik}} = \boldsymbol{\rho} \mathbf{V_i V_k} + \mathbf{p}\boldsymbol{\delta}_{\mathbf{ik}}$ is the Reynolds stress tensor for an ideal fluid and represents the mechanical momentum flux $\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial R_{ik}}{\partial x_k}$

This is possible for MHD too

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + (1/4\pi)(\vec{\nabla}\times\vec{B})\times\vec{B}$$

Make use of the identity $(\vec{\nabla}\times\vec{B})\times\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2$

The double vector prod can be written as the divergence of a tensor

$$[(\vec{\nabla}\times\vec{B})\times\vec{B}]_i = B_j\nabla_j B_i - \frac{1}{2}\nabla_i B^2$$

Now $\nabla_j(B_jB_i) = B_i\nabla_j B_j + B_j\nabla_j B_i$ but $\nabla_j B_j = \vec{\nabla}\cdot\vec{B} = 0$

$\Rightarrow$ $\nabla_j(B_jB_i) = B_j\nabla_j B_i$ $\Rightarrow$ $[(\vec{\nabla}\times\vec{B})\times\vec{B}]_i = \nabla_j(B_jB_i) - \frac{1}{2}\nabla_i B^2 = \nabla_j(B_jB_i) - \nabla_j\frac{1}{2}B^2\delta_{ij}$

$$= \nabla_j[(B_jB_i) - \frac{1}{2}B^2\delta_{ij}]$$

$\Rightarrow$ $$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + (1/4\pi)\nabla_j[(B_jB_i) - \frac{1}{2}B^2\delta_{ij}]$$

Conservative form

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i(\rho v_i v_j + p\delta ij) = 0$$

$R_{ik}= \rho V_i V_k + p\delta_{ik}$

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial R_{ik}}{\partial x_k}$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \frac{1}{4\pi}\nabla_j[(B_j B_i) - \frac{1}{2}B^2\delta_{ij}]$$

The Euler eqn can be put therefore in the same form as for hydrodynamics case defining

$$M_{ij} = -\frac{1}{4\pi}[(B_j B_i) - \frac{1}{2}B^2\delta_{ij}]$$

And $T_{ij} = R_{ij} + M_{ij} = \rho v_i v_j + p\delta ij - \frac{1}{4\pi}[(B_j B_i) - \frac{1}{2}B^2\delta_{ij}]$ $\Rightarrow$ $\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial T_{ik}}{\partial x_k}$

M_{ij} describes flux per time unit of the i-th component of the B field momentum through a surface oriented along the j-th direction and it's called Maxwell stress tensor

The relative importance in concrete astrophysical situations, as pulsar winds and black hole jets, of the mechanical and magnetic momentum flux is, at the moment, uncertain and of great interest

Conservative form

It is well known that elm field transports energy too

The energy density of the field is $\epsilon_B = B^2/8\pi$

The energy flux is described by Poynting vector $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}$

We have to eliminate the E field

Consider the Ohm's law $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}/c)$

In the ideal limit $\sigma \to \infty$, to avoid infinite currents we must have

$$\vec{E} + \vec{v} \times \vec{B}/c = 0$$

$$\Longrightarrow \vec{E} = -\vec{v} \times \vec{B}/c \qquad \Longrightarrow \vec{S} = -\frac{1}{4\pi}(\vec{v} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}\vec{B} \times (\vec{v} \times \vec{B})$$

Conservative form

$\epsilon_B = B^2/8\pi$

$$\vec{S} = -\frac{1}{4\pi}(\vec{v} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}\vec{B} \times (\vec{v} \times \vec{B})$$

We have to add these terms into the energy equation of the hydro case

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\text{eff}}$$

$h = \varepsilon + P/\rho$

$$\mathcal{H}_{\text{eff}} \equiv \mathcal{H} + \rho\,\frac{\partial \Phi}{\partial t}$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e + \rho\Phi + \frac{1}{8\pi}B^2\right) + \vec{\nabla}\cdot\left[\vec{v}\left(\frac{1}{2}\rho v^2 + \rho h + \rho\Phi\right) + \frac{1}{4\pi}\vec{B}\times(\vec{v}\times\vec{B})\right] = H_{eff}$$

With no irreversible losses and no time dependent gravitational potentail $H_{\text{eff}}=0$

Summary in the ideal limit

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0$$ Mass conservation law

M density M flux

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i[\rho v_i v_j + p\delta ij - \frac{1}{4\pi}(B_j B_i - \frac{1}{2}B^2\delta_{ij})] = 0$$ Momentum conservation law

p density ***p*** flux

Energy conservation law

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2 + \rho e + \rho\Phi + \frac{1}{8\pi}B^2) + \vec{\nabla}\cdot[\vec{v}(\frac{1}{2}\rho v^2 + \rho e + P + \rho\Phi) + \frac{1}{4\pi}\vec{B}\times(\vec{v}\times\vec{B})] = 0$$

E density E flux

Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.

→ ***Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.***

→ ***Then all variables can be expressed by one, say the magnetic field.***

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys:

$$\boxed{|\delta \mathbf{B}| \ll B_0}$$

In the uniform plasma with straight field lines, the field provides the only ***symmetry axis*** which may be chosen as z-axis of the coordinate system such that: $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_{||}$.

Small perturbations

We will treat the small perturbations of a magnetohydrodynamical equilibrium

Let choose an equilibrium configuration of an omogeneous, infinite medium at rest for sake of simplicity (but the method can be applied to any equilibrium configuration, even non omogeneous and time dependent)

Let therefore consider a configuration in which $p=p_o$, $\rho=\rho_o$, $\mathbf{v}=0$ and $\mathbf{B}=B_o\mathbf{z}$

Consider now a situation in which the quantities are perturbed from equilibrium, that is $p=p_o+\delta p$, $\rho=\rho_o+\delta\rho$, $v=\delta v$ and $B=B_o z+\delta B$

Our goal is find perturbations corresponding to acoustic waves, therefore let assume that perturbations are iso-entropic, so that we have a relation between pressure and density:

$$\delta p=c^2_s\delta\rho$$

With $c_s=\gamma\ p_o/\rho_o$ iso-entropic sound speed

Small perturbations

The motion equations, under the assumption that perturbations are small, can be linearized

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 & \text{(i)} \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\dfrac{\vec{\nabla}p}{\rho} + \dfrac{1}{4\pi\rho}(\vec{\nabla}\times\vec{B})\times\vec{B} & \text{(ii)} \\ \dfrac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) & \text{(iii)} \end{cases}$$

$$\rho = \rho_0 + \delta\rho \quad p = p_0 + \delta p \quad \vec{v} = \delta\vec{v} \quad \vec{B} = B_0\hat{z} + \delta\vec{B}$$

$$\delta\rho \ll \rho_0 \quad \delta p \ll p_0 \quad \delta\vec{v} \ll 1 \quad \delta\vec{B} \ll B_0\hat{z}$$

$$\delta p = c_s^2\,\delta\rho \qquad c_s^2 = \frac{\gamma p_0}{\rho_0}$$

$$\text{(i)}\quad \frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \vec{\nabla}\cdot\left[(\rho_0+\delta\rho)\delta\vec{v}\right] = \frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v} + \vec{\nabla}\cdot(\delta\rho\,\delta\vec{v}) \approx \frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v} \Longrightarrow \frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_0}\right) \approx -\vec{\nabla}\cdot\vec{v}$$

$$\text{(ii)}\quad \frac{\partial\delta\vec{v}}{\partial t} + (\delta\vec{v}\cdot\vec{\nabla})\delta\vec{v} = -\frac{\vec{\nabla}(p_0+\delta p)}{(\rho_0+\delta\rho)} + \frac{1}{4\pi(\rho_0+\delta\rho)}\left[\vec{\nabla}\times(B_0\hat{z}+\delta\vec{B})\right]\times(B_0\hat{z}+\delta\vec{B})$$

$$\frac{\partial\delta\vec{v}}{\partial t} \approx -\frac{\vec{\nabla}\delta p}{\rho_0+\delta\rho} + \frac{1}{4\pi(\rho_0+\delta\rho)}\Big[\underbrace{(\vec{\nabla}\times B_0\hat{z})}_{=0}\times B_0\hat{z} + \underbrace{(\vec{\nabla}\times B_0\hat{z})}_{=0}\times\delta\vec{B} + (\vec{\nabla}\times\delta\vec{B})\times B_0\hat{z} + \underbrace{(\vec{\nabla}\times\delta\vec{B})\times\delta\vec{B}}_{\ll 1}\Big] \approx$$

$$\approx -\frac{\vec{\nabla}\delta p}{\rho_0} + \frac{1}{4\pi\rho_0}(\vec{\nabla}\times\delta\vec{B})\times B_0\hat{z} = -c_s^2\,\vec{\nabla}\left(\frac{\delta\rho}{\rho_0}\right) + \frac{1}{4\pi\rho_0}(\vec{\nabla}\times\delta\vec{B})\times B_0\hat{z}$$

$$\text{(iii)}\quad \frac{\partial\delta\vec{B}}{\partial t} = \vec{\nabla}\times\left[\delta\vec{v}\times(B_0\hat{z}+\delta\vec{B})\right] = \vec{\nabla}\times(\delta\vec{v}\times B_0\hat{z}) + \vec{\nabla}\times(\delta\vec{v}\times\delta\vec{B}) \approx \vec{\nabla}\times(\delta\vec{v}\times B_0\hat{z})$$

(iv) Ds/Dt=0 in automatically satisfied since entropy is assumed constant

The magnetic field equation can be furtherly simplified noting that Bo has null all the spatial derivatives. Using the vectorial identity:

NB: $\boldsymbol{B} = B_0\mathbf{z}$

$$\vec{\nabla}\times(\delta\vec{v}\times\vec{B}) = (\vec{B}\cdot\nabla)\delta\vec{v} - (\delta\vec{v}\cdot\nabla)\vec{B} - (\vec{B}\nabla)\cdot\delta\vec{v} + \delta\vec{v}(\nabla\cdot\vec{B})$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$= (\vec{B}\cdot\nabla)\delta\vec{v} - (\vec{B}\nabla)\cdot\delta\vec{v} \quad = B_o\frac{\partial}{\partial z}\delta\vec{v} - B_o\hat{z}\nabla\cdot\delta\vec{v}$$

$$(\vec{v}\cdot\vec{\nabla})\vec{B} = 0$$

$$\frac{\partial}{\partial t}\delta\vec{B} = B_o\frac{\partial}{\partial z}\delta\vec{v} - B_o\hat{z}\nabla\cdot\delta\vec{v}$$

Small perturbations

$$\frac{\partial}{\partial t}(\frac{\delta\rho}{\rho_o}) + \nabla\cdot\delta\vec{v} = 0$$

This is a system of partial derivative equations

$$\frac{\partial\vec{v}}{\partial t} = -c_s^2\nabla(\frac{\delta\rho}{\rho_o}) + \frac{1}{4\pi\rho_o}(\nabla\times\delta\vec{B})\times B_o\hat{z}$$

Since the coefficients are constant we can develop the unknowns in Fourier series and find a solution for each Fourier amplitude

$$\frac{\partial\delta\vec{B}}{\partial t} - \vec{\nabla}\times(\delta\vec{v}\times\vec{B_o}\hat{z}) = 0$$ or $$\frac{\partial}{\partial t}\delta\vec{B} = B_o\frac{\partial}{\partial z}\delta\vec{v} - B_o\hat{z}\nabla\cdot\delta\vec{v}$$

Derive wrt t the momentum equation

$$\frac{\partial^2\vec{v}}{\partial t^2} = -c_s^2\nabla(\frac{\partial}{\partial t}[\frac{\delta\rho}{\rho_o})] + \frac{1}{4\pi\rho_o}(\nabla\times\frac{\partial}{\partial t}\delta\vec{B})\times B_o\hat{z}$$

Use 1 and 3 to eliminate time deriv in RHS

$$\frac{\partial^2\vec{v}}{\partial t^2} = c_s^2\nabla(\nabla\cdot\delta\vec{v}) + \frac{1}{4\pi\rho_o}(\nabla\times(\vec{\nabla}\times(\delta\vec{v}\times B_o\hat{z})))\times B_o\hat{z}$$

Introducing the Alven speed $\vec{v}_A \equiv \frac{\vec{B}_o}{\sqrt{4\pi\rho_o}}$ We get

$$\frac{\partial^2\vec{v}}{\partial t^2} = c_s^2\nabla(\nabla\cdot\delta\vec{v}) + (\nabla\times(\vec{\nabla}\times(\delta\vec{v}\times\vec{v}_A)))\times\vec{v}_A$$

This is the wave equation for MHD

Small perturbations

$$\frac{\partial^2 \delta v}{\partial t^2} = c_s^2 \nabla(\nabla \cdot \delta\vec{v}) + (\nabla \times (\vec{\nabla} \times (\delta\vec{v} \times \vec{v}_A))) \times \vec{v}_A$$

$$\vec{v}_A \equiv \frac{\vec{B}_o}{\sqrt{4\pi\rho_o}}$$

In absence of B field, it reduces to the acoustic wave equation:
Infact taking 1-dim case, ie δ**v** directed along the x-axis (δv,0,0), we get

$$\nabla \cdot \delta\vec{v} = \frac{\partial \delta v}{\partial x} \qquad \nabla = \frac{\partial}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \delta v}{\partial t^2} = c_s^2 \frac{\partial^2 \delta v}{\partial x^2}$$

When B field is present, the situation is much more complicated:
i) There are two speeds involved in the wave propagation, c_s and v_A
ii) There are waves types which are not acoustic, since a speed completely dependent on the unperturbed B field appears

Small perturbations

Therefore let look for solutions of type

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi} \qquad \delta\vec{v} = \vec{V}e^{i\Phi} \qquad \frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \qquad \Phi = \vec{k}\cdot\vec{r} - \omega t$$

With k and ω completely arbitrary at this moment

Inserting these solutions into the equations we get

$$\omega r = \vec{k}\cdot\vec{V}$$

$$\omega\vec{V} = c_s^2 r\vec{k} - \frac{(\vec{k}\times\vec{b})\times\hat{z}B_o^2}{4\pi\rho_o}$$

$$\omega\vec{b} = -k_z\vec{V} + \hat{z}\vec{k}\cdot\vec{V}$$

In this way the coupled equation system becomes a simple omogeneous linear system in the unknowns r,***V*** and ***b***

The system admits only the trivial solution r=***V***=***b*** =0, unless the determinant of the system is zero ➔ the condition for non zero solutions to exist is that det=0

Small perturbations

It is convenient to decompose the wave vector ***k*** in its components || and perp to ***B***

After a little bit of algebra, we find the determinant of the system

$$\det\begin{pmatrix} \omega^2 - v_A^2 k_{\parallel}^2 & 0 & 0 \\ 0 & \omega^2 - v_A^2 k_{\parallel}^2 - (c_s^2 + v_A^2) k_{\perp}^2 & -c_s^2 k_{\parallel} k_{\perp} \\ 0 & -c_s^2 k_{\parallel} k_{\perp} & \omega^2 - c_s^2 k_{\parallel}^2 \end{pmatrix} = 0$$

$$(\omega^2 - v_A^2 k_{\parallel}^2)(\omega^4 - \omega^2 (c_s^2 + v_A^2) k^2 + c_s^2 v_A^2 k_{\parallel}^2 k^2) = 0$$

This establishes a relation between the wave frequency ω and the wave vector k, which is called dispersion relation

$v_f=\omega/k$ is not the real wave speed (but only the phase speed), which is the group speed $v_g=\partial\omega/\partial k$

There is a dispersion (ie v depends on ω) every time phase and group velocity differ

Small perturbations

Occasionally it may happens that, for a given k, ω has an imaginary part (ie is complex)

Remembering that all the physical quantities depend on time as $e^{i\omega t}$, this means that the phenomenon either is strongly dumped or is strongly amplified

When even only one solution is amplified, one speaks of instabilities of the 0th order solutions: perturbations, even if initially small, tend to increase without bound with time

This is not the case for previous relation dispersion

Small perturbations

$$(\omega^2 - v_A^2 k_{\parallel}^2)(\omega^4 - \omega^2 (c_s^2 + v_A^2) k^2 + c_s^2 v_A^2 k_{\parallel}^2 k^2) = 0$$

This equation has three distinct solutions in ω^2 (ie six independent solutions in ω) but the pure k^2 dependence in the dispersion relation tells us that couniugated solutions +- ω represent the same wave propagating in the +B and −-B directions

The first is $\omega^2 = v_A^2 k_{||}$ and it is called Alfven's wave

Its peculiarity is that it is transmitted exclusively along the B field direction with the characteristic speed v_A, as shown by the presence of the || k component only in the spatial part

The 2nd and 3rd are $$\omega_{ms}^2 = \frac{k^2}{2}\left\{c_{ms}^2 \pm \left[\left(v_A^2 - c_s^2\right)^2 + 4 v_A^2 c_s^2 \frac{k_\perp^2}{k^2}\right]^{1/2}\right\}$$ $c^2_{ms} = c^2_s + c^2_A$

Called magneto-sonic waves, fast or slow, depending on the sign in front of square root

The fact that the dispersion relation is even in k, that depends exclusively on k^2 tells us that the two propagation directions are completely equivalent

Alfvén waves

Inspection of the determinant shows that the fluctuation in the *z*-direction decouples from the other two components and has the linear dispersion

$$\boxed{\omega_A = \pm k_{\|} v_A}$$

This ***transverse wave*** travels parallel to the field. It is called ***shear Alfvén wave***. It has no density fluctuation and a constant group velocity, $\mathbf{v}_{gr,A} = \mathbf{v}_A$, which is always oriented along the background field, along which the wave energy is transported.

*The transverse velocity and **magnetic field components are (anti)-correlated** according to:* $\delta v_z/v_A = \pm\, \delta B_z/B_0$, *for **parallel (anti-parallel) wave propagation**. The wave electric field points in the x-direction:* $\delta E_x = \delta B_z/v_A$

Magnetosonic waves

The remaing four matrix elements couple the fluctuation components, $\delta v_{||}$ and $\delta v_{\perp}$. The corresponding determinant reads:

$$\boxed{\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k^2 k_{\|}^2 = 0}$$

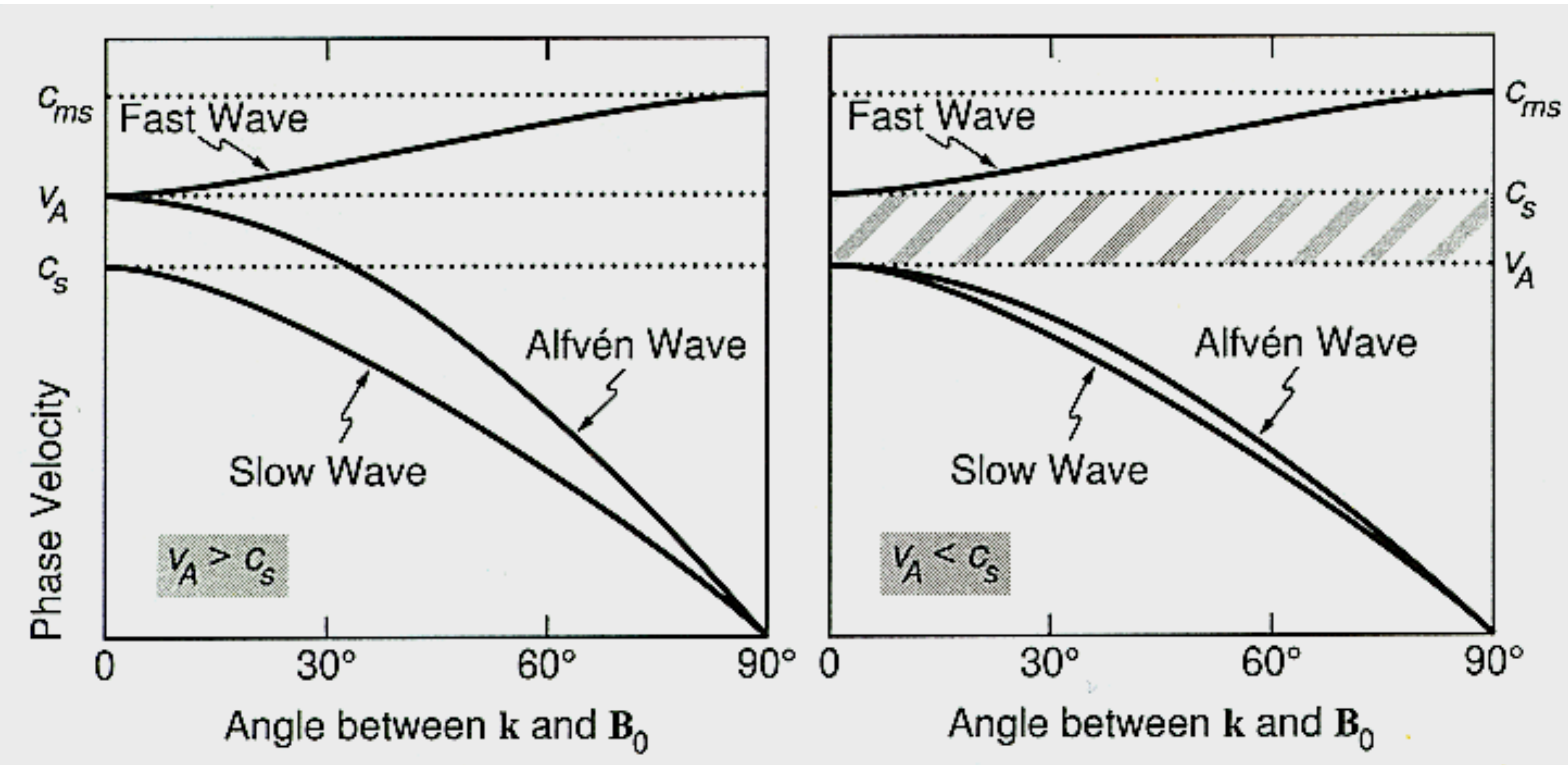
This bi-quadratic equation has the roots:

$$\boxed{\omega_{ms}^2 = \frac{k^2}{2}\left\{c_{ms}^2 \pm \left[\left(v_A^2 - c_s^2\right)^2 + 4 v_A^2 c_s^2 \frac{k_\perp^2}{k^2}\right]^{1/2}\right\}}$$

which are the phase velocities of the compressive ***fast and slow magnetosonic waves***. They depend on the propagation angle θ, with $k_\perp^2/k^2 = sin^2\theta$. For $\theta = 90^0$ we have: $\omega = kc_{ms}$, and $\theta = 0^0$:

$$\boxed{\omega^2 = \frac{1}{2}k^2\left[c_s^2 + v_A^2 \pm \left(c_s^2 - v_A^2\right)\right]}$$

Dependence of phase velocity on propagation angle



Small perturbations

$\omega^2 = v_A^2 k_\parallel$ Alfven's wave

$\omega^2 = \frac{k^2}{2}(c_s^2 + v_A^2 \pm \sqrt{\frac{(c_s^2 - v_A^2)^2 + 4c_s^2 v_A^2 k_\perp^2}{k^2}})$ magneto-sonic waves, fast or slow

Since there are three independent solution there must be three distinct modes of oscillation

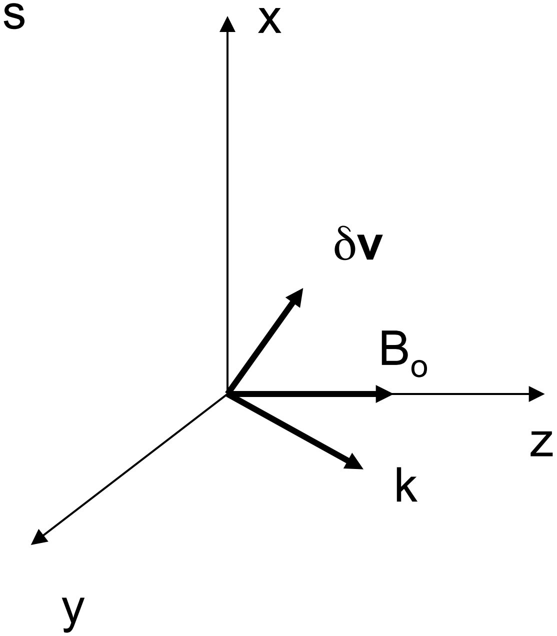
In principle, these modes could be identified finding the solutions, linearly independent and orthogonal of the system, but there is a simpler, more physical way to follow

Small perturbations

Let choose a suitable reference frame: z axis oriented along the B field, k in the yz plane (this simplifies the representation with no loss of generality

The speed perturbation has three components

With the reference choice k_n lies in the xy plane and $k_{||}$ is along z-axis



Small perturbations

Let consider the perturbed momentum equation $\frac{\partial \delta v}{\partial t} = -c_s^2 \nabla(\frac{\delta\rho}{\rho_o}) + \frac{1}{4\pi\rho_o}(\nabla \times \delta\vec{B}) \times B_o\hat{z}$

Get the z component, along the unperturbed field direction, and put the solutions in

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi} \quad \delta\vec{v} = \vec{V}e^{i\Phi} \quad \frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \quad \Phi = \vec{k}\cdot\vec{r} - \omega t$$

Then $\partial(\delta \boldsymbol{v})/\partial t = i\omega\delta\boldsymbol{v}$

We find that $\omega\delta v_z = k_z\frac{\delta p}{\rho_o}$

since the double vector product is by construction perp to z and $\delta p = c^2_s\delta\rho$

So, every perturbation of the speed <u>along the field direction</u> is exclusively due to pressure gradient and is not connected to any perturbation of the B field and can be considered as an acoustic wave

Let consider now the y comp of the equation $\omega\rho_0\delta v_y = k_\perp\delta p + \frac{B_o}{4\pi}(k_\perp\delta B_z - k_\parallel\delta B_y)$

The div equation divB=0 for the field becomes $k_\parallel\delta B_z = -k_\perp\delta B_y$

Small perturbations

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi}$$

$$\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla\left(\frac{\delta\rho}{\rho_o}\right) + \frac{1}{4\pi\rho_o}(\nabla \times \delta\vec{B}) \times B_o \hat{z}$$

$$\frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \qquad \delta\vec{v} = \vec{V}e^{i\Phi}$$

$$\Phi = \vec{k}\cdot\vec{r} - \omega t$$

Z comp

$$\omega \delta v_z = k_z \frac{\delta p}{\rho_o}$$

Y comp

$$\omega\rho_0\delta v_y = k_\perp \delta p + \frac{B_o}{4\pi}(k_\perp \delta B_z - k_\parallel \delta B_y)$$

$$k_\parallel \delta B_z = -k_\perp \delta B_y$$

Insert this condition in y comp eqn

$$\omega\rho_o k_\perp \delta v_y = k_\perp^2 \delta p + k^2 \frac{B_o \delta B_z}{4\pi} = k_\perp^2 \delta p + k^2 \frac{\delta B_z^2}{8\pi}$$

Use now the z comp eqn and finally we get

$$\vec{k}\cdot\delta\vec{v} = \frac{k^2}{\omega\rho_o}\delta(p + B^2/8\pi)$$

From this we see this wave has as restoring force the total pressure of the fluid, that is mechanical + magnetic pressure

For this reason they are called magneto-sonic waves

The reason why there are slow and fast modes depends on the fact that p and $B^2/8\pi$ may have signs equal or opposite: when with same sign, the restoring force is greater and the waves are faster, when with opposite signs the two restoring forces δp and δB^2 tend to cancel and the wave will propagate more slowly

Small perturbations

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi}$$

$$\frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \qquad \delta\vec{v} = \vec{V}e^{i\Phi}$$

$$\Phi = \vec{k}\cdot\vec{r} - \omega t$$

$$\frac{\partial\vec{v}}{\partial t} = -c_s^2\nabla(\frac{\delta\rho}{\rho_o}) + \frac{1}{4\pi\rho_o}(\nabla\times\delta\vec{B})\times B_o\hat{z}$$

To examine the 3rd wave type, consider the 3rd component of the motion equation

This is directed along x direction, perp B_o and k

$$\omega\rho_o\delta v_x = -\delta B_x\frac{k_\parallel B_o}{4\pi}$$

From this we see that the pressure is not involved at all in the Alfven wave, neither the mechanical or the magnetic pressure: the restoring force is entirely due to the magnetic tension and the wave is transverse

(i) Sound Waves (B_0 = 0)

Uniform medium of pressure p_0, density ρ_0

Disturbance v=$\mathbf{v_1}$, $p = p_0 + p_1, \rho = \rho_0 + \rho_1$

Linearise eqns motion, continuity, energy

$$(p/\rho^\gamma = c)$$

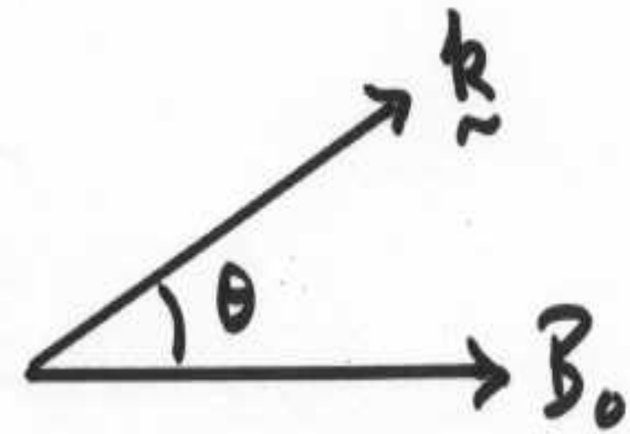
Fourier analys $\mathbf{v_1}, p_1, \rho_1 \approx \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$

$\rightarrow$ $\omega^2 = k^2 c_s{}^2$ **Dispersion Relation**

$\rightarrow$ **Waves propagate with speed** $\boxed{\omega / k = c_s}$

(ii) Magnetic Waves ($p_0 = 0$)

Repeat, but uniform (B_0)
- include **j** x **B** force
- assume wave propagates at an angle to B_0



Either $\boxed{\omega^2 = k^2 v_A{}^2 \cos^2\theta}$ *

Alfvén Waves *

Incompressible - due to magnetic tension

Or $\boxed{\omega^2 = k^2 v_A{}^2}$ **Compressional Alfvén Waves**

Compressible - due to magnetic pressure

- propagate at speed v_A

(iii) MHD Waves (p_0 and B_0 nonzero)

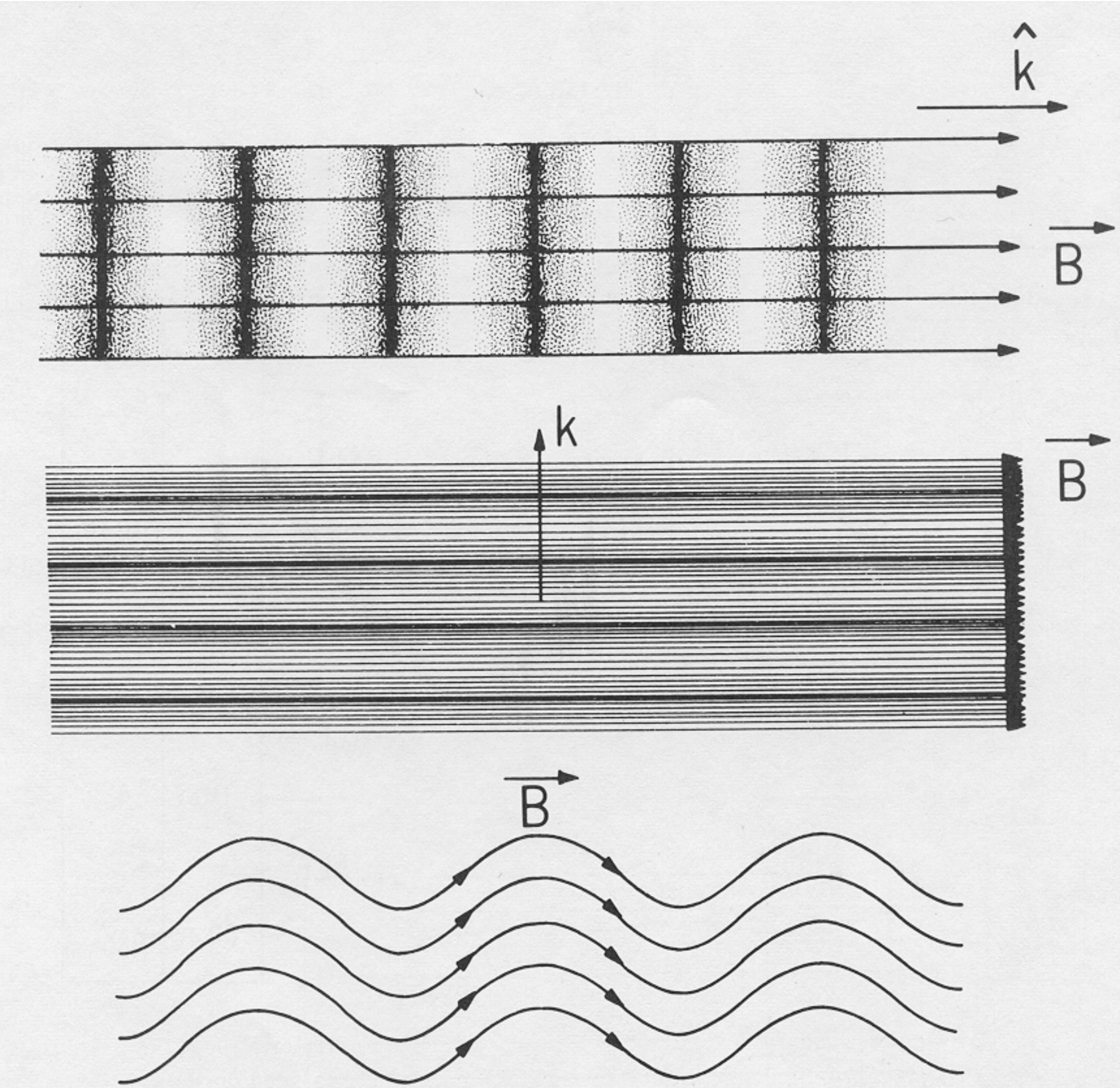
- Alfvén Wave is unaffected

- Compressional Alfvén Wave and Sound Wave are coupled:

  Slow Magnetoacoustic Wave (Slow-Mode)
  + **Fast Magnetoacoustic Wave** (Fast-Mode)

  Propagate slower/faster than Alfvén Wave

Magnetohydrodynamic waves



• Magnetosonic waves

compressible

- parallel slow and fast

- perpendicular fast

$c_{ms} = (c_s{}^2 + \mathrm{v_A}{}^2)^{1/2}$

• Alfvén wave

incompressible

parallel and oblique

$\mathrm{v_A} = B/(4\pi\rho)^{1/2}$