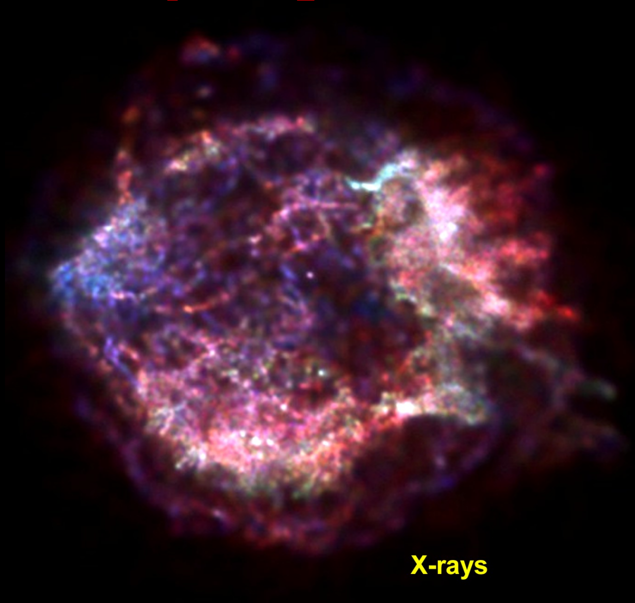
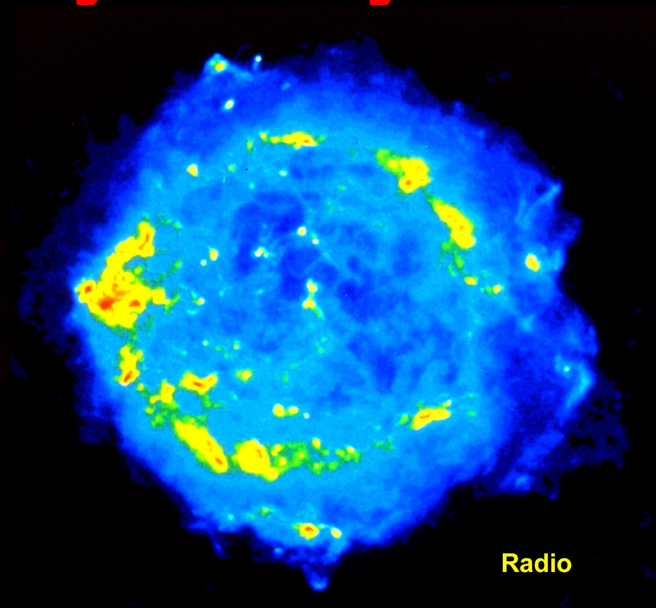
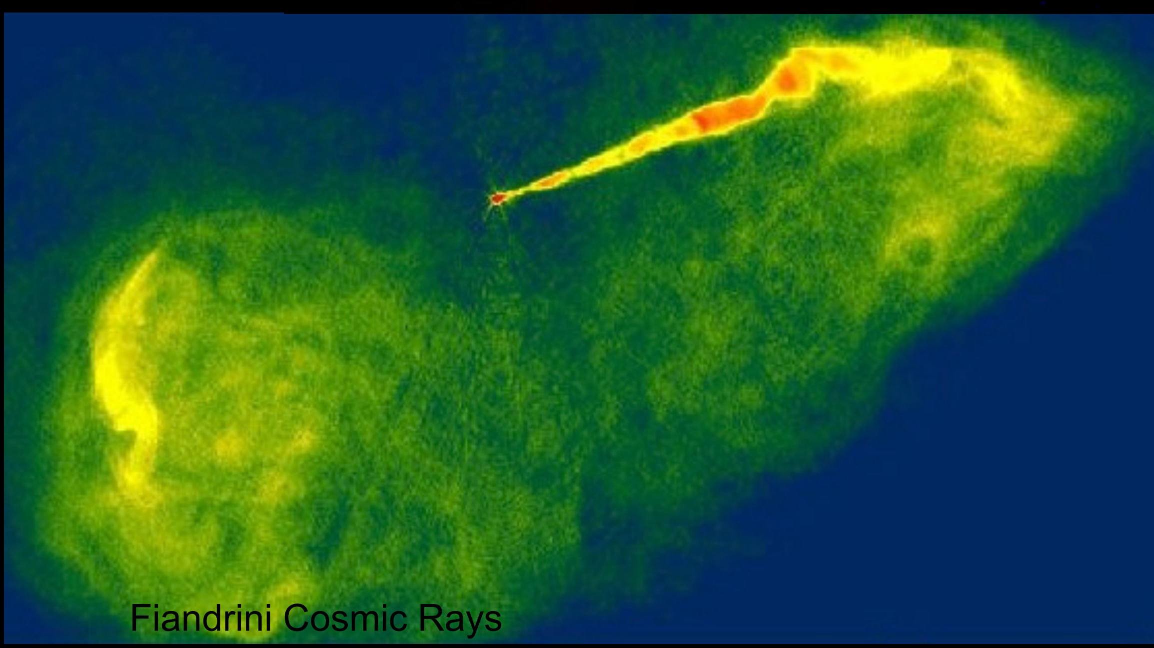
Lecture 12 141119

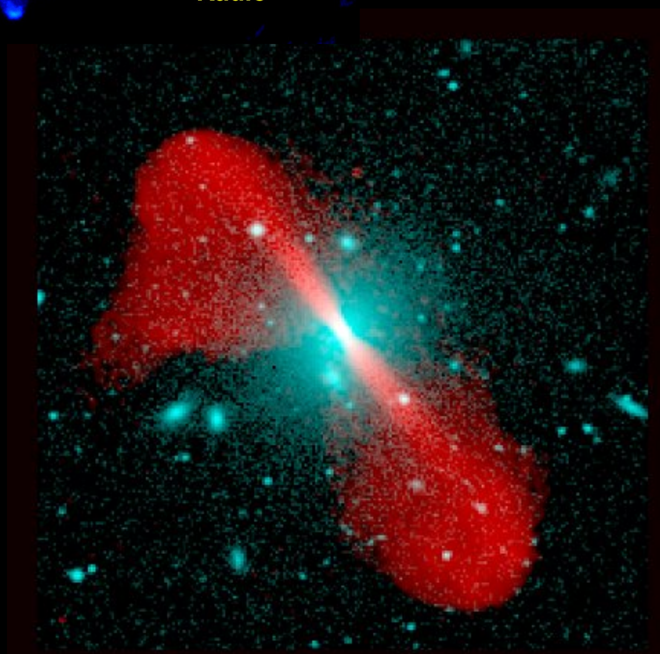
- Il pdf delle lezioni puo’ essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1920/

Astrophysical Hydrodynamics









Idrodinamica (non relativistica)

La maggior parte dei fenomeni astrofisici concerne il rilascio di energia all'interno di stelle, o al loro esterno, nel mezzo interstellare.

In entrambi i casi, il mezzo in cui avviene il rilascio di energia comincia a muoversi, a espandersi o contrarsi, a riscaldarsi o raffreddarsi

Le proprieta' della radiazione emessa (fotoni, particelle cariche), e rivelata a Terra, dipendono in dettaglio dalle condizioni termodinamiche e di moto del fluido in questione

Ne segue che un requisito necessario per l'astrofisica delle alte energie e' lo studio dell'idrodinamica e della magnetoidrodinamica

Idrodinamica (non relativistica)

In idrodinamica il fluido e' considerato un sistema termodinamico macroscopico.
Viene idealizzato come un mezzo continuo

La descrizione del fluido in quiete richiede la conoscenza delle sue proprieta' termodinamiche locali (p, ρ, T) → occorre quindi avere un'equazione di stato che lo caratterizza

Lo stato di moto di un fluido generico, non in quiete, sara' descritto dalla velocita' istantanea dell'elemento di fluido $\mathbf{v}(\mathbf{x},t)$

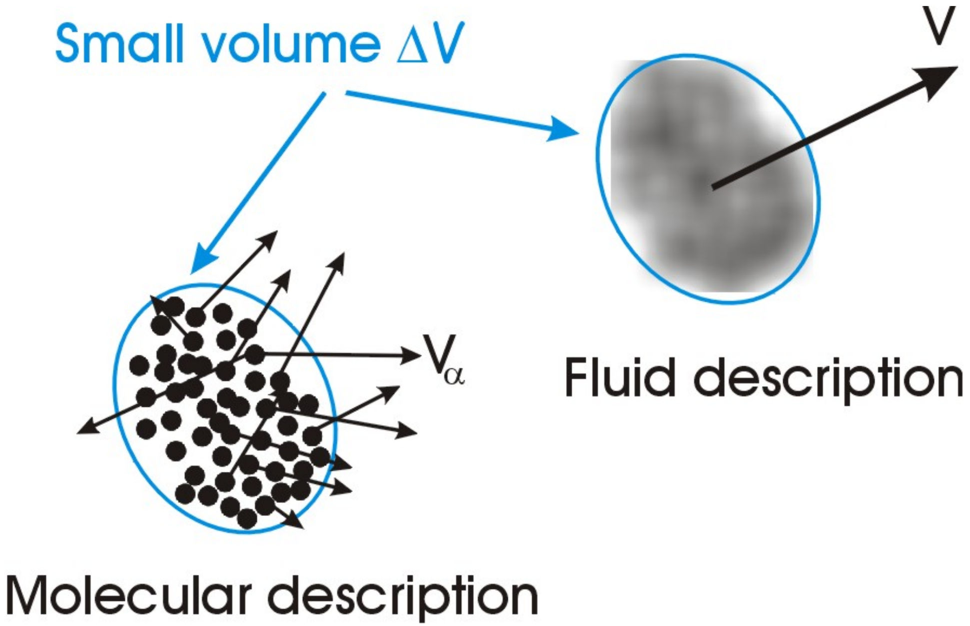
→ lo stato del fluido e' quindi determinato dalla conoscenza di:

$p(\mathbf{x},t)$, $\rho(\mathbf{x},t)$, $T(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$

Classical Mechanics vs. Fluid Mechanics

Single-particle (classical) Mechanics	Fluid Mechanics
Deals with single particles with a fixed mass	Deals with a continuum with a variable mass-density
Calculates a single particle trajectory	Calculates a collection of flow lines (flow field) in space
Uses a position *vector* and velocity *vector*	Uses a *fields* : Mass density, velocity field..
Deals only with externally applied forces (e.g. gravity, friction etc)	Deals with internal AND external forces
Is formally linear (superposition principle for solutions)	Is intrinsically non-linear No superposition principle

Mass, mass-density and velocity



$$\rho(\boldsymbol{x}, t) = \lim_{\Delta\mathcal{V}\to 0} \frac{\Delta m}{\Delta\mathcal{V}}$$

$$\Delta m = \sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha$$

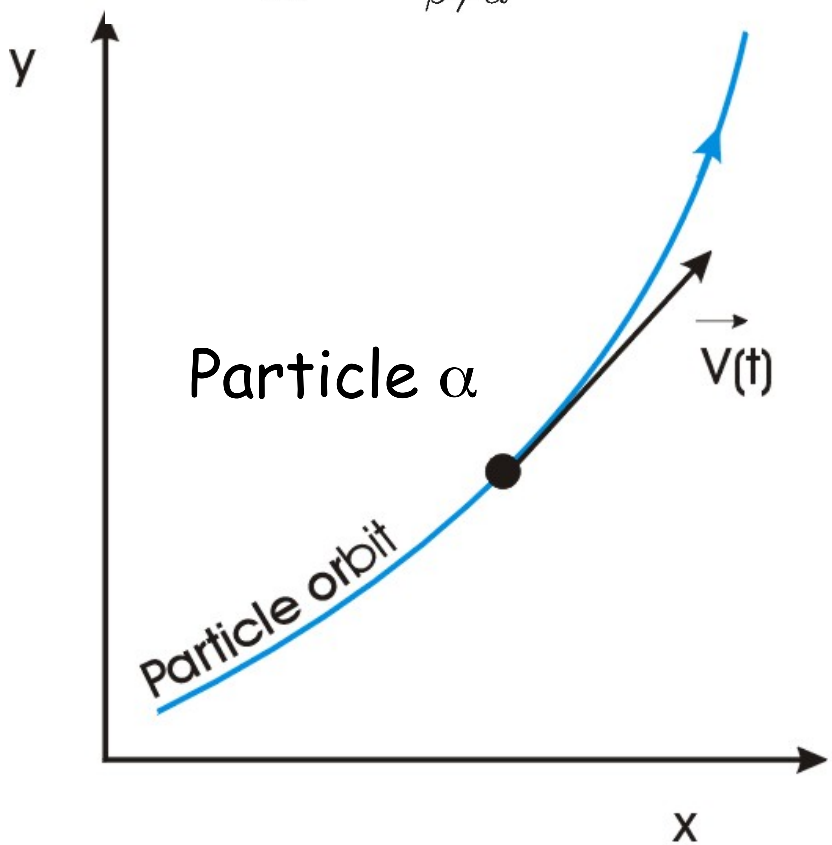
$$\boldsymbol{V} = \frac{\sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha \boldsymbol{V}_\alpha}{\Delta m}$$

Dal punto di vista dei costituenti la velocita' ***V*** di un elemento di fluido e' la velocita' media delle particelle contenute nel volume (che sono tante in modo che la media statistica abbia senso)

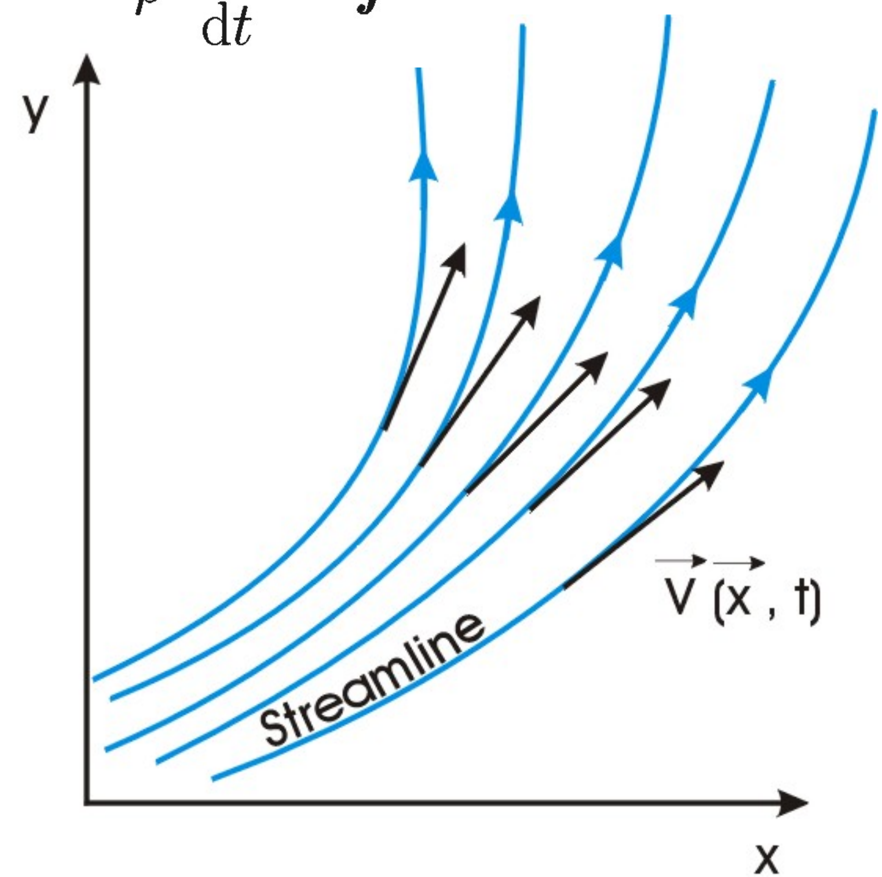
Equation of Motion: from Newton to Navier-Stokes/Euler

$$m_\alpha \frac{\mathrm{d}\boldsymbol{V}_\alpha}{\mathrm{d}t} = \sum_{\beta \neq \alpha} \boldsymbol{F}_{\alpha\beta}$$

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$



Single-particle dynamics



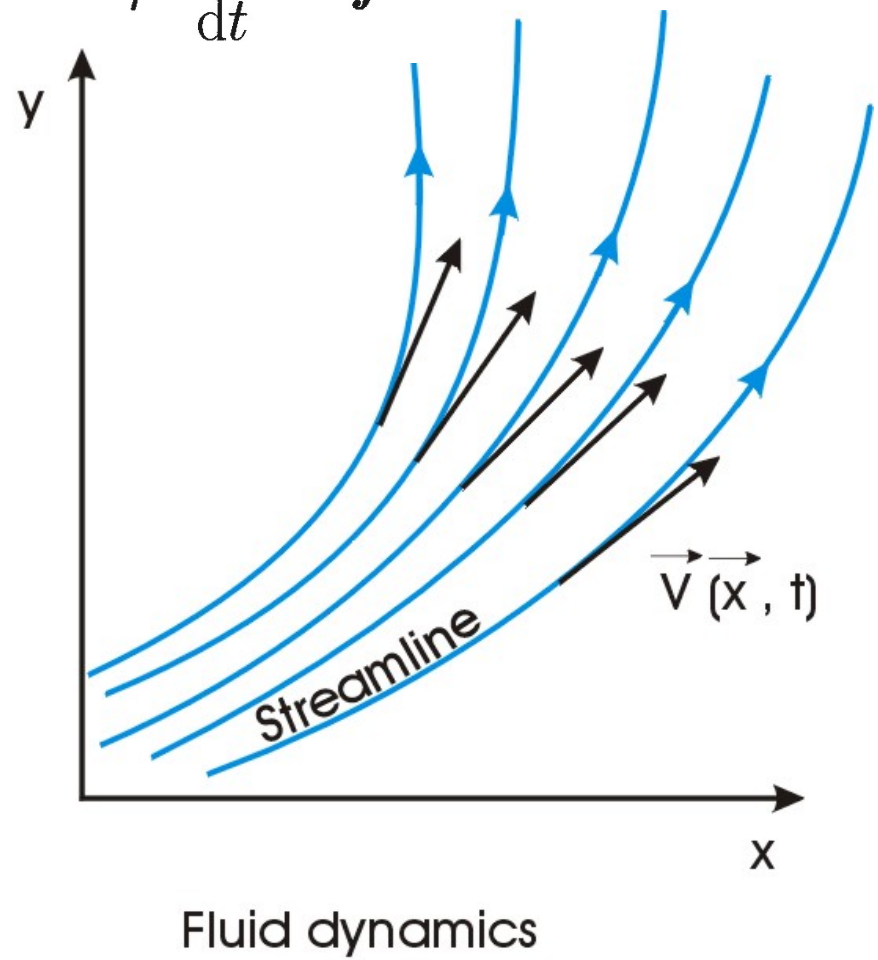
Fluid dynamics

Equation of Motion: from Newton to Navier-Stokes/Euler

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$

You have to work with a velocity <u>field</u> that depends on position *and* time!

$$\boldsymbol{V} = (V_{\mathrm{x}}\,,\, V_{\mathrm{y}}\,,\, V_{\mathrm{z}}) = \boldsymbol{V}(\boldsymbol{x}\,,\, t)$$



Derivatives, derivatives…

Eulerian change:
fixed position

$$\delta Q = Q(\boldsymbol{x}\ ,\ t + \Delta t) - Q(\boldsymbol{x}\ ,\ t) \approx \frac{\partial Q}{\partial t}\,\Delta t$$

Lagrangian change:
shifting position

$$\Delta Q = Q(\boldsymbol{x} + \Delta \boldsymbol{x}\ ,\ t + \Delta t) - Q(\boldsymbol{x}\ ,\ t) \approx \frac{\mathrm{d}Q}{\mathrm{d}t}\,\Delta t$$

Shift along
streamline:

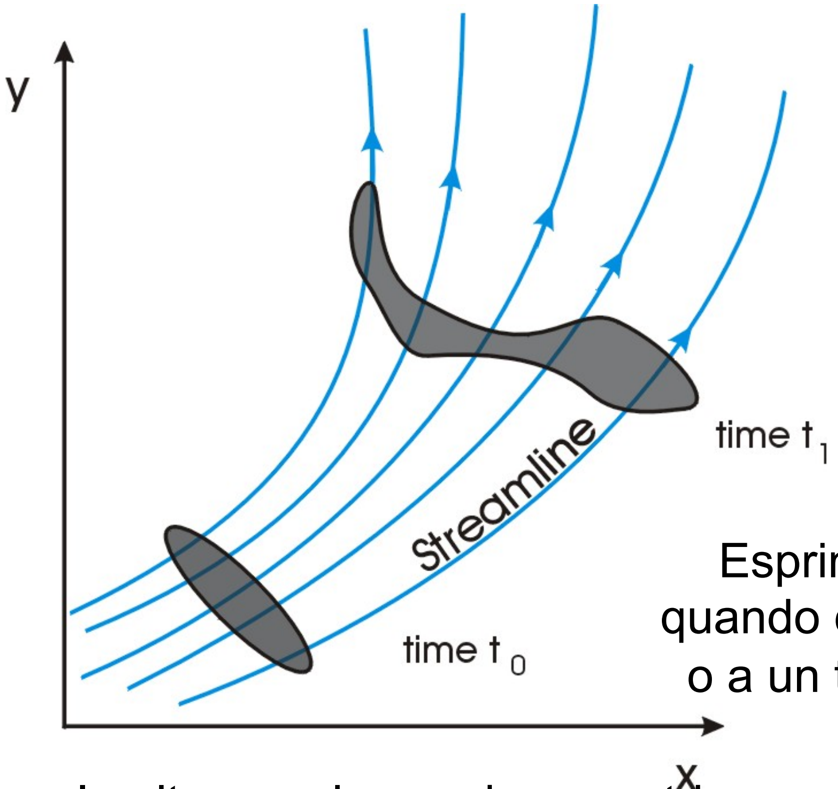
$$\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t$$

Comoving derivative d/dt

$$\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t$$

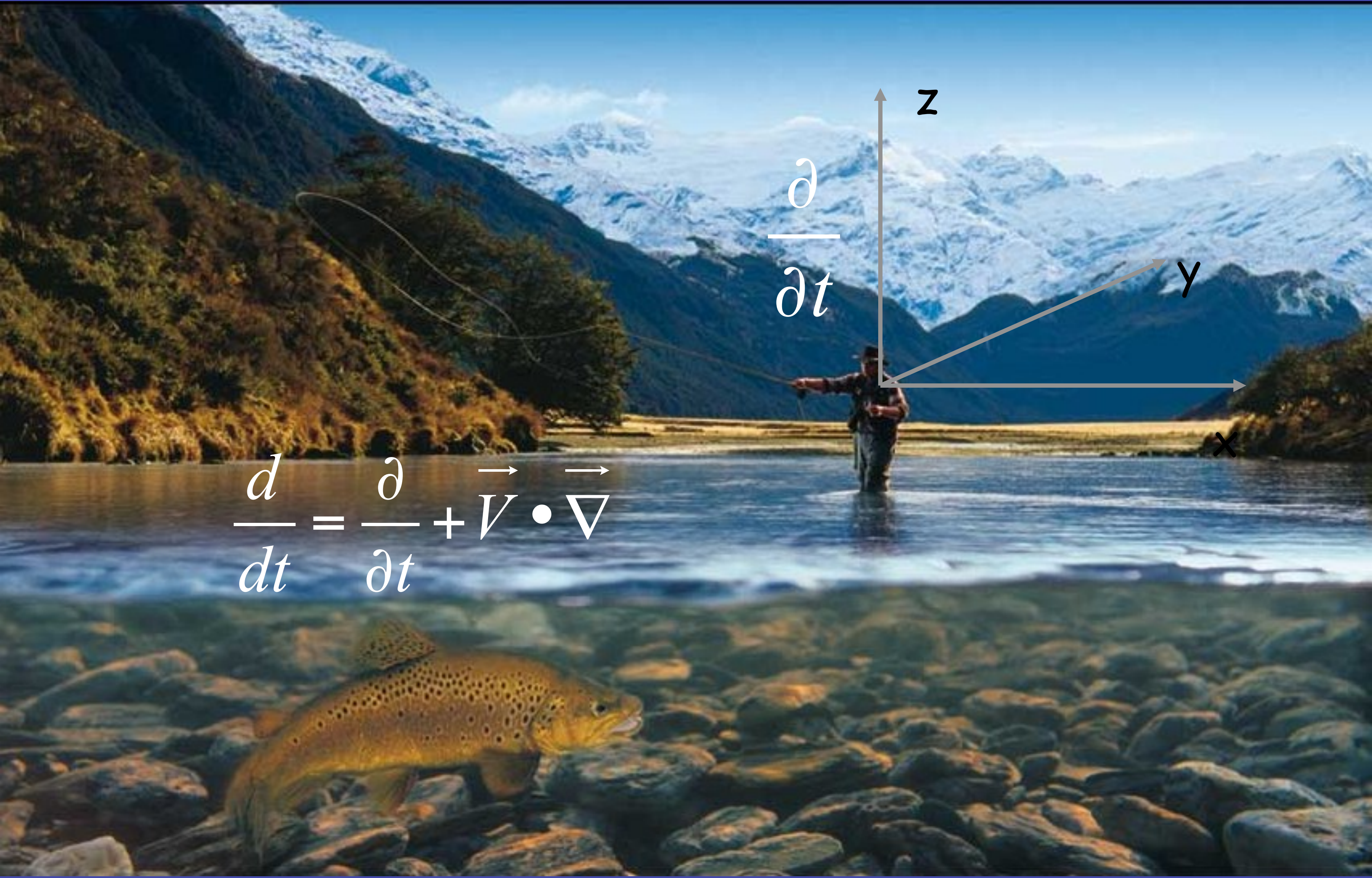
$$\begin{aligned} \Delta Q &= Q(t+\Delta t\,,\ \boldsymbol{x}+\Delta\boldsymbol{x}) - Q(t\,,\ \boldsymbol{x}) \\ &\approx \frac{\partial Q}{\partial t}\Delta t + (\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla})Q \\ &= \left[\frac{\partial Q}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})Q\right]\Delta t \\ &\equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)\ \Delta t\ . \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})$$



Esprime come varia una quantita' fisica nel tempo quando consideriamo la sua variazione non in un posto o a un tempo fissato ma a elemento di massa fissato

In altre parole, se ci concentriamo su un certo elemento di massa e lo seguiamo nel suo moto, d/dt esprime la variazione col tempo come la vedremmo se ci trovassimo a cavalcioni dell'elemento di massa in questione



$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \vec{\nabla}$$

Equazione del moto per un fluido

Variazione di p di un elemento = F totale applicata sull'elemento

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right] = \boldsymbol{f}$$

<u>termine non lineare!</u>

Rende molto piu' difficile trovare soluzioni "semplici"

E' il prezzo da pagare per lavorare con un <u>campo</u> di velocita'

$$\boldsymbol{V} = (V_{\mathrm{x}}\,,\, V_{\mathrm{y}}\,,\, V_{\mathrm{z}}) = \boldsymbol{V}(\boldsymbol{x}\,,\, t)$$

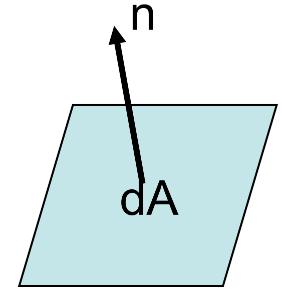
<u>Densita' di forza</u>

Puo' essere:
- interna:
  - pressione
  - viscosita' (frizione)
  - self-gravity
- esterna

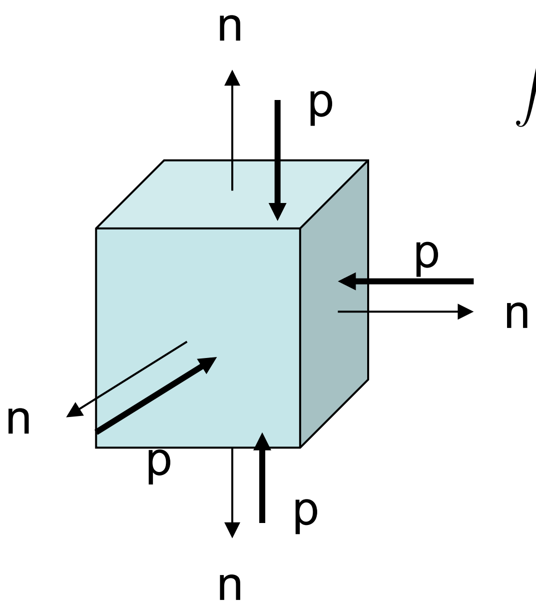
Equazione del moto per un fluido ideale

Un fluido ideale e’ uno in cui gli sforzi sono sempre normali alle superfici dell’elemento di volume del fluido

In un fluido su un elemento di superficie arbitrario viene esercitata una forza d**F**=pd**A**=pdA**n**



→ la forza a cui e' soggetto un elemento di fluido in un volume V e' $\vec{F} = -\int_S p d\vec{A}$



$$\int_S p d\vec{A} = \int_V \nabla p dV$$

→ la densita' di forza e' $\vec{f} = \nabla p$

$$\rho \frac{d\vec{V}}{dt} = -\nabla p \qquad \rho(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla})\vec{v} = -\nabla p$$

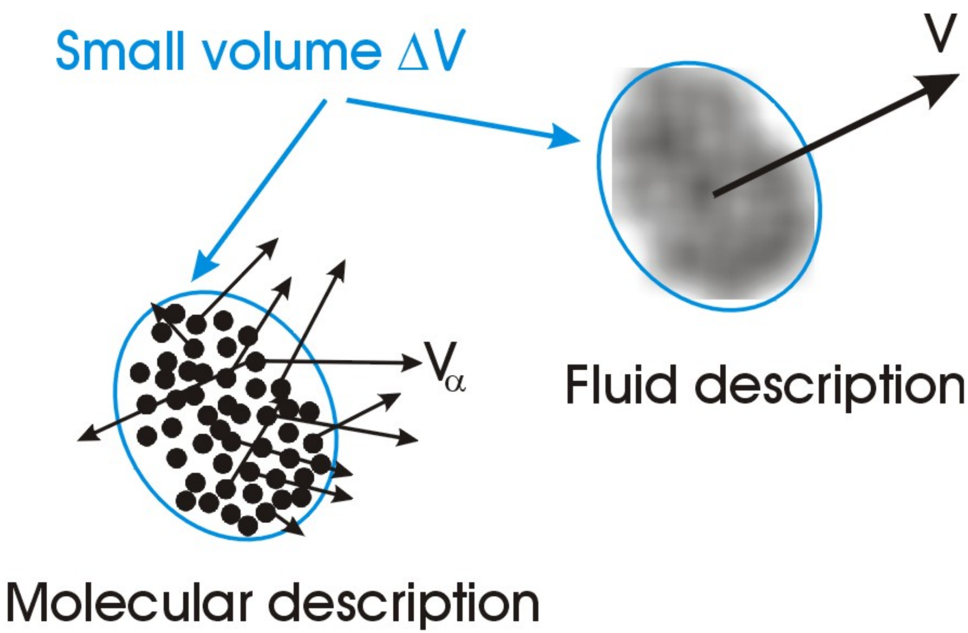
$$(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla})\vec{v} = -\frac{\nabla p}{\rho} + \frac{\vec{f}_{ext}}{\rho}$$

E' l'equazione del moto del fluido ideale (detta di Eulero)

P. Es **f** puo' essere forza di gravita' $\vec{f} = -\rho\nabla\Phi$

Connessione microscopica

La pressione deriva dal fatto che i costituenti (ie particelle) sono in agitazione termica per cui le singole particelle possiedono una distribuzione di velocita' (p. Es. Maxwelliana all'equilibrio) intorno al valore medio V della velocita' dell'elemento di fluido



Questo significa che l'impulso esatto istantaneo di una particella NON coincide con quello medio del fluido

Questa differenza "genera", in ultima analisi, una forza: la pressione

Pressure force and thermal motions

Split velocity into the <u>average velocity</u> V(**x**, t),

and an

Isotropically distributed deviation from average, the <u>random velocity:</u> σ(**x**, t)

Individual particle:

$$\boldsymbol{v}_\alpha = \boldsymbol{V}(\boldsymbol{x}\ ,\ t) + \boldsymbol{\sigma}_\alpha(\boldsymbol{x}\ ,\ t)\ .$$

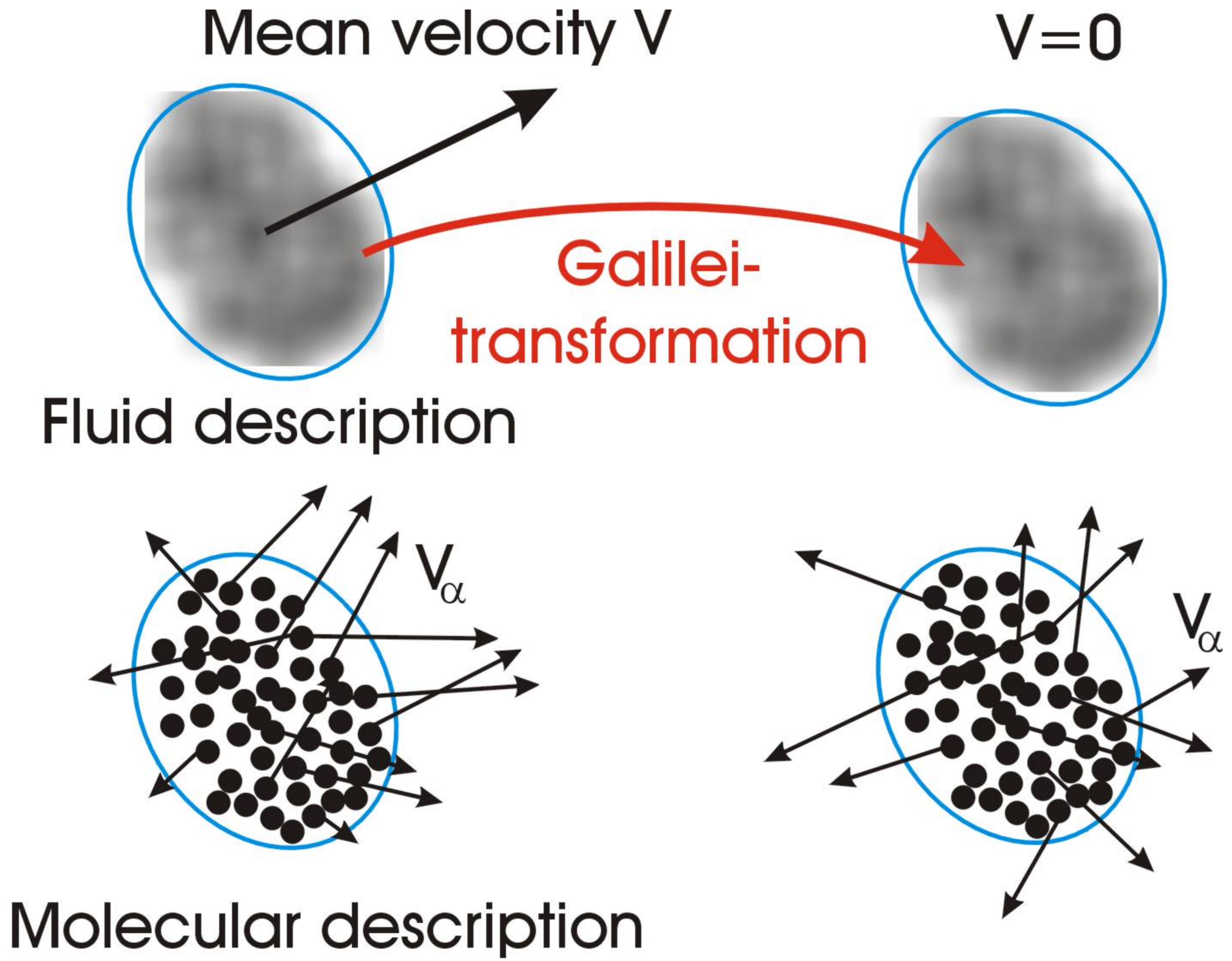
Average properties of random velocity $\boldsymbol{\sigma}$:

$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{v}} - \boldsymbol{V} = \boldsymbol{0}\ ;$$

$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = \tfrac{1}{3}\overline{\sigma^2}\ ,$$

and

$$\overline{\sigma_x\sigma_y} = \overline{\sigma_x\sigma_z} = \overline{\sigma_y\sigma_z} = \cdots = 0\ .$$



Nel sistema di quiete dell'elemento di fluido le molecole sono in moto a causa dell'agitazione termica

Acceleration of particle α

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{v}_\alpha}{\mathrm{d}t} &= \frac{\partial \boldsymbol{v}_\alpha}{\partial t} + (\boldsymbol{v}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{v}_\alpha \\
&= \frac{\partial(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)}{\partial t} + ((\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \cdot \boldsymbol{\nabla})(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \\
&= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\frac{\partial \boldsymbol{\sigma}_\alpha}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}_\alpha}_{\text{linear in } \boldsymbol{\sigma}}} + \boxed{\underbrace{(\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}_\alpha}_{\text{quadratic in } \boldsymbol{\sigma}}}
\end{aligned}$$

Effect of average over many particles in small volume:

$$\begin{aligned}
\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} &= \overline{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}} \\
&= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\right)\overline{\boldsymbol{\sigma}}}_{\text{vanishes: } \overline{\boldsymbol{\sigma}}=0!}} + \boxed{\underbrace{\overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}}}_{\text{remains: quadratic in } \boldsymbol{\sigma}}}
\end{aligned}$$

Average equation of motion

$$\rho \, \overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\boldsymbol{f}}$$

$$\rho \left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right) = \underbrace{\overline{\boldsymbol{f}}}_{\text{mean ext. force}} - \rho \, \overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \, \boldsymbol{\sigma}}$$

For isotropic fluid: $\rho \, \overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \boldsymbol{\sigma})} = \boldsymbol{\nabla} \left(\frac{\rho \overline{\sigma^2}}{3} \right) \equiv \boldsymbol{\nabla} P$

Some tensor algebra:

Vector

$$\boldsymbol{A} \equiv A_i\,\boldsymbol{e}_i = A_x\,\boldsymbol{e}_1 + A_y\boldsymbol{e}_2 + A_z\,\boldsymbol{e}_3 = \begin{pmatrix} A_x \\ \\ A_y \\ \\ A_z \end{pmatrix}$$

Three notations for the same animal!

the divergence of a vector in cartesian coordinates

Scalar

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Rank 2 Tensor

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Rank 2 Tensor and Tensor Divergence

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Vector

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = \left(\frac{\partial T_{ij}}{\partial x_i}\right) \boldsymbol{e}_j = \begin{pmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{pmatrix}$$

Special case:
Dyadic Tensor = Direct Product of two Vectors

$$\boldsymbol{A} \otimes \boldsymbol{B} \equiv A_i B_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{A} \otimes \boldsymbol{B} \right) = \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} \right) \, \boldsymbol{B} + \left(\boldsymbol{A} \cdot \boldsymbol{\nabla} \right) \boldsymbol{B}$$

Application: Pressure Force

Tensor divergence

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot\,(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\;\boldsymbol{\sigma}$$

Isotropy of Random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

Second term = scalar x vector!

This <u>must</u> vanish upon averaging!!

Application: Pressure Force

Tensor divergence

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot\,(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\;\boldsymbol{\sigma}$$

Isotropy of Random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

$$\overline{\sigma_i\sigma_j} = \tfrac{1}{3}\overline{\sigma^2}\,\delta_{ij} = \begin{cases} \tfrac{1}{3}\overline{\sigma^2} & \text{when } i=j \\ 0 & \text{when } i\neq j \end{cases}$$

$$\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} = \rho\begin{pmatrix} \tfrac{1}{3}\overline{\sigma^2} & 0 & 0 \\ 0 & \tfrac{1}{3}\overline{\sigma^2} & 0 \\ 0 & 0 & \tfrac{1}{3}\overline{\sigma^2} \end{pmatrix} = \frac{\rho\overline{\sigma^2}}{3}\,\boldsymbol{I}$$

Diagonal Pressure Tensor

Pressure force, continued

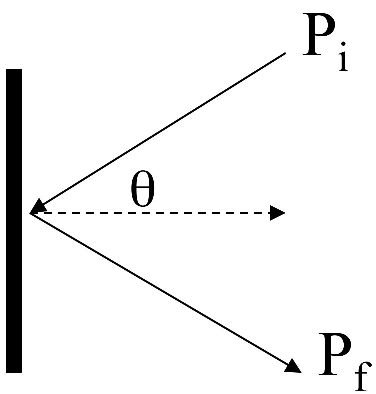
$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}) = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P$$

Equation of motion for frictionless ('ideal') fluid:

$$\boxed{\begin{gathered}\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = -\boldsymbol{\nabla}P + \text{other (external) forces} \\[2em] P(\boldsymbol{x}\;,\;t) \equiv \tfrac{1}{3}\rho\,\overline{\sigma^2}\end{gathered}}$$

Pressure, statistical derivation

We have to find the momentum transfer to a wall from a gas of particles with number density n in the hypothesis of elastic collisions $|p_i|=|p_f|$



A single particle with momentum p arriving from direction θ with respect to the normal n to the wall gets a $\Delta p = 2p\cos\theta$

In the time dt, the # of particles coming from θ direction impinging the wall are $dN=nv\cos\theta\, Sdt$, ie all the particles in the volume $(v\cos\theta)dtS$

Then the net momentum transfer to the wall is dP = (single part Δp)x(# of part imping. the wall) = $2p\cos\theta \times nv\cos\theta\, Sdt = 2nvp\cos^2\theta\, Sdt$

Pressure

The force is $F=dP/dt=2nvp\cos^2\theta\, S$ The pressure is $p=dF/dS$

So $p(\theta)=2nvp\cos^2\theta$ is the pressure due to to particles arriving from direction θ

If the distribution of directions is isotropic, then the probability for a particle to arrive in a solid angle $d\Omega$ along θ is $d\Omega/4\pi$, so the total pressure is

$$P_{tot}=\int p(\theta)\frac{d\Omega}{4\pi} \quad =\int_0^1 p(\theta)\frac{2\pi d cos\theta}{4\pi} \quad = nvp\int_0^1 \cos^2\theta d\cos\theta = nvp/3$$

non relativistic particle p=mv $\quad P_{tot}=nmv^2/3 \quad =\frac{2}{3}u \quad u=\frac{1}{2}nmv^2$

From equipartition theorem <E> = 3kT/2 Energy density

$P_{tot}=nkT$ Ideal gas law

Relativistic particle (as photons) E_{tot}' cp and v' c

$P_{tot}=ncp/3 \quad =u/3$ $\quad u=ncp$ Energy density

Summary:

- We know how to treat the time-derivative
- We know what the equation of motion looks like
- We know where the pressure term comes from
- We still need:
  - A way to link the pressure to density and temperature
  - A way to calculate how the density of the fluid changes

Connection with thermodynamics: Ideal Gas Law

Isotropic gas of point particles in Thermodynamic Equilibrium:

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_\mathrm{b}T$$

Temperature is defined in terms of kinetic energy of the thermal motions!

Connection with thermodynamics: Ideal Gas Law

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$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\rm b}T$$

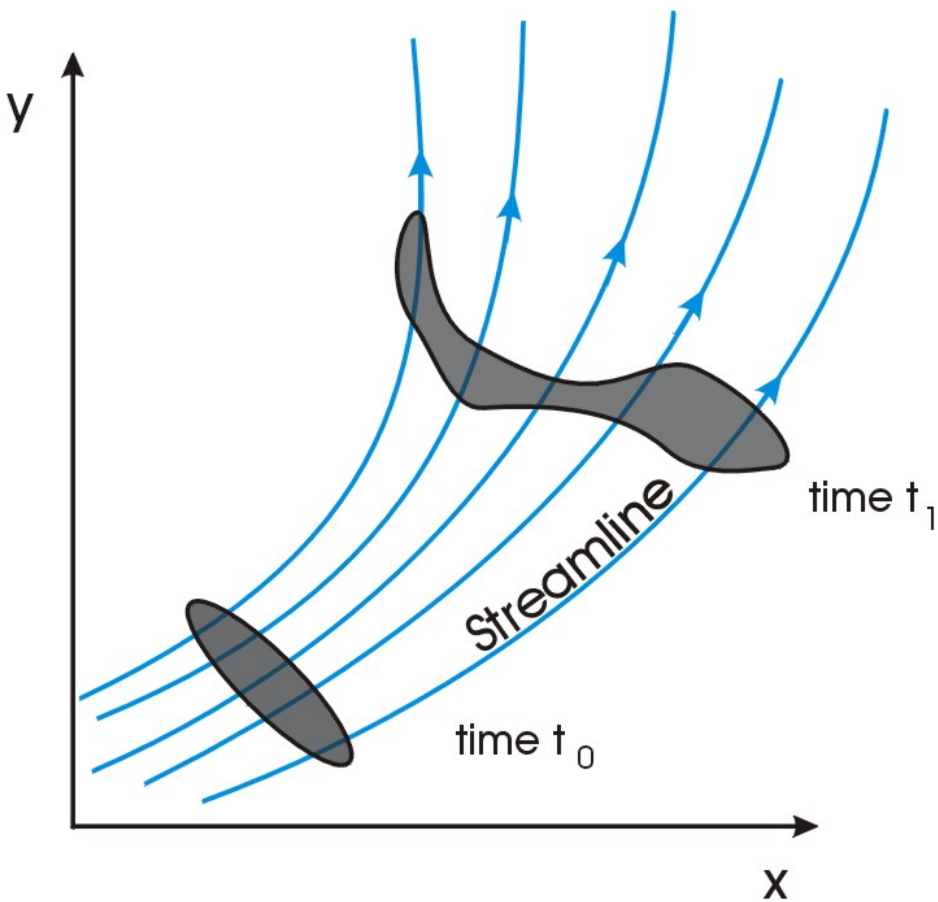
Ideal Gas Law: in terms of temperature T and number-density n: (ρ = nm = μnm$_H$, $R = k_b / m_H$)

$$p = \frac{1}{3}\rho\sigma^2$$

$$\downarrow$$

$$P(\rho\,,\,T) = nk_{\rm b}T = \frac{\rho\mathcal{R}T}{\mu}$$

Density Changes and Mass Conservation

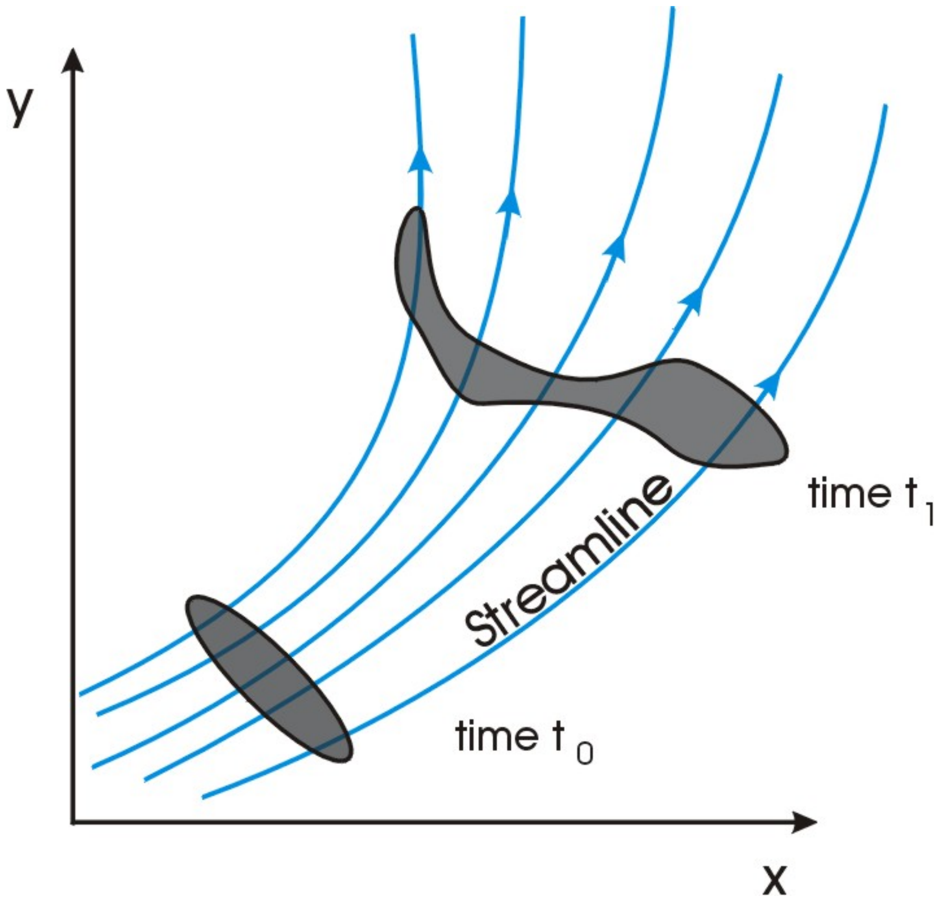


Two-dimensional example:

A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

Density Changes and Mass Conservation



Fiandrini Cosmic Rays

Two-dimensional example:

A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

<u>So:</u> the mass density must change in response to the flow!

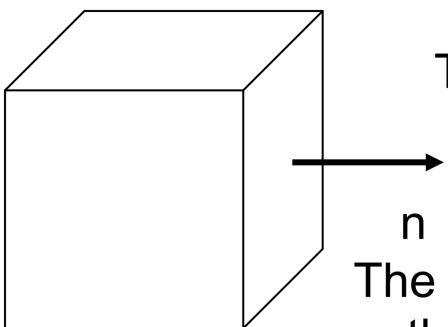
Mass conservation law

The mass cannot be created nor destroyed (in non-relativistic classical dynamics)

Therefore in a volume V the mass can change only because some of it leaves or enters the volume

The amount of mass through d**A** per time unit is $dF = \rho\vec{v}\cdot d\vec{A}$

Convention: dF>0 if outgoing



The flux $(\frac{dM}{dt})_{out} = \int_S \rho\vec{v}\cdot d\vec{A}$

The outgoing flux must be balanced by the change of mass in the volume $\frac{\partial M_V}{\partial t} = -(\frac{dM}{dt})_{out}$

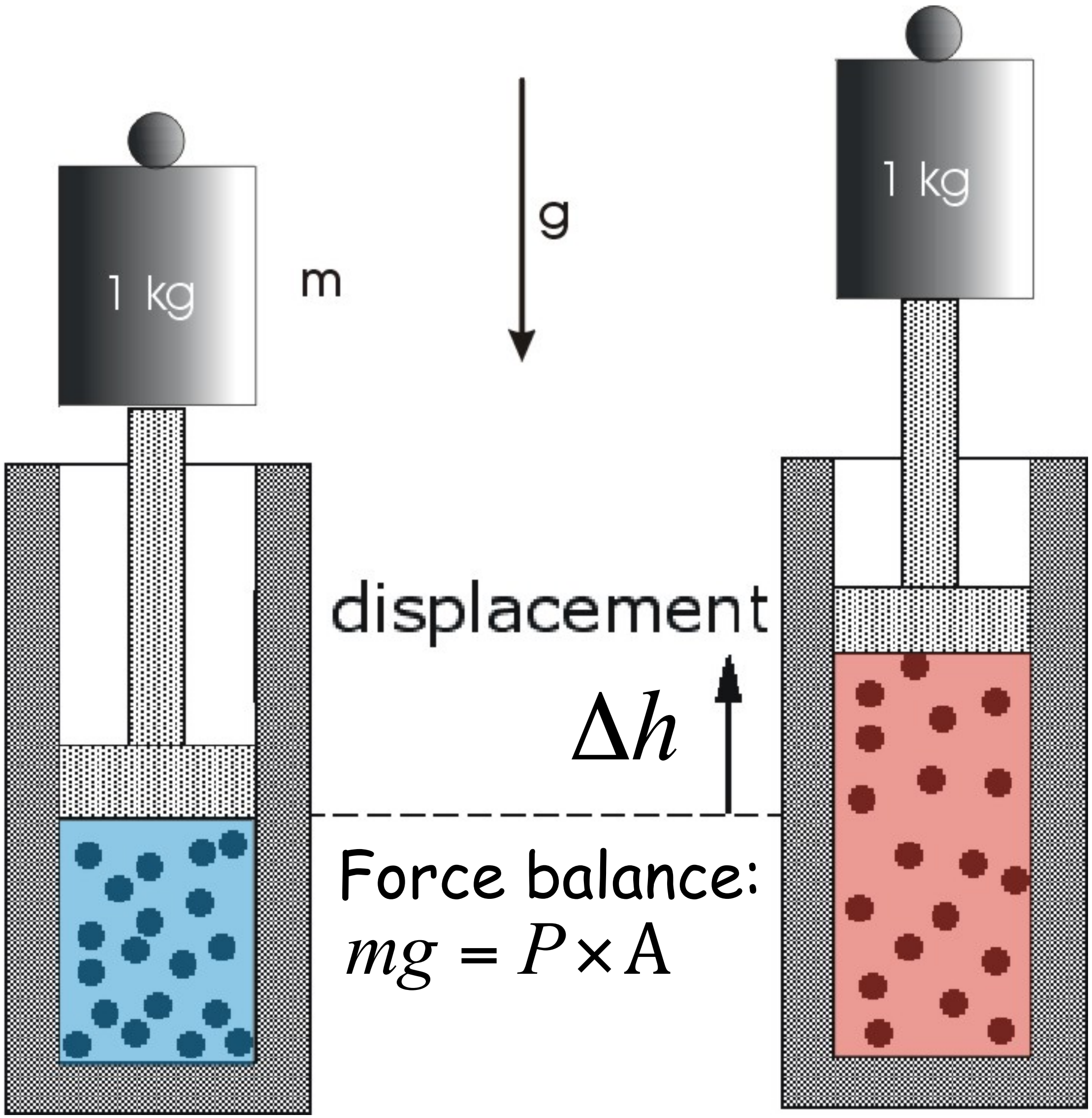
$$\frac{\partial M_V}{\partial t} = \frac{\partial}{\partial t}(\int_V \rho dV)$$

$$\frac{\partial}{\partial t}(\int_V \rho dV) = -\int_S \rho\vec{v}\cdot d\vec{A} \quad = -\int_V \nabla\cdot(\rho\vec{v})dV$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{v})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

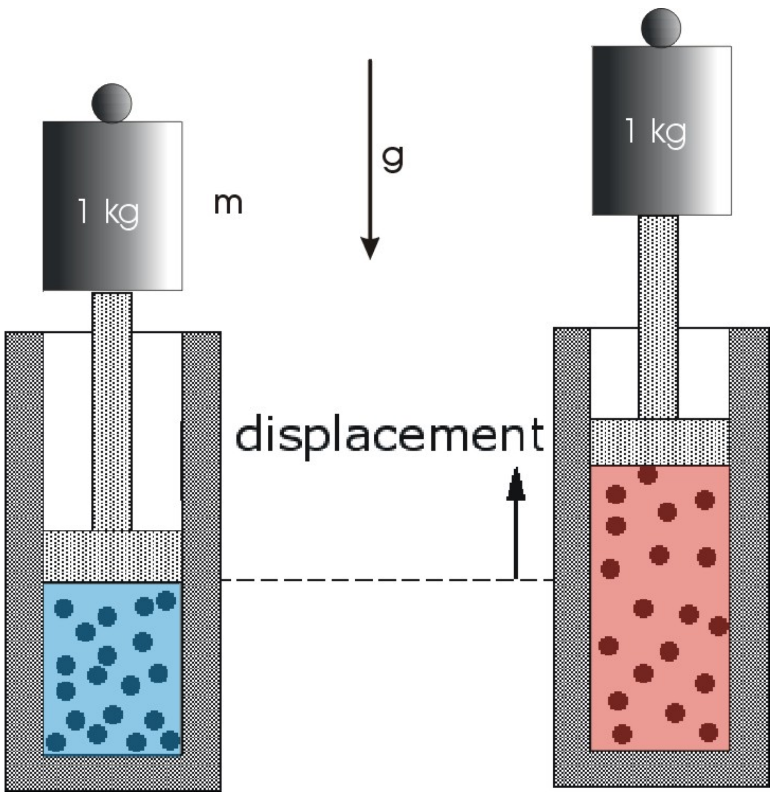
Thermodynamics



$$\Delta W = mg\Delta h$$

$$\Delta U_{gas} = \Delta Q - \Delta W$$
$$= \Delta Q - mg\Delta h$$
$$= \Delta Q - PA\ \Delta h$$
$$= \Delta Q - P\ \Delta V$$

First law of thermodynamics:



$$dU_{gas} = dQ - pdV$$

Change in internal energy
=
heat added by <u>external</u> sources
-
work done by gas

Entropy:

A measure of
'disorder'

Second Thermodynamic law:

$$dQ = TdS$$

$$dS \geq 0$$

$$\frac{dS}{dt} \geq 0$$

For an isolated system

The Adiabatic Gas Law: the behaviour of pressure

Thermodynamics:

$$\mathrm{d}Q \equiv T\,\mathrm{d}S = \mathrm{d}U + P\,\mathrm{d}\mathcal{V}$$

Special case: adiabatic change

$$\mathrm{d}Q = T\,\mathrm{d}S = 0$$

U = internal energy, *T* = temperature, *S* = entropy and V = volume

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Thermodynamics:

$$dQ \equiv T\,dS = dU + P\,d\mathcal{V}$$

Special case: adiabatic change

$$dQ = T\,dS = 0$$

Gas of point particles of mass *m* :

Internal energy: $$U = n\,\mathcal{V} \times \left[\tfrac{1}{2}m\left(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}\right)\right] = \tfrac{1}{2}\,\rho\,\mathcal{V}\overline{\sigma^2}$$

Pressure: $$P = \tfrac{1}{3}nm\,\left(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}\right) = \tfrac{1}{3}\,\rho\,\overline{\sigma^2}$$

Thermal equilibrium: $$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\mathrm{b}}T$$

$$\rightarrow\quad P = \frac{\rho\mathcal{R}T}{\mu} \quad , \quad U = \frac{3}{2}\,\frac{\rho\mathcal{R}T\,\mathcal{V}}{\mu}$$

Thermal equilibrium:

$$\tfrac{1}{2}m\overline{\sigma_x^2} = \tfrac{1}{2}m\overline{\sigma_y^2} = \tfrac{1}{2}m\overline{\sigma_z^2} = \tfrac{1}{6}\,m\overline{\sigma^2} = \tfrac{1}{2}\,k_{\mathrm{b}}T$$

$$P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \tfrac{3}{2}\,\frac{\rho \mathcal{R} T\,\mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P\,\mathrm{d}\mathcal{V} = 0 \longrightarrow \mathrm{d}\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\mathcal{V} = 0$$

Adiabatic Gas Law: a polytropic relation

Thermal equilibrium:
$$\tfrac{1}{2}m\overline{\sigma_x^2} = \tfrac{1}{2}m\overline{\sigma_y^2} = \tfrac{1}{2}m\overline{\sigma_z^2} = \tfrac{1}{6}\, m\overline{\sigma^2} = \tfrac{1}{2}\, k_{\rm b}T$$

$$P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \tfrac{3}{2}\,\frac{\rho \mathcal{R} T\, \mathcal{V}}{\mu}$$

U = (3/2)PV

Adiabatic change:

$$\mathrm{d}U + P\,\mathrm{d}\mathcal{V} = 0 \longrightarrow \mathrm{d}\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\mathcal{V} = 0$$

Chain rule for 'd'-operator:

$$\mathrm{d}(f\,g) = (\mathrm{d}f)\,g + f\,(\mathrm{d}g) \longrightarrow \tfrac{5}{3}\,P\,\mathrm{d}\mathcal{V} + \mathcal{V}\,\mathrm{d}P = 0\,.$$

(just like differentiation!)

$$\frac{\mathrm{d}P}{P} + \tfrac{5}{3}\,\frac{\mathrm{d}\mathcal{V}}{\mathcal{V}} = \mathrm{d}\log\left(P\,\mathcal{V}^{5/3}\right) = 0$$

Adiabatic Gas Law: a polytropic relation

Adiabatic pressure change:

$$P \times \mathcal{V}^{5/3} = \mathrm{constant}$$

For <u>small</u> volume:
mass conservation!

$$M = \rho\,\mathcal{V} = \mathrm{constant}$$

$$P\,\rho^{-5/3} = \mathrm{constant}$$

Specific Heat and Entropy

Specific Volume: contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics of unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific Heat and Entropy

Specific Volume contains <u>unit</u> mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics of a unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific energy *e* and pressure *P* :

$$e \equiv \tfrac{3}{2}\,\frac{\mathcal{R}T}{\mu} = \tfrac{3}{2}\,\frac{k_{\mathrm{b}}T}{m}\ , \quad P = \frac{\rho\,\mathcal{R}\,T}{\mu}$$

Specific heat coeff. at constant volume

$$\boxed{c_{\mathrm{v}} = \frac{\partial e}{\partial T} = \tfrac{3}{2}\,\frac{k_{\mathrm{b}}}{m}}$$

ρ is kept constant!
d(1/ρ) =0

Specific Heat and Entropy

Specific Volume contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics of a unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific energy e and pressure P :

$$e \equiv \tfrac{3}{2}\frac{\mathcal{R}T}{\mu} = \tfrac{3}{2}\frac{k_\mathrm{b}T}{m}\ , \quad P = \frac{\rho\,\mathcal{R}\,T}{\mu}$$

Specific heat coeff. at constant volume

$$\boxed{c_\mathrm{v} = \frac{\partial e}{\partial T} = \tfrac{3}{2}\frac{k_\mathrm{b}}{m}}$$

$$\boxed{c_\mathrm{p} - c_\mathrm{v} = \frac{k_\mathrm{b}}{m} = \frac{\mathcal{R}}{\mu}}$$

$$\mathrm{d}q = \mathrm{d}\left(e + \frac{P}{\rho}\right) - \frac{\mathrm{d}P}{\rho}$$

Specific heat coeff. at constant pressure: dP =0

$$\boxed{c_\mathrm{p} = \frac{\partial(e + P/\rho)}{\partial T} = \tfrac{5}{2}\frac{k_\mathrm{b}}{m}}$$

$$
\begin{aligned}
\mathrm{d}q &= c_{\mathrm{v}}\,\mathrm{d}T + \left(\frac{\rho \mathcal{R} T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right) \\
&= c_{\mathrm{v}}\,\mathrm{d}T - \left(\frac{\mathcal{R} T}{\rho\mu}\right)\,\mathrm{d}\rho \\
&= c_{\mathrm{v}} T\,\left[\frac{\mathrm{d}T}{T} - \boxed{\left(\frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} - 1\right)}\,\frac{\mathrm{d}\rho}{\rho}\right]
\end{aligned}
$$

Thermodynamic law for a unit mass,
rewritten in terms of specific heat coefficients

Definition specific entropy *s*

$$
\begin{aligned}
\mathrm{d}q &= c_\mathrm{v}\,\mathrm{d}T + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right) \\
&= c_\mathrm{v}\,\mathrm{d}T - \left(\frac{\mathcal{R}T}{\rho\mu}\right)\,\mathrm{d}\rho \\
&= c_\mathrm{v}T\,\left[\frac{\mathrm{d}T}{T} - \boxed{\left(\frac{c_\mathrm{p}}{c_\mathrm{v}} - 1\right)}\,\frac{\mathrm{d}\rho}{\rho}\right]
\end{aligned}
$$

$$T\,\mathrm{d}s = c_\mathrm{v}T\,\left[\frac{\mathrm{d}T}{T} - \boxed{(\gamma - 1)}\,\frac{\mathrm{d}\rho}{\rho}\right]$$

$$s = c_\mathrm{v}\,\log\left(\frac{T}{\rho^{\gamma-1}}\right) + \mathrm{constant}$$

$$s = c_\mathrm{v}\,\log\left(P\,\rho^{-\gamma}\right) + \mathrm{constant}$$

γ is the <u>specific heat ratio</u>
= 5/3 for ideal gas of point particles

$$\mathrm{d}q = c_{\mathrm{v}}\,\mathrm{d}T + \left(\frac{\rho \mathcal{R} T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

$$= c_{\mathrm{v}}\,\mathrm{d}T - \left(\frac{\mathcal{R} T}{\rho\mu}\right)\,\mathrm{d}\rho$$

$$= c_{\mathrm{v}} T \left[\frac{\mathrm{d}T}{T} - \boxed{\left(\frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} - 1\right)} \frac{\mathrm{d}\rho}{\rho}\right]$$

Definition specific entropy s

$$T\,\mathrm{d}s = c_{\mathrm{v}} T \left[\frac{\mathrm{d}T}{T} - \boxed{(\gamma - 1)} \frac{\mathrm{d}\rho}{\rho}\right]$$

$$s = c_{\mathrm{v}}\ \log\left(\frac{T}{\rho^{\gamma-1}}\right) + \text{constant}$$

$$s = c_{\mathrm{v}}\ \log\left(P\,\rho^{-\gamma}\right) + \text{constant}$$

Case of <u>constant</u> entropy (adiabatic gas) : $ds = 0$

$$\log T - (\gamma - 1)\ \log\rho = \text{constant}$$

with

$$\gamma \equiv \frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} = \frac{5}{3}\,.$$

$$T\rho^{-(\gamma-1)} = \text{constant} \quad , \quad P\rho^{-\gamma} = \text{constant}$$

(Self-)gravity

$$\boldsymbol{f}_{\mathrm{gr}} = \rho\, \boldsymbol{g} = -\rho\, \boldsymbol{\nabla}\Phi$$

$$\boldsymbol{g}(\boldsymbol{x}, t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x}, t) = -\begin{pmatrix} \dfrac{\partial \Phi}{\partial x} \\[2ex] \dfrac{\partial \Phi}{\partial y} \\[2ex] \dfrac{\partial \Phi}{\partial z} \end{pmatrix}$$

$$\rho\, \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla}\Phi$$

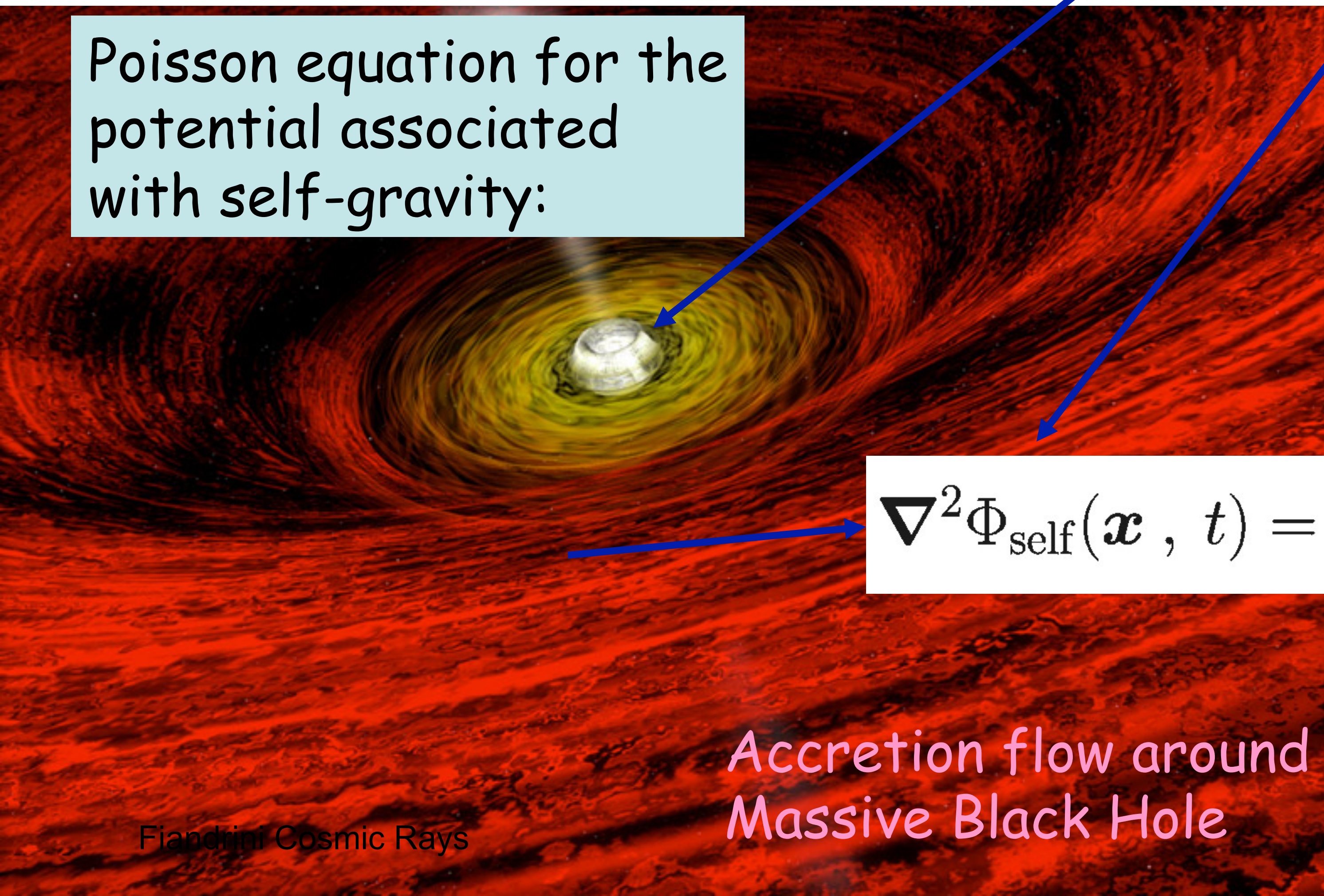
Self-gravity and Poisson's equation

Potential: two contributions!

$$\Phi(\boldsymbol{x}\,,t) = \Phi_{\rm ext}(\boldsymbol{x}\,,t) + \Phi_{\rm self}(\boldsymbol{x}\,,t)$$

Poisson equation for the potential associated with self-gravity:

$$\boldsymbol{\nabla}^2\Phi_{\rm self}(\boldsymbol{x}\,,\;t) = 4\pi G\,\rho(\boldsymbol{x}\,,\;t)$$

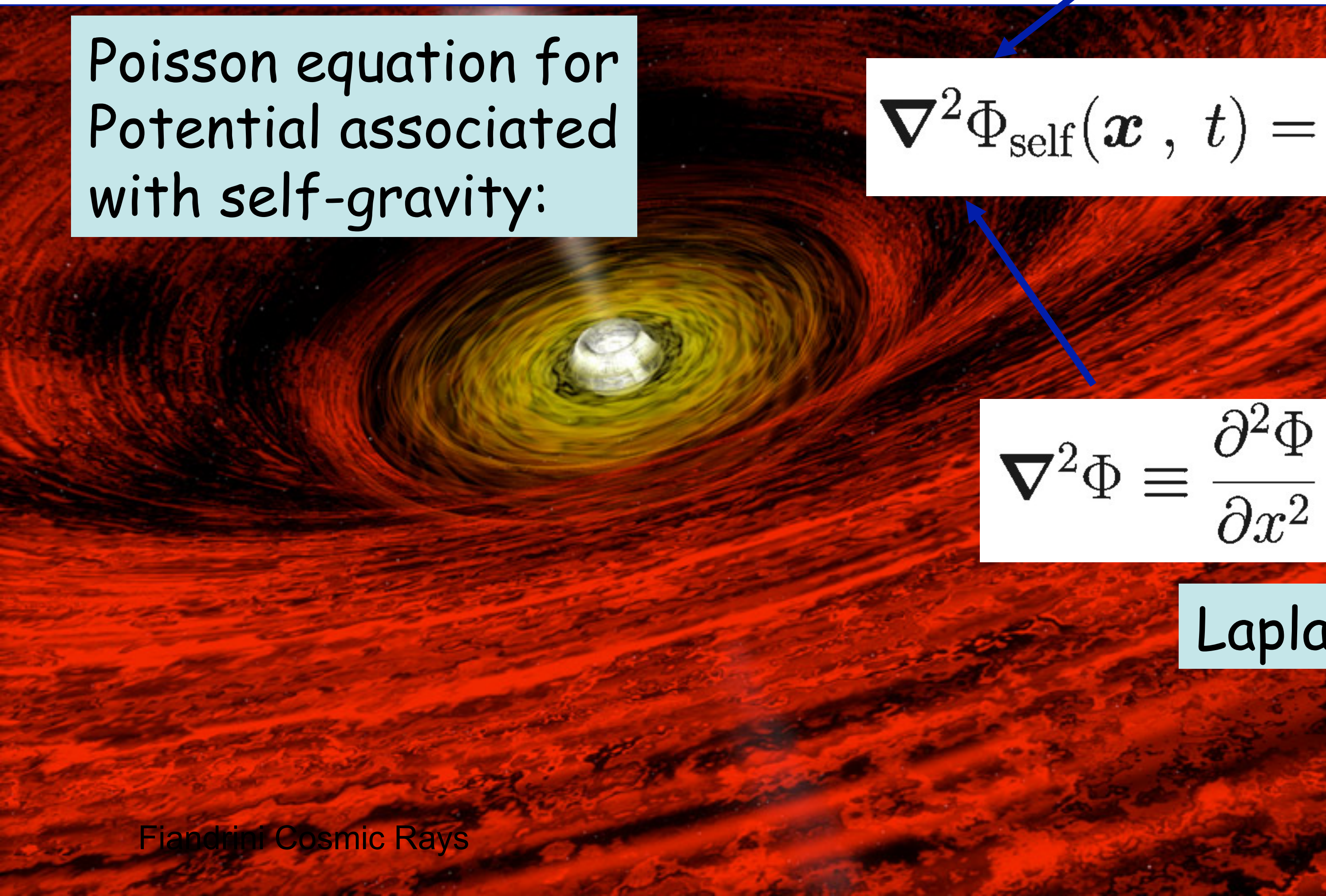


Accretion flow around Massive Black Hole

Self-gravity and Poisson's equation

Potential: two contributions!

$$\Phi(\boldsymbol{x}\,,t) = \Phi_{\rm ext}(\boldsymbol{x}\,,t) + \Phi_{\rm self}(\boldsymbol{x}\,,t)$$



Poisson equation for Potential associated with self-gravity:

$$\boldsymbol{\nabla}^2\Phi_{\rm self}(\boldsymbol{x}\,,\;t) = 4\pi G\,\rho(\boldsymbol{x}\,,\;t)$$

$$\boldsymbol{\nabla}^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Laplace operator

Summary: Equations describing ideal (self-)gravitating fluid

Equation of Motion:

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P-\rho\,\boldsymbol{\nabla}\Phi$$

Continuity Equation:
behavior of mass-density

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V})=0$$

Ideal gas law
&
Adiabatic law:
Behavior of pressure
and temperature

$$P(\rho\,,\,T)=nk_{\mathrm{b}}T=\frac{\rho\mathcal{R}T}{\mu}$$

$$P\,\rho^{-5/3}=\mathrm{constant}$$

Poisson's equation: self-gravity

$$\boldsymbol{\nabla}^2\Phi_{\mathrm{self}}(\boldsymbol{x}\,,\,t)=4\pi G\,\rho(\boldsymbol{x}\,,\,t)$$

Conservative Form of the Equations

Aim: To cast all equations in the same *generic form:*

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

<u>Reasons</u>:

1. Allows quick identification of conserved quantities
2. This form works best in constructing numerical codes for *Computational Fluid Dynamics*
3. Shock waves are best studuied form a conservative point of view

Generic Form: $$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of} \\ \\ \text{quantity}\end{pmatrix} + \boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that} \\ \\ \text{quantity}\end{pmatrix} = \begin{pmatrix}\text{external sources} \\ \\ \text{per unit volume}\end{pmatrix}$$

$$\frac{\partial S}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = q(\boldsymbol{x}\ ,\ t)$$

Transported quantity is a scalar *S*, so flux **F** must be a vector!

Component form: $$\frac{\partial S}{\partial t} + \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = q$$

Generic Form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Transported quantity is a vector ***M*** , so the flux must be a tensor ***T*** .

$$\frac{\partial \boldsymbol{M}}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{T}=\boldsymbol{Q}(\boldsymbol{x}\,,\,t)$$

Component form:

$$\frac{\partial}{\partial t}\begin{pmatrix}M_x\\ \\ M_y\\ \\ M_z\end{pmatrix}+\begin{pmatrix}\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{yx}}{\partial y}+\frac{\partial T_{zx}}{\partial z}\\ \\ \frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{zy}}{\partial z}\\ \\ \frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{yz}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\end{pmatrix}=\begin{pmatrix}Q_x\\ \\ Q_y\\ \\ Q_z\end{pmatrix}$$

The fact the the flux of a vector field is a rank 2 tensor can be understood as follows: the transported quantity is a vector with 3 arbitrary components, each of them can be transported in 3 indipendent directions → so there are 3x3 indipendent quantitites...exactly the nbr of components of a rank 2 tensor

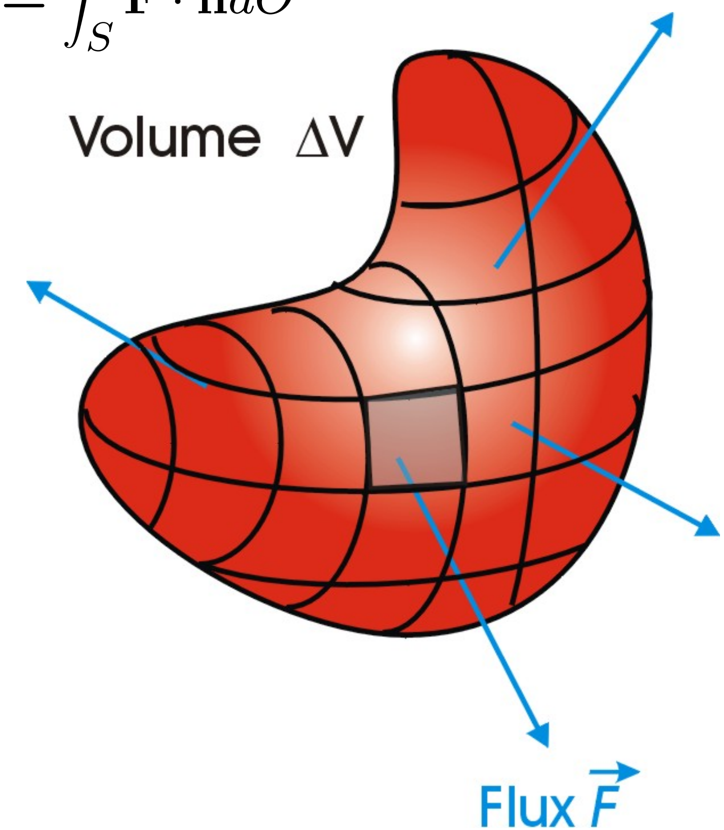
Integral properties: Stokes Theorem

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix} + \boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix} = \begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Let integrate the equation over a volume V and use the Stokes theorem

$$\int_V \nabla\cdot\mathbf{F}dV = \int_S \mathbf{F}\cdot\mathbf{n}dO$$

$$\frac{\partial}{\partial t}\left(\int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, S\right) = \int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, q(\boldsymbol{x}\,,\,t) - \oint_{\partial\mathcal{V}} \mathrm{d}\boldsymbol{O}\cdot\boldsymbol{F}$$



The integral relation states the amount of quantity S in a volume can change only due to sources in V or by a flux of S into or out from V

Examples: mass and momentum conservation

Mass conservation: already in conservation form!

Continuity Equation:
transport of the scalar ρ

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Excludes 'external mass sources' due to processes like two-photon pair production etc.

Examples: mass- and momentum conservation

Mass conservation: already in conservation form!

Continuity Equation:
transport of the scalar ρ

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{V}) = 0$$

Momentum conservation: transport of a vector!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \, \boldsymbol{V} \right] = -\boldsymbol{\nabla} P - \rho \, \boldsymbol{\nabla} \Phi$$

Algebraic Manipulation

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{V} \otimes \boldsymbol{V} + P \, \boldsymbol{I} \right) = -\rho \boldsymbol{\nabla} \Phi$$

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla} \Phi$$

Starting point: Equation of Motion

As advertised: Algebraic Manipulation!

$$\boxed{\rho\left[\frac{\partial \mathbf{V}}{\partial t}\right.}+(\mathbf{V}\cdot\boldsymbol{\nabla})\,\mathbf{V}\Big] = -\boldsymbol{\nabla}P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{V}) = 0}$$

$$\begin{aligned} \rho\frac{\partial \mathbf{V}}{\partial t} &= \frac{\partial(\rho\mathbf{V})}{\partial t} - \mathbf{V}\boxed{\frac{\partial \rho}{\partial t}} \\ &= \frac{\partial(\rho\mathbf{V})}{\partial t} + \mathbf{V}\,(\boldsymbol{\nabla}\cdot(\rho\mathbf{V})) \end{aligned}$$

Use:

1. chain rule for differentiation
2. continuity equation for density

As advertised: Algebraic Manipulation!

$$\boxed{\rho\left[\frac{\partial \mathbf{V}}{\partial t}\right.} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\mathbf{V}\Big] = -\boldsymbol{\nabla} P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{V}) = 0}$$

$$\begin{aligned}\rho\frac{\partial \mathbf{V}}{\partial t} &= \frac{\partial(\rho\mathbf{V})}{\partial t} - \mathbf{V}\boxed{\frac{\partial \rho}{\partial t}}\\ &= \frac{\partial(\rho\mathbf{V})}{\partial t} + \mathbf{V}\,(\boldsymbol{\nabla}\cdot(\rho\mathbf{V}))\end{aligned}$$

$$\frac{\partial(\rho\mathbf{V})}{\partial t} + (\boldsymbol{\nabla}\cdot(\rho\mathbf{V}))\mathbf{V} + \rho(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} = -\boldsymbol{\nabla} P - \rho\boldsymbol{\nabla}\Phi$$

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boxed{(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi$$

$$\boxed{(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\, \boldsymbol{V} + \rho\, (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = \boldsymbol{\nabla} \cdot\, (\rho \boldsymbol{V} \otimes \boldsymbol{V})}$$

Use divergence chain rule for dyadic tensors

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P-\rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t}+\boxed{(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V}+\rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}}=-\boxed{\boldsymbol{\nabla}P}-\rho\boldsymbol{\nabla}\Phi$$

$$\boxed{(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\,\boldsymbol{V}+\rho\,(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}\otimes\boldsymbol{V})}$$

$$\boxed{\boldsymbol{\nabla}P=\boldsymbol{\nabla}\cdot(P\,\boldsymbol{I})}$$

Rewrite pressure gradient as a divergence

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\,\boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boxed{(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}} = -\boxed{\boldsymbol{\nabla} P} - \rho\boldsymbol{\nabla}\Phi$$

$$\boxed{(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\,\boldsymbol{V} + \rho\,(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = \boldsymbol{\nabla} \cdot\, (\rho \boldsymbol{V} \otimes \boldsymbol{V})}$$

$$\boxed{\boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot (P\,\boldsymbol{I})}$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot\, (\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\,\boldsymbol{I}) = -\rho\boldsymbol{\nabla}\Phi$$

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P-\rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t}+(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V}+\rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=-\boldsymbol{\nabla}P-\rho\boldsymbol{\nabla}\Phi$$

Momentum density

Stress tensor = momentum flux

Momentum source: gravity

$$\frac{\partial\boxed{(\rho\boldsymbol{V})}}{\partial t}+\boldsymbol{\nabla}\cdot\boxed{(\rho\boldsymbol{V}\otimes\boldsymbol{V}+P\,\boldsymbol{I})}=\boxed{-\rho\boldsymbol{\nabla}\Phi}$$

The tensor $\mathbf{R}_{ik} = \rho \mathbf{V}_i \mathbf{V}_k + p\delta_{ik}$ is the Reynolds stress tensor for an ideal fluid and represents the momentum flux

Energy Conservation

The total energy of a fluid element is given by the mechanical energy (kinetic + potential) and the internal energy (thermodynamical)

$$\epsilon_{tot} = \epsilon_{mech} + \epsilon_{int} = \epsilon_{kin} + \epsilon_{pot} + \epsilon_{int}$$

The mechanical part can be obtained by the motion equation

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right] = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi$$

Using the vectorial identity

$$(\vec{V} \cdot \nabla)\vec{V} = \nabla(\frac{V^2}{2}) - \vec{V} \times (\nabla \times \vec{V})$$

And multiplying scalarly the motion equation by **V**, we get

$$\rho\vec{V}\cdot(\frac{\partial\vec{V}}{\partial t}+\nabla(\frac{V^2}{2})-\vec{V}\times(\nabla\times\vec{V})) = -\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

The right hand side corresponds to the power of the force densities acting on the system

Energy Conservation

$$\rho\vec{V}\cdot\left(\frac{\partial\vec{V}}{\partial t}+\nabla\left(\frac{V^2}{2}\right)-\vec{V}\times(\nabla\times\vec{V})\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

The scalar prod with 2nd term in the lefthand side is = 0 because the vectors are perp →

$$\boxed{\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}}+\rho\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

$$\boxed{\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}=\frac{\partial}{\partial t}(\rho V^2/2)-\frac{V^2}{2}\frac{\partial\rho}{\partial t}\quad=\frac{\partial}{\partial t}(\rho V^2/2)+\frac{V^2}{2}\nabla\cdot(\rho\vec{V})}$$

Using the mass conservation law to eliminate ∂ρ/∂t

$$\frac{\partial}{\partial t}(\rho V^2/2)+\boxed{\frac{V^2}{2}\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)}=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

$$\boxed{=\nabla\cdot\left(\frac{\rho V^2\vec{V}}{2}\right)}$$

$$\Rightarrow\quad\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot\left(\frac{\rho V^2\vec{V}}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

This equation shows how the kinetic energy of the fluid changes due to work done by pressure forces and by gravitational force: they act as sources of kinetic energy

The energy flux merely redistributes kinetic energy over space

Energy Conservation

$$\frac{\partial}{\partial t}(\rho V^2/2) + \nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p - \boxed{\rho\vec{V}\cdot\nabla\Phi}$$

Now it's the turn of potential energy

$$\boxed{\rho\vec{V}\cdot\nabla\Phi = \nabla\cdot(\rho\Phi\vec{V}) - \Phi\nabla\cdot(\rho\vec{V}) \quad = \nabla\cdot(\rho\Phi\vec{V}) + \Phi\frac{\partial\rho}{\partial t}}$$

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p-\nabla\cdot(\rho\Phi\vec{V})-\boxed{\Phi\frac{\partial\rho}{\partial t}}$$

$$\boxed{\Phi\frac{\partial\rho}{\partial t} = \frac{\partial(\rho\Phi)}{\partial t} - \rho\frac{\partial\Phi}{\partial t}}$$

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p-\nabla\cdot(\rho\Phi\vec{V})-\frac{\partial(\rho\Phi)}{\partial t}+\rho\frac{\partial\Phi}{\partial t}$$

Rearranging terms:

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2})+\nabla\cdot(\rho\Phi\vec{V})+\frac{\partial(\rho\Phi)}{\partial t} = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

This equation states how the mechanical energy changes due to the work done by the pressure forces and due to non conservative gravitational field (ie explicitly time dependent)

Energy Conservation

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

The thermodynamics is in the pressure term

Let rewrite the I principle of TD in terms of unit mass variables (specific variables)

$$dq = Tds = d\epsilon + pdv$$

Where dq is the specific heat exchanged, ds the specific entropy, dε is the specific internal energy and v the specific volume **v= V/m=1/ρ**

a) $Tds = d\epsilon + pd(1/\rho) \equiv d(\epsilon + p/\rho) - \frac{dp}{\rho}$ But ε +p/ρ = h = specific enthalpy ➔

b) $Tds = dh - \frac{dp}{\rho}$ ➔ $\rho dh - \rho Tds = dp$ ➔ The pressure gradient is $\rho\nabla h - \rho T\nabla s = \nabla p$

Energy Conservation

To complete the energy balance we have to add the internal energy to the equation

The explicit time variation of internal energy density is $\dfrac{\partial(\rho\epsilon)}{\partial t} = \rho\dfrac{\partial \epsilon}{\partial t} + \epsilon\dfrac{\partial \rho}{\partial t}$

Taking the time derivative of a), we get

$$T\frac{\partial s}{\partial t} = \frac{\partial \epsilon}{\partial t} + p\frac{\partial(1/\rho)}{\partial t} = \frac{\partial \epsilon}{\partial t} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} \quad \Longrightarrow \quad \rho T\frac{\partial s}{\partial t} = \rho\frac{\partial \epsilon}{\partial t} - \frac{p}{\rho}\frac{\partial \rho}{\partial t}$$

$$\rho\frac{\partial \epsilon}{\partial t} = \rho T\frac{\partial s}{\partial t} + \frac{p}{\rho}\frac{\partial \rho}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\nabla\cdot(\rho\vec{V}) \qquad \frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{V})$$

$$\frac{\partial(\rho\epsilon)}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\nabla\cdot(\rho\vec{V}) - \epsilon\nabla\cdot(\rho\vec{V}) = \rho T\frac{\partial s}{\partial t} - (\frac{p}{\rho} + \epsilon)\nabla\cdot(\rho\vec{V}) = \rho T\frac{\partial s}{\partial t} - h\nabla\cdot(\rho\vec{V})$$

We add ∂ε/∂t on both sides of mech E equation and substitute the grad(p)

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2} + \rho\Phi) + \nabla\cdot[(\frac{\rho V^2}{2} + \rho\Phi)\vec{V}] = \boxed{-\vec{V}\cdot\nabla p} + \rho\frac{\partial \Phi}{\partial t} \qquad \rho\nabla h - \rho T\nabla s = \nabla p$$

$$\boxed{\frac{\partial(\rho\epsilon)}{\partial t}} + \frac{\partial}{\partial t}(\frac{\rho V^2}{2} + \rho\Phi) + \nabla\cdot[(\frac{\rho V^2}{2} + \rho\Phi)\vec{V}] = \boxed{\rho T\frac{\partial s}{\partial t} - h\nabla\cdot(\rho\vec{V})} \boxed{-\vec{V}\cdot(\rho\nabla h - \rho T\nabla s)} + \rho\frac{\partial \Phi}{\partial t}$$

Energy Conservation

To complete the energy balance we have to add the internal energy to the equation

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = \rho T\boxed{(\frac{\partial s}{\partial t}+\vec{V}\cdot\nabla s)}-\boxed{(h\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla h)}+\rho\frac{\partial\Phi}{\partial t}$$

$$= \frac{Ds}{Dt} \qquad\qquad = \nabla\cdot(\rho h\vec{V})$$

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = \rho T\frac{Ds}{Dt}-\nabla\cdot(\rho h\vec{V})+\rho\frac{\partial\Phi}{\partial t}$$

Move the grad on right hand side to left side and get

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi+\rho h)\vec{V}] = \rho T\frac{Ds}{Dt}+\rho\frac{\partial\Phi}{\partial t}$$

Energy density — Energy flux — "Net heating rate density"

The 1st term in RHS in the true heating (or cooling) due to "external" irreversible processes as radiation losses. The 2nd "gravitational heating" $\rho\partial\Phi/\partial t$ corresponds to the process known as violent relaxation in a time-varying gravitational potential, which plays an important role in the dynamics of galaxies, where it acts in a way analogous to a heating mechanism

Energy Conservation

Energy density is a scalar! → the energy flux is a vector

Kinetic energy density

Internal energy density

Gravitational potential energy density

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + \rho e + \rho\Phi\right) + \nabla\cdot\left[\rho \boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\rm eff}$$

$$\mathcal{H}_{\rm eff} \equiv \mathcal{H} + \rho\,\frac{\partial\Phi}{\partial t}$$

Irreversibly lost/gained energy per unit volume

Energy Conservation

$$e = \frac{P}{(\gamma - 1)\,\rho} \text{ if } P\,\rho^{-\gamma} = \text{constant}$$

Internal energy per unit mass

$$h = e + \frac{P}{\rho} = \frac{\gamma P}{(\gamma - 1)\,\rho}$$

Specific enthalpy

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho\,V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\text{eff}}$$

$$\mathcal{H}_{\text{eff}} \equiv \mathcal{H} + \rho\,\frac{\partial\Phi}{\partial t}$$

Irreversible gains/losses, e.g. radiation losses

"Dynamical Friction"

Conservazione dell'energia

Nei fluidi ideali, in cui sono assenti fenomeni di dissipazione dovuti ad attrito "interno" (cioe' viscosita') e nell'ipotesi di assenza di conduzione termica (che puo' trasferire calore da una regione all'altra), l'unico fenomeno di scambio di energia non meccanica (ie non esprimibile come pdV) puo' essere solo attraverso l'irraggiamento

Nel caso in cui anche i processi radiativi siano assenti o trascurabili, il processo e' adiabatico (se reversibile) → dQ = 0 → TdS = 0

In tal caso la conservazione dell'energia di un elemento di massa e' equivalente alla conservazione dell'entropia del sistema dello stesso elemento

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Conservazione dell'energia

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

La densita' dei fluidi astrofisici e' tipicamente molto bassa (~1 idrogeno/cm^3) per cui un fotone emesso dalle particelle del fluido non viene mai riassorbito (a meno che vi siano certe condizioni (cfr. Autoassorbimento e spessore ottico) e lascia il sistema, facendo perdere energia attraverso processi radiativi

In tal caso l'entropia non e' piu' conservata

Anche in questo caso pero' l'equazione di sopra ha un suo ambito di validita': infatti i processi radiativi hanno tempi scala caratteristici e se l'evoluzione del sistema avviene su scale di tempo << di quelli radiativi, il processo puo' essere considerato adiabatico

Conservazione dell'energia

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Vi sono situazioni in cui conduzione e viscosita' giocano un ruolo importante (p es all'interno delle stelle, in dischi di accrescimento e piu' in generale in oggetti compatti e/o densi in cui il libero cammino medio diventa "piccolo" e quindi lo spessore ottico diventa "grande" (cfr. Autoassorbimento)

Ma in genere il meccanismo per cui un sistema si discosta dalla isoentropia e' il fatto che il fluido viene riscaldato o raffreddato da una varieta' di processi radiativi

Se definiamo Γ e Λ i coefficienti di riscaldamento e raffreddamento per unita' di massa e di tempo

$$\frac{Dq}{Dt} = T\frac{Ds}{Dt} = T(\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s) = \Gamma - \Lambda$$

Steady Flows: <u>no</u> explicit time-dependence:

$$\boxed{\frac{\partial}{\partial t} = 0}$$

$$\text{mass conservation:} \quad \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}) = 0 \; ;$$

$$\text{momentum conservation:} \quad \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V} \otimes \boldsymbol{V} + P\, \boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi \; ;$$

$$\text{energy conservation:} \quad \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{V} \left(\tfrac{1}{2} V^2 + h + \Phi \right) \right] = 0 \; .$$

Energy conservation

mass conservation: $\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V})=0\ ;$

momentum conservation: $\boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V}\otimes\boldsymbol{V}+P\,\boldsymbol{I})=-\rho\boldsymbol{\nabla}\Phi\ ;$

energy conservation: $\boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{V}\left(\tfrac{1}{2}V^2+h+\Phi\right)\right]=0\ .$

$$\boxed{\begin{gathered}\boldsymbol{\nabla}\cdot(f\boldsymbol{A})=f(\boldsymbol{\nabla}\cdot\boldsymbol{A})+(\boldsymbol{A}\cdot\boldsymbol{\nabla})f\\ \boldsymbol{A}=\rho\boldsymbol{V}\quad,\quad f=\tfrac{1}{2}V^2+h+\Phi\end{gathered}}$$

$\nabla\cdot\mathbf{A}=0$

For mass conservation law

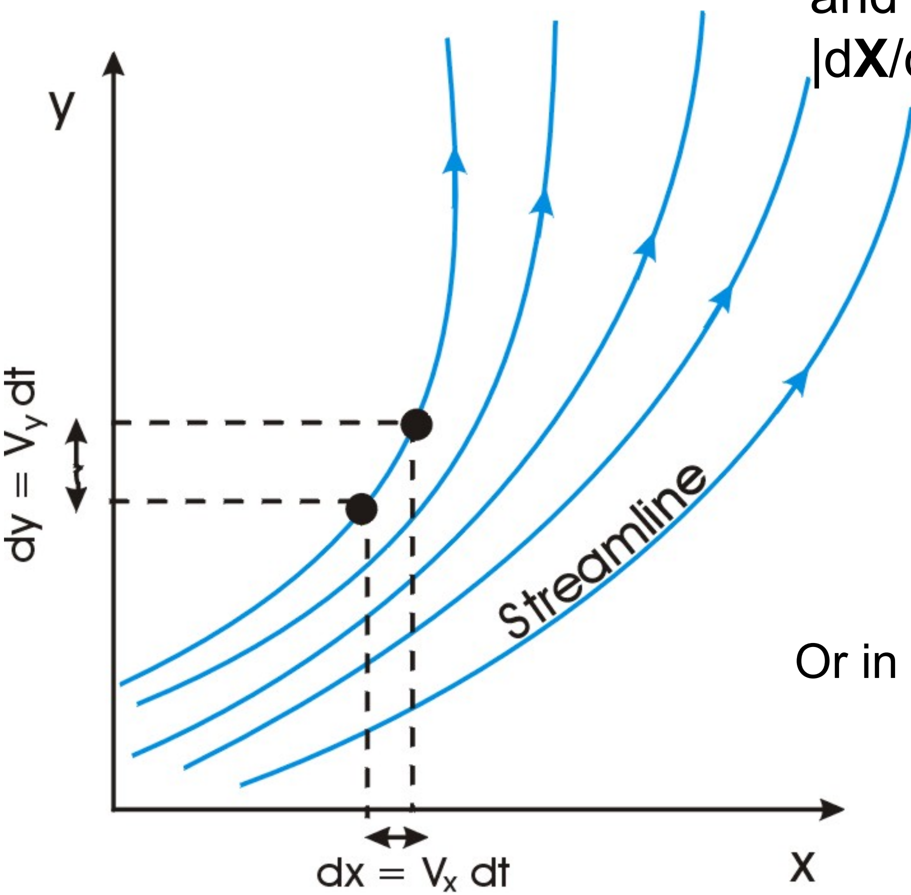
$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left(\tfrac{1}{2}V^2+h+\Phi\right)=0$$

The quantity f is (obviously) the specific energy (energy/mass)

Variation along flow lines in steady flows

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left(\tfrac{1}{2}V^2+h+\Phi\right)=0$$

Consider the flowlines defined as a trajectory **x**=**X**(l) such that the tangent vector (versor), d**X**/dl, is || the local **V,** d**X**/dl || **V**(**X**) and it can be choosen such that |d**X**/dl| = 1 → dl/dt=V



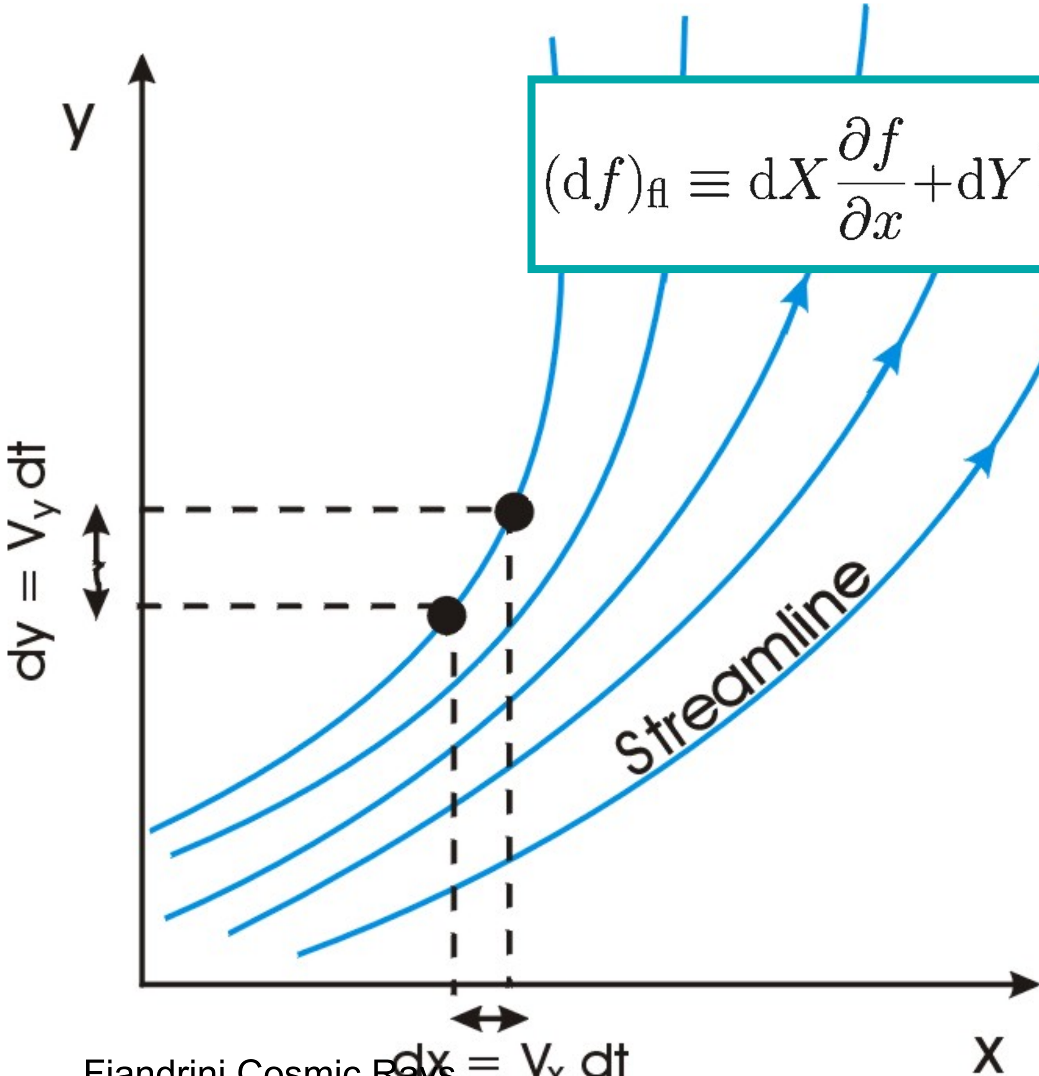
The coordinates of points on a given flow line satisfy the relation

$$\mathrm{d}\boldsymbol{X}=\boldsymbol{V}(\boldsymbol{x}=\boldsymbol{X})\,\mathrm{d}t$$

Or in components $\dfrac{\mathrm{d}X}{V_x(\boldsymbol{X})}=\dfrac{\mathrm{d}Y}{V_y(\boldsymbol{X})}=\dfrac{\mathrm{d}Z}{V_z(\boldsymbol{X})}=\mathrm{d}t$

When is a function $f(x,y,z)$ constant along flow lines ?

$$(\mathrm{d}f)_{\mathrm{fl}} \equiv \mathrm{d}X\frac{\partial f}{\partial x}+\mathrm{d}Y\frac{\partial f}{\partial y}+\mathrm{d}Z\frac{\partial f}{\partial z} = (\mathrm{d}\boldsymbol{X}\cdot\boldsymbol{\nabla} f) = 0$$



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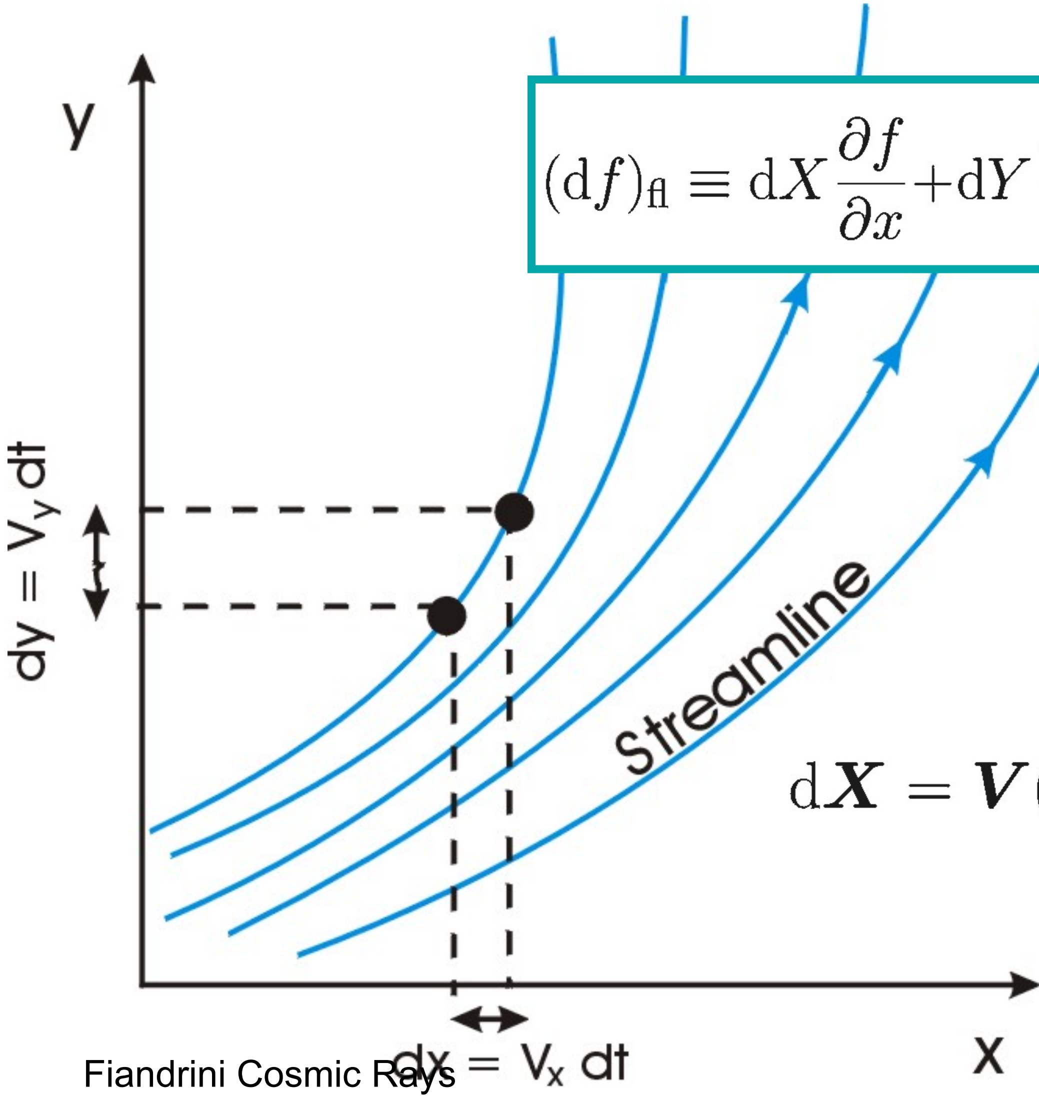


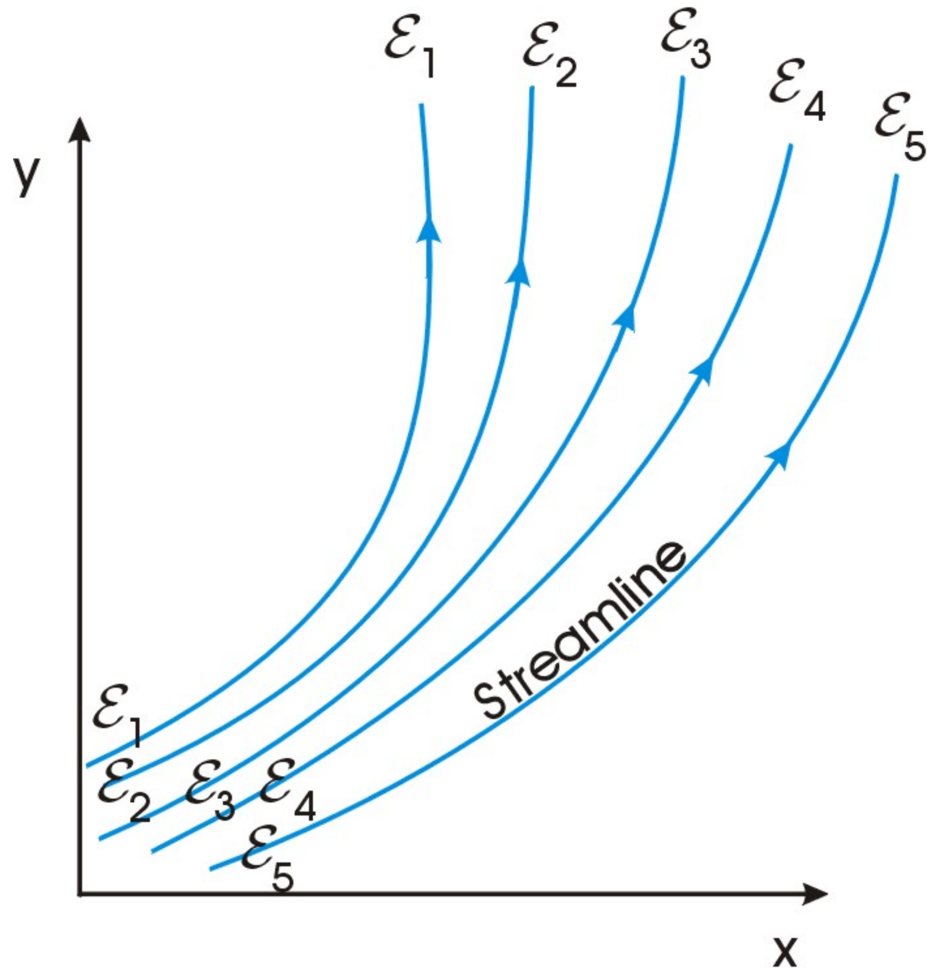
$$\mathrm{d}\boldsymbol{X} = \boldsymbol{V}(\boldsymbol{x}=\boldsymbol{X})\,\mathrm{d}t$$

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})f(\boldsymbol{x}) = 0$$

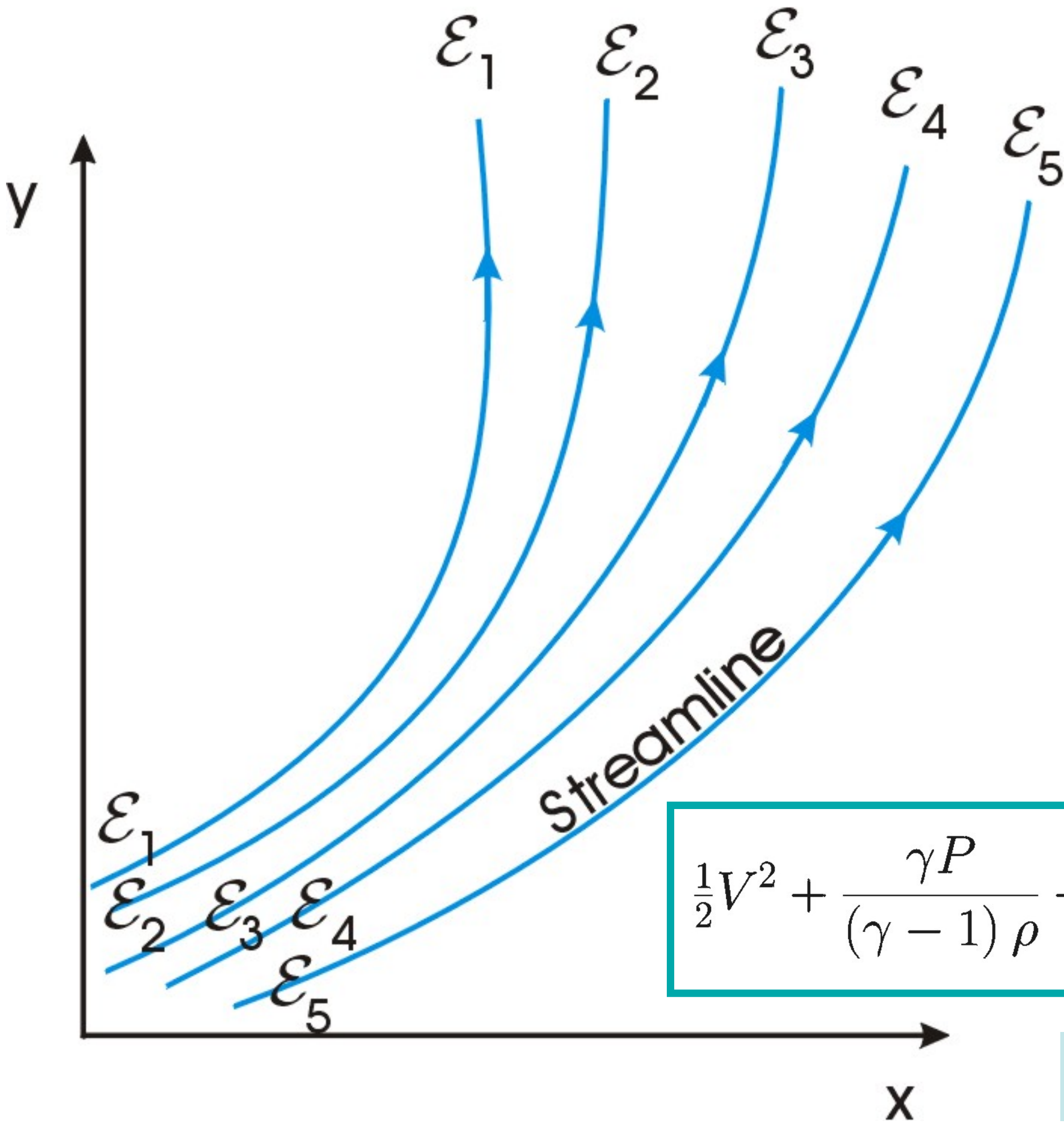
Bernouilli's Law for steady flows:

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left(\tfrac{1}{2}V^2+h+\Phi\right)=0 \longrightarrow \tfrac{1}{2}V^2+\frac{\gamma P}{(\gamma-1)\,\rho}+\Phi=\text{constant along flowlines}$$

NB: bernoulli law dont say anything about the variation of E_{spec} across the flow lines



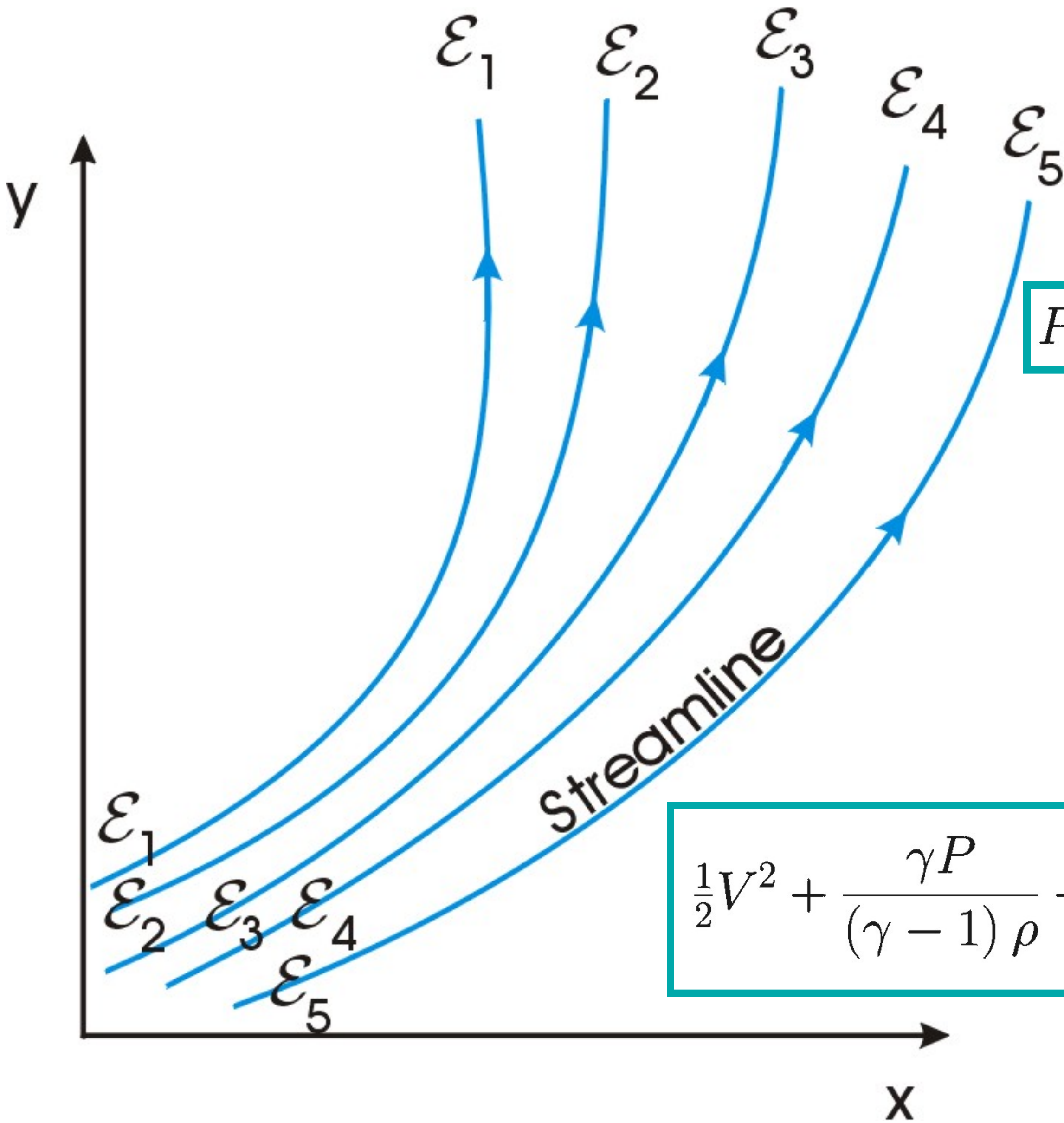
In general the constant may differ from flowline to flowline → it is not a global constant over all space



Bernoulli's Law

$$\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

(Energy conservation)



Adiabatic fluid

$$P\rho^{-\gamma} = \text{constant along flowlines}$$

(Entropy conservation)

Bernoulli's Law

$$\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

(Energy conservation)