Lecture 6 241019

- Il pdf delle lezioni puo' essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1920/

1st invariant: gyromotion

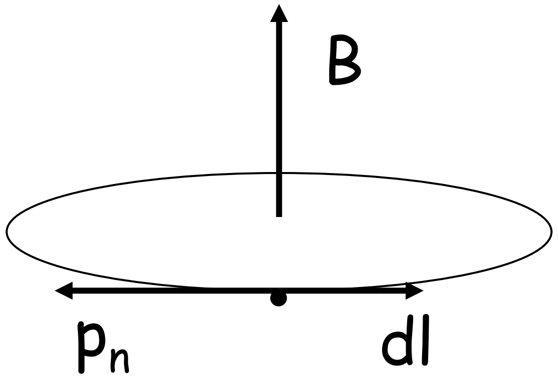
The so-called first adiabatic invariant is obtained by integrating P from equation (4.10) around the gyration orbit, where dl is an element of the particle path around the orbit.

$$
\begin{aligned}
J_1 &= \oint [\mathbf{p} + q\mathbf{A}] \cdot d\mathbf{l} \\
&= p_\perp \cdot 2\pi\rho + q\oint \mathbf{A} \cdot d\mathbf{l} \\
&= p_\perp \cdot 2\pi \frac{p_\perp}{Bq} + q\oint \nabla \times \mathbf{A} \cdot d\mathbf{S} \qquad (4.11)
\end{aligned}
$$

where dS is an element of the area enclosed by the path. Therefore,

$$
\begin{aligned}
J_1 &= \frac{2\pi p_\perp^2}{Bq} + q\oint \mathbf{B} \cdot d\mathbf{S} \\
&= \frac{2\pi p_\perp^2}{Bq} - qB\pi\rho^2 \\
&= \frac{2\pi p_\perp^2}{Bq} - \frac{\pi p_\perp^2}{Bq} = \frac{\pi p_\perp^2}{qB} \qquad (4.12)
\end{aligned}
$$

The second term in (4.12) is negative because dS as defined by the particle orbit points in the opposite direction to **B**.



1st invariant: gyromotion

$$\frac{d}{dt}\left(\frac{p_n^2}{B}\right)=\left(\frac{dp_n^2}{Bdt}-\frac{p_n^2}{B^2}\frac{\partial B}{\partial t}\right) \qquad \frac{B}{p_n^2}\frac{d}{dt}\left(\frac{p_n^2}{B}\right)=\left(\frac{dp_n^2}{p_n^2dt}-\frac{1}{B}\frac{\partial B}{\partial t}\right)$$

Let |B| vary in time uniformly. Because B varies in time, there an induction field such that

$$\oint \nabla \times \mathbf{E}\cdot d\mathbf{S} = \oint \mathbf{E}\cdot d\mathbf{l} = -\oint \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = -\pi\rho^2\frac{\partial B}{\partial t} \qquad (4.14)$$

The energy change in one revolution or in one gyroperiod τ_g is therefore

$$\Delta W = -q\oint \mathbf{E}\cdot d\mathbf{l} = q\pi\rho^2\frac{\partial B}{\partial t} \qquad (4.15)$$

<u>The fundamental assumption is that dB/dt does not vary over a gyroperiod</u>

Hence

$$\frac{dW}{dt} = \frac{\Delta W}{\tau_g} = q\pi\rho^2\frac{\partial B}{\partial t}\cdot\frac{Bq}{2\pi m} = \frac{p_\perp^2}{2mB}\frac{\partial B}{\partial t} \qquad (4.16)$$

$\tau_g=2\pi/\omega=2\pi m/qB$

1st invariant: gyromotion

Also,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 c^2) = m_0 c^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t} \tag{4.17}$$

where

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[1 + \frac{p_\perp^2}{m_0^2 c^2}\right]^{1/2}$$

$$= \frac{1}{2 m_0^2 c^2 \gamma} \frac{\mathrm{d}p_\perp^2}{\mathrm{d}t} \tag{4.18}$$

$\gamma = E_n / m_o c^2$
$= [p_n^2 + (m_o c^2)^2]^{1/2} / m_o c^2 =$
$= [(p_n / m_o c^2)^2 + 1]^{1/2}$

Equate (4.16) and (4.17) using (4.18) for $\mathrm{d}\gamma/\mathrm{d}t$ to obtain

$$\frac{1}{B}\frac{\partial B}{\partial t} = \frac{1}{p_\perp^2}\frac{\mathrm{d}p_\perp^2}{\mathrm{d}t}.$$

Therefore

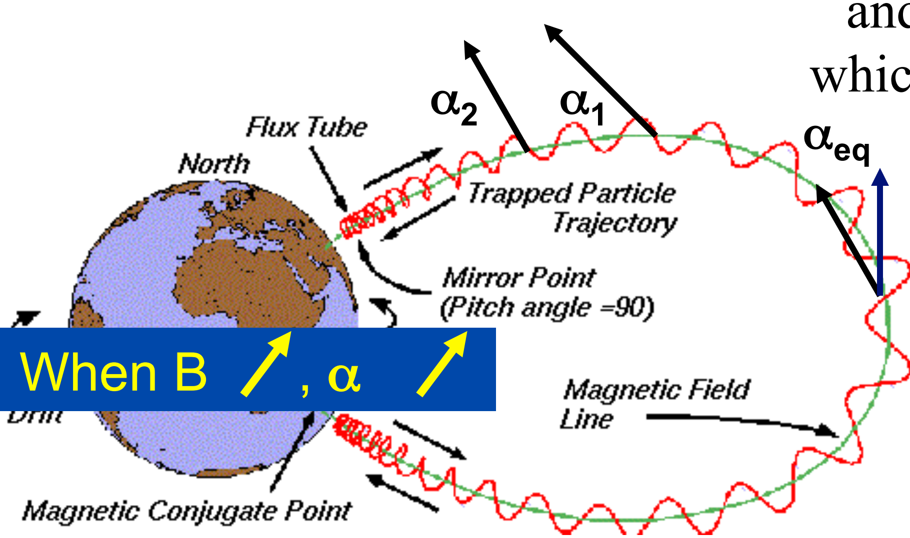
$$\frac{p_\perp^2}{B} = \text{constant} \tag{4.19}$$

and it follows that $\mu = p_\perp^2 / 2 m_0 B$ is also constant.

Adiabatic invariants in B fields: 1st invariant

If B field varies only weakly in 1 gyroradius, i.e. $dB/Bdt \ll w_L/2\pi$, or $B d\rho/dB \ll \rho$, then

$$\mu = \frac{p_\perp^2}{2m_o B} \approx \mathbf{const.} \quad \text{or} \quad I_1 = J_1/p = \sin^2\alpha/B \approx const.$$



- When α=90°, mirroring occurs,

and $B_m = B/\sin^2\alpha$ defines the mirror field value which is the same at all the mirror points along the ticle trajectory, i.e. particle reflection occurs always at B_m =constant.

At magn equator, α is minimum and

$$\sin^2\alpha_{eq} = B_{eq}/B_m$$
$$\sin^2\alpha = B/B_m$$

any field value→<u>α depends only on the field</u>

B_m is an adiabatic invariant because identical to an adiabatic invariant and because B_{eq} is a constant α_{eq} is an adiabatic invariant too

Adiabatic invariants in B fields: 2nd invariant

Bouncing→ $\mathbf{J}_2 = \oint \left(\vec{p} + q\vec{A}\right)\cdot \mathbf{d}\vec{s}$ with ds element of path along the field line

If B field varies only weakly on a scale comparable with the distance traveled along the field by the particle during one gyration → $\nabla \mathbf{B_p}/\mathbf{B} << \omega_L/2\pi v_p$ then J_2~const.

The 2nd term gives

$$\begin{aligned}\oint q\mathbf{A}\cdot d\mathbf{s} &= q\int \nabla\times\mathbf{A}\cdot d\mathbf{S} \\ &= q\int \mathbf{B}\cdot d\mathbf{S} \\ &= 0 \end{aligned} \quad (4.30)$$

since the integration path along the field line encloses a negligible area and no magnetic flux.

Therefore

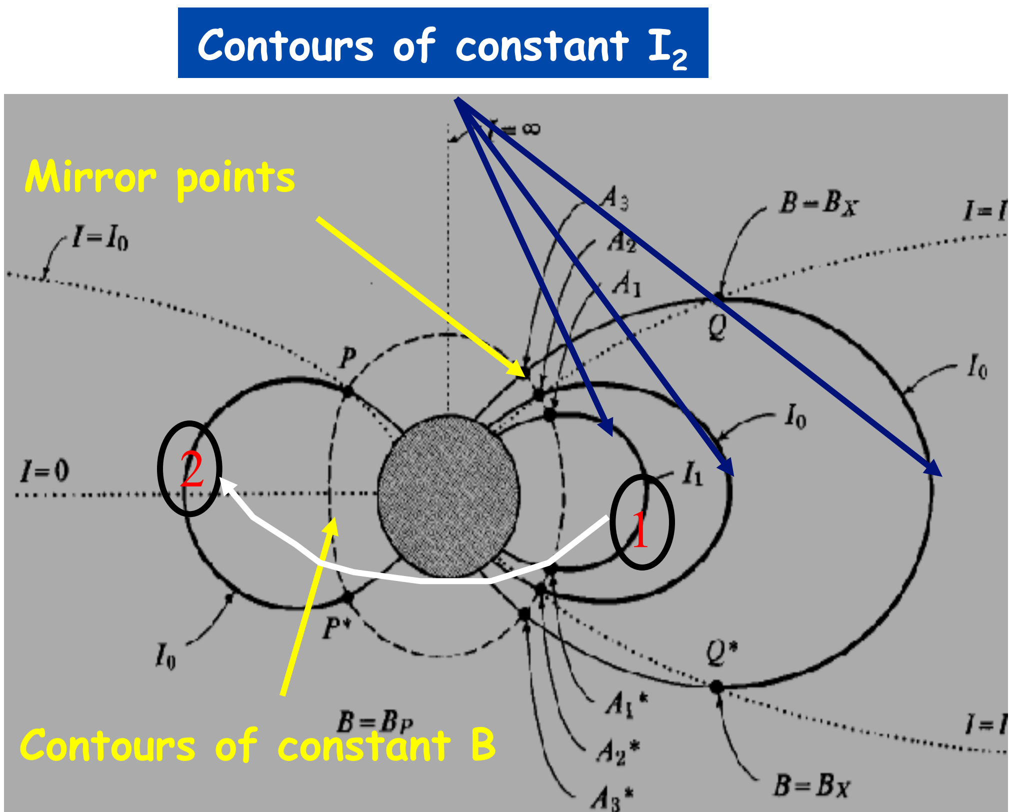
$$J_2 = \oint \mathbf{p}\cdot d\mathbf{s} = \oint p\cos\alpha\, ds = \oint p_{\parallel}\, ds = \text{constant} \quad (4.31)$$

Da dimostrare..

Adiabatic invariants in B fields: 2nd invariant

J_2 does not depend on particle properties but only on field structure, because
$\cos\alpha=(1-\sin^2\alpha)^{1/2}=[1-B(s)/B_m]^{1/2}$

$$\mathbf{J_2 = p\int_{s'}^{s}\sqrt{1-B(s)/B_m}\,ds \rightarrow I_2 = \frac{J_2}{2p} \approx const}$$



The primary use of I_2 is to find surfaces mapped out during bouncing and drifting. A particle initially on curve 1, with a given I, will drift on curve 2 (with the same I) and return to 1, mirroring at B_m in both the hemispheres throughout the drifting.
At each longitude there is ONLY one curve –or field line segment- having the required value of I. The particle will follow a trajectory made of field line segments such that I is constant.

Adiabatic invariants in B fields: 3rd invariant

Drifting→ $\mathbf{J}_3 = \oint \left(\vec{p} + q\vec{A}\right) \cdot \mathbf{d}\vec{l}_D$ with dl_D element along the long drift path

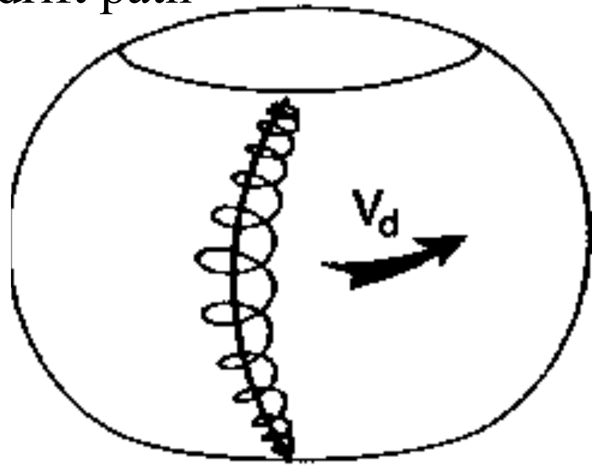
If B field varies only weakly in the area encircled by particle during the gyration or drift motion i.e. $\nabla \mathbf{B}_n / \mathbf{B} << \omega_L / 2\pi v_n$ or $\nabla \mathbf{B}_n / \mathbf{B} << \omega_L / 2\pi v_D$

then $J_3 \sim$const.

$$\mathbf{J_3} = \oint_{drift} (\mathbf{q}\vec{\mathbf{A}} + \vec{\mathbf{p}}) \cdot \mathbf{d}\bar{\mathbf{l}}_\mathbf{D} \approx \oint_{drift} \mathbf{q}\vec{\mathbf{A}} \cdot \mathbf{d}\bar{\mathbf{l}}_\mathbf{D} = \int (\nabla \times \vec{A}) \cdot dS = \int \vec{B} \cdot d\vec{S} = \mathbf{q\Phi} \approx \mathbf{const}$$

The 3rd invariant is prop to magnetic flux Φ enclosed by drift path

Important to describe drifts paths during slow changes of B.
In slowly changing fields 1st and 2nd invariant are conserved but E can change, e.g. due to slow compression/expansion of field or secular variations of the field.
Conservation of Φ requires particles to move inward/outward reversibly on the orbit during changes.
Rapid changes, i.e. dB/dt>>B$\mathbf{w_D}$**,** will cause permanent changes in **Φ** and therefore in particle orbits, e.g. solar storms, CME,…



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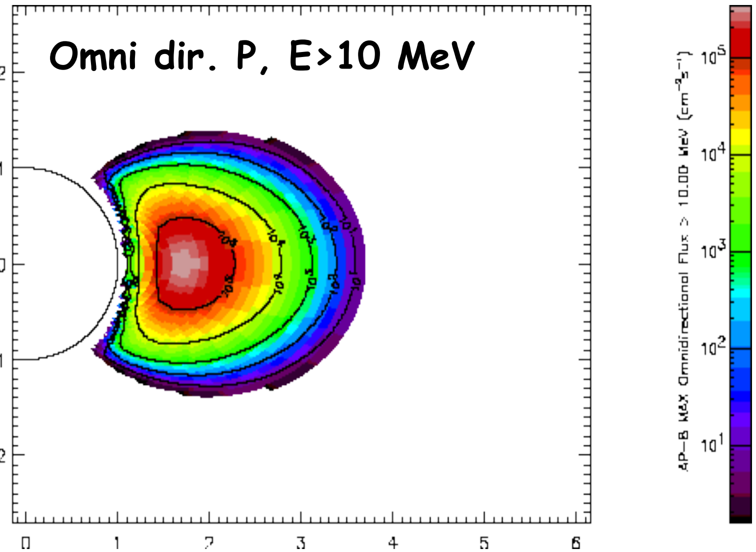
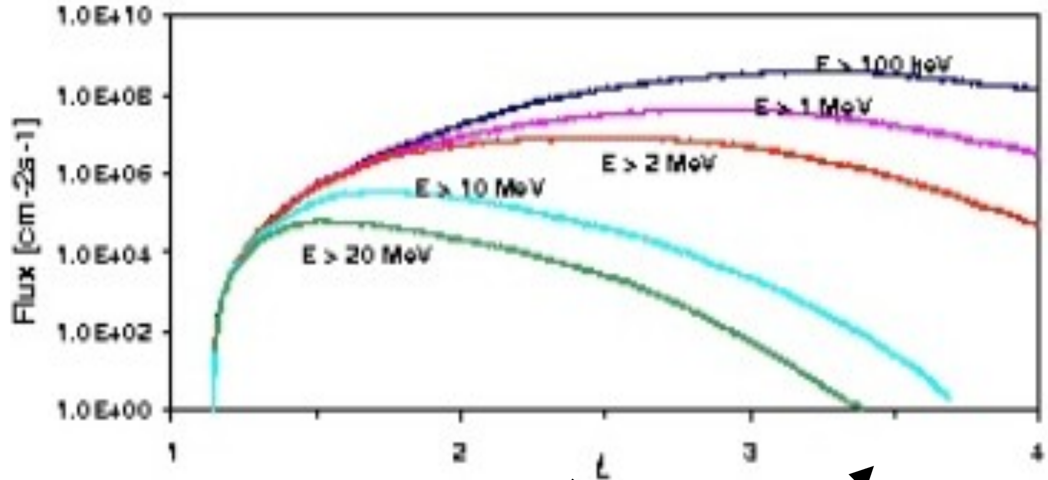
The Radiation Belts: static model





Single maxima L_{max} ↘ when E ↗

Inner belts, high energy up to ~400MeV

Average structure of the belts:

→ Inner belts centered at ~1.5 R_E

→ Outer belts centered at ~4 R_E

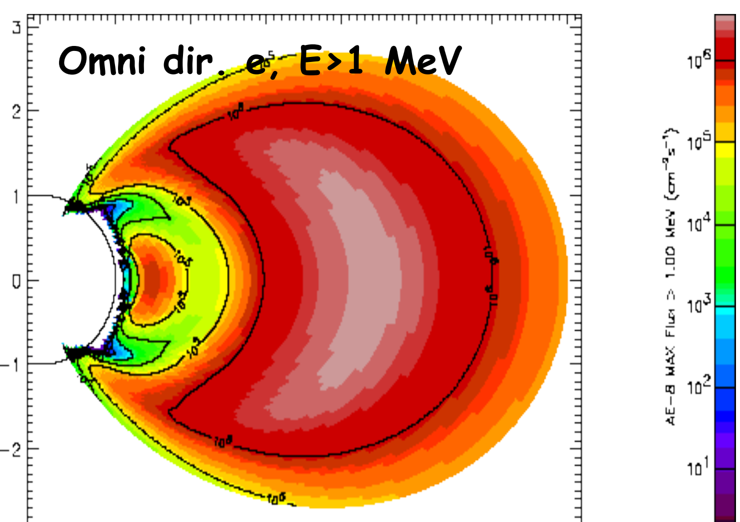
p only in inner belts

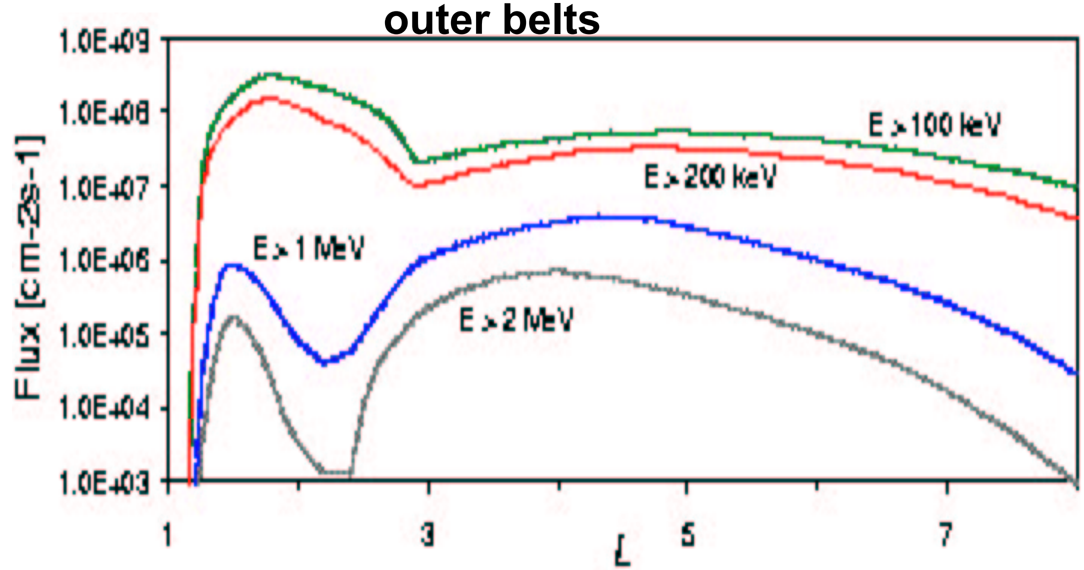
e^- found in both

Two maxima, L ↘ when E ↗

Low E (< 7MeV)

Slot Region in between Inner and outer belts





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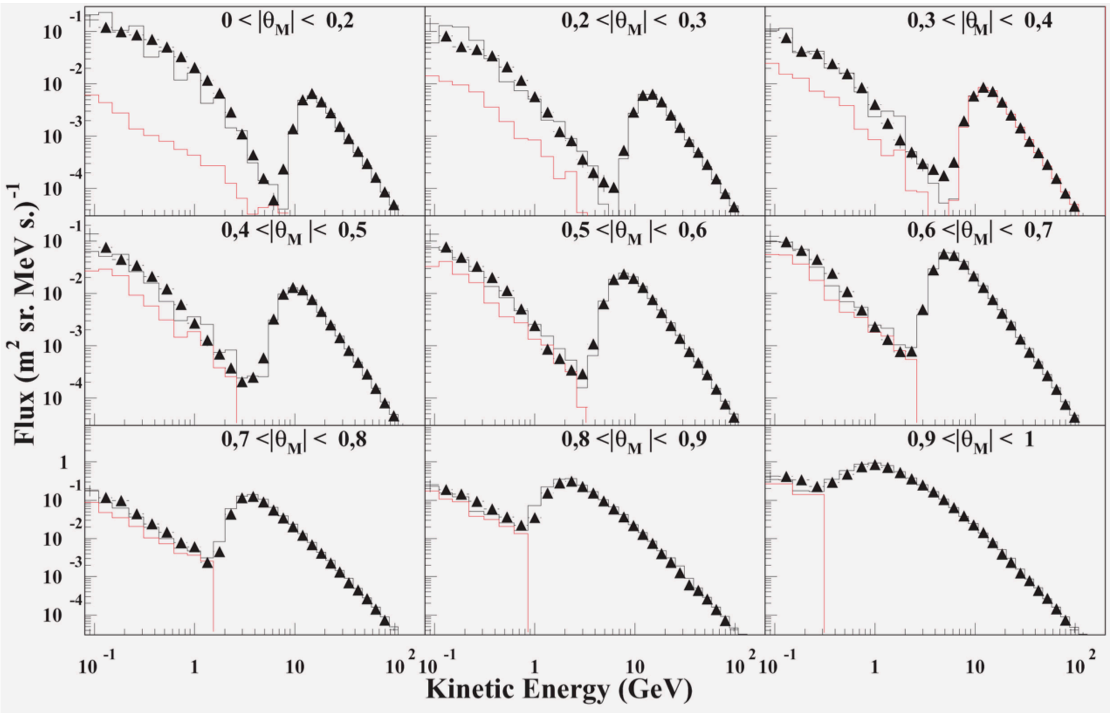
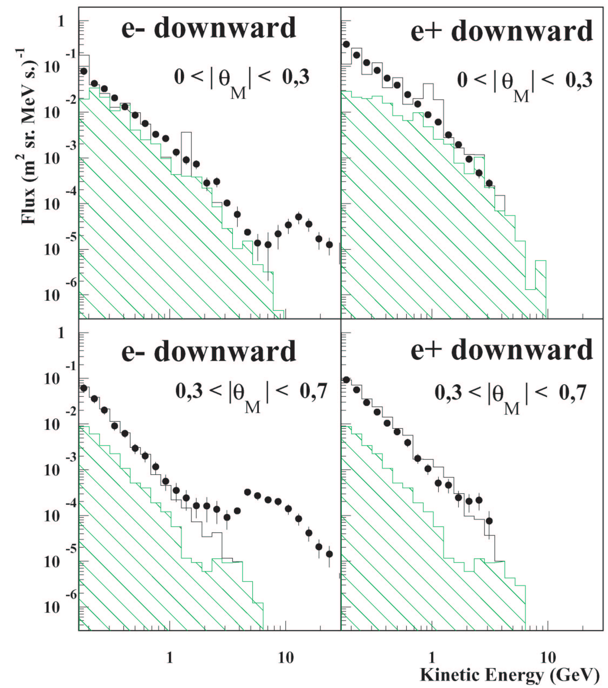
MonteCarlo Approach:Fluka 2000





Key points:

→accurate primaty CR fluxes

→realistic model of atmosphere (MSIS-E90)

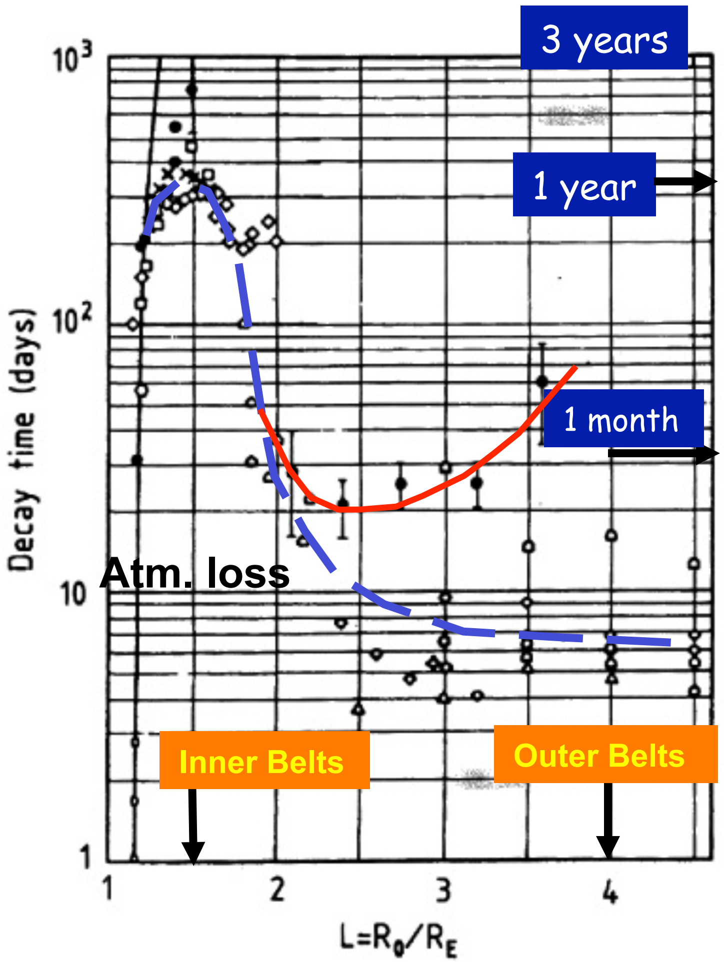
→interaction model (FLUKA 2000)

intrinsically 3-D

continously checked vs exp data

theory driven approach

The Radiation Belts: residence times



Time for ele flux to decay by a factor 1/e

Lifetimes of low E particles range from few days to years for energies not exceeding few tens of MeV

→Atmospheric interactions are the dominant losses for L<1.25, as shown by continous line

→For L>1.3, lifetimes are much shorter than those expected from atmosph.

→Loss due to plasma-wave interactions with pitch angle and radial diffusion

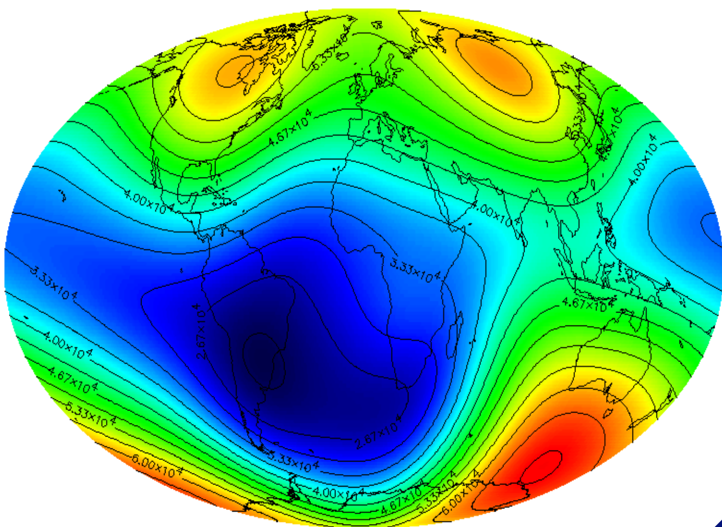
→Most of exp data shown was obtained following the decay of e- flux produced in high altitude nuclear explosions in ’60’s (e.g. StarFish event in ’62)

This ‘*events*’ form very likely a heavy background to AP-8 models developped during that period

→Confirmed also from studies on decays of e- injected during magn storms and substorms

Adiabatic invariants in B fields: coordinates

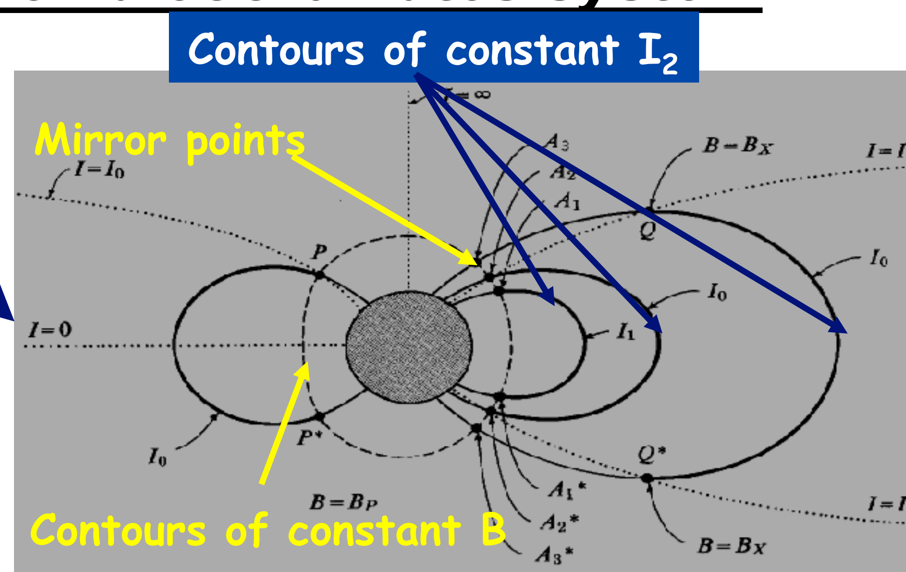
Any reference system based on geocentric coordinates does not allow insights into the relationships between the particle distributions at different locations due to lack of simmetry in the irregular geomagnetic field



What is needed is a coord system based on trapped particle motion which will have naturally identical values for equivalent magnetic positions

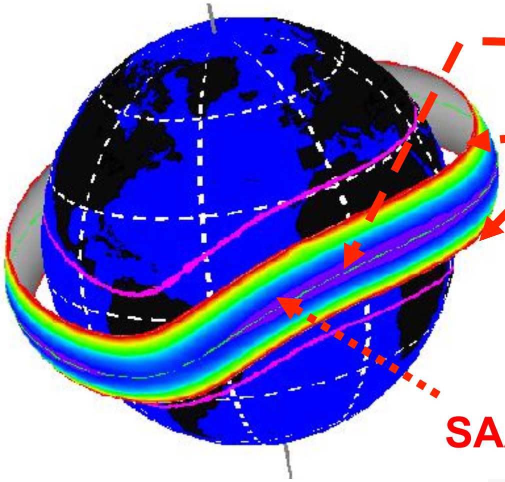
<u>Adiabatic invariants provide such a coordinates system</u>

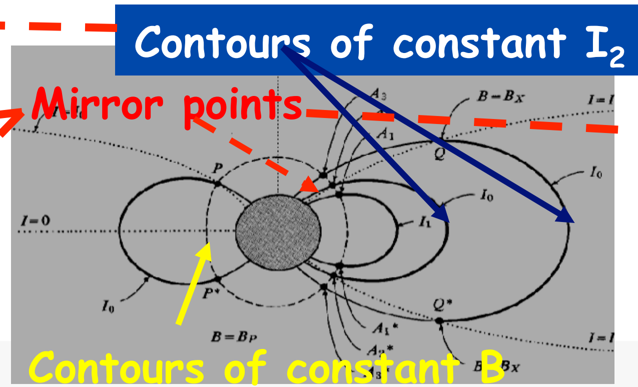
To conserve invariants particles will move following segments of field lines such that B_m (or α_o), L, Φ are conserved

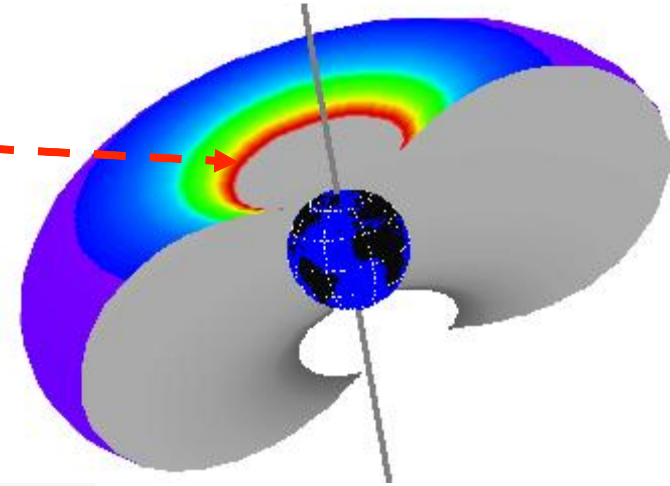


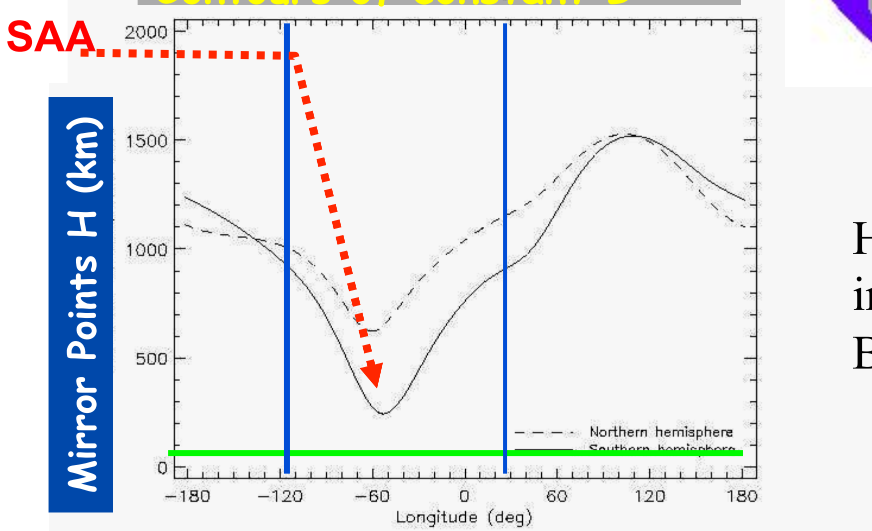
The ensemble of field lines segments of constant invariants forms the surface mapped out by the guiding center of a particle during its motion:

the drift shell









H_{mir} not const due to irregularities in earth's B field

<u>All the particles with the same invariants map out the same drift shell, i.e. are equivalent from magnetic point of view</u>

Adiabatic invariants in B fields:drift shells (2)

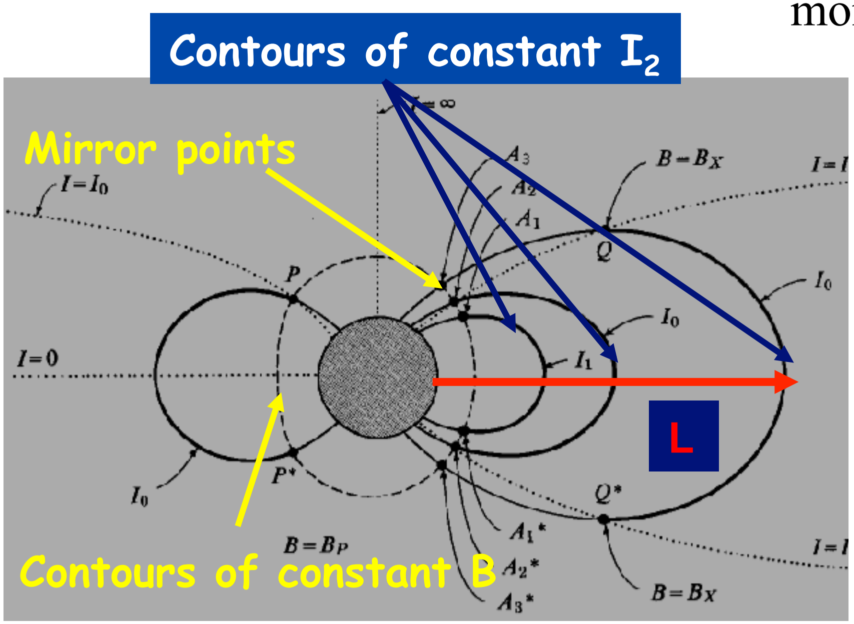
Adiabatic Invariants are difficult to visualize and interpret in a simple way, due to their complicate definition what is needed is to build more easily readable coords derived from AI:

μ ➔ B_m or $\boldsymbol{\alpha}_{eq}$, because are very easy to interpret and are still AI

→all the particles with same B_m and $\boldsymbol{\alpha}_{eq}$ will mirror at same location

For I_2 a dipole analogy: in a dipole field, all particles with same AI will cross magn. equator at same distance R_o from dipole axis, i.e. particles will remain on field lines having the same R_0

➔ $R_0=f_D(I_D,B_D,M_D)$ with f_D known function of dipole AI of the particle and magn moment of dipole

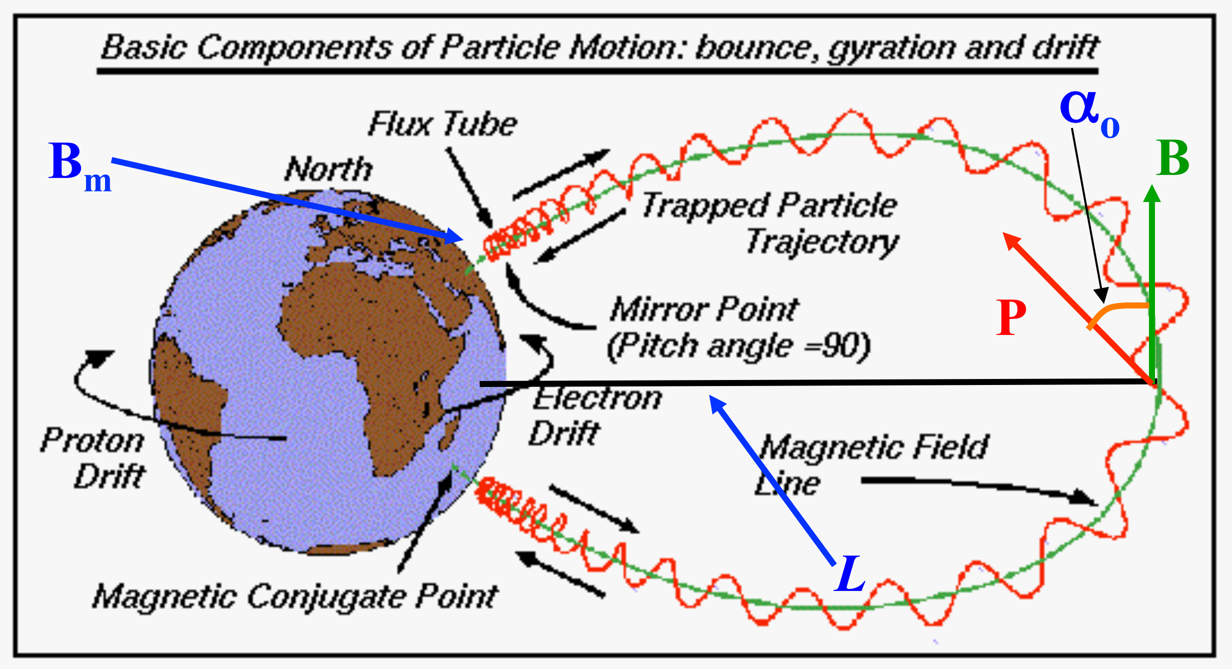


For real earth's field a new variable is defined based on dipole f_D: by definition the equivalent equatorial radius, L, called McIllwain parameter, is given by $LR_E=f_D(I,B,M_E)$

Particles will follow paths such that L=const.

NOTE: L=const. does not imply R const.!!!

Motion in Earth's Magnetic Field



***3 quasi-periodic* motion comp.**		***Adiab. Invariants***
➤ ***Gyration* with Larmor freq.**	⇔	***B_m* or α_o**
➤ ***Bouncing* betw. mirror points**	⇔	***Shell Par. L***
➤ ***East-West* drift**	⇔	***Mag. Flux Φ***

Particles with the same adiabatic invariants (L,a_o) or (L,B_m) have same motion in the Earth's field

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Back to cosmic rays

Cosmic Rays propagation as function of:

1) Diffusion $D = lv/3$

2) Rate of change of particle energy $b(E) = dE/dt$

3) Particle loss term due to interactions and decays

4) Particle gain from sources and all interactions and decays

Equazione di propagazione (1)

$$\frac{\partial N_i}{\partial t} - \vec{\nabla}\cdot(\hat{D}_i \vec{\nabla} N_i] + \frac{\partial}{\partial E}(b_i N_i) + n v \sigma_i N_i + \frac{N_i}{\tau_i} = q_i + \sum_{j<i} n v \sigma_{ij} N_j + \sum_j \frac{N_j}{\tau_{ij}}$$

- I CR non sono accelerati nell' ISM, sono accelerati da sorgenti puntiformi
- La loro potenza e distribuzione spazio-temp. e' descritta dalle funzioni $q_i(t, \vec{r}, E)$ (per la specie i)

* $\hat{D}_i(\vec{r}, E)$ e' il tensore di diffusione

* $b_i(\vec{r}, E)$ caratterizza le perdite contine di energia delle <u>singole</u> particelle così che $dE/dt = b_i$ (≡ ioniz. + brems + sincrotr + Compton inverso)

* $\sigma_i(E)$ e' la sez. d'urto inelastica del nucleo i con i nuclei dell' ISM

* $n(\vec{r})$ e' la densità dell'ISM

* σ_{ij} e' la sez. d'urto di prod. di nuclei di tipo i da nuclei più pesanti

* τ_i e' la vita media rispetto a decad. radioatt.

* N_j/τ_{ij} descrive l'apparizione di nuclei di tipo i a causa del decad. di altri nuclei

Equazione di propagazione (2)

$$\frac{\partial N_i}{\partial t} - \vec{\nabla}\cdot\left(\hat{D}_i \vec{\nabla} N_i\right) + \frac{\partial}{\partial E}\left(b_i N_i\right) + n v \sigma_i N_i + \frac{N_i}{T_i} = q_i + \sum_{j<i} n v \sigma_{ij} N_j + \sum_j \frac{N_j}{T_{ij}}$$

* Una sol. completa richiede l'uso di equazioni di questo tipo per tutti i tipi di nuclei, i.e. un sistema di equ. accoppiate
* Dobbiamo assumere che le densità osservabili N_i, le σ di frammentazione e le vite medie T siano note
* Dobbiamo anche conoscere la forma e il volume delle regioni di propagazione dei CR nella galassia, l'intensità e direzione del campo magnetico e la distribuzione del gas interstellare e delle sorgenti

Equazione di propagazione:esempio

$$\frac{\partial n}{\partial t} = D\nabla^2 n - \Gamma^{sp} n + Q$$

• $\Gamma^{sp} = \frac{1}{\tau_{sp}} = n_{ISM}\,\sigma\, v$

• Condizione al contorno

$n(E, t, z = \pm H) = 0$

• Diffusione isotropa

• Perdite di energia trascurabili

• Sorgenti Q uniformemente distribuite nel disco: SNR che esplodono con un rate $\mathcal{R}$

Soluzione: introduco il propagatore (o funzione di Green) $G(\vec{r}, t; \vec{r}', t')$ t.c.

$$n(\vec{r}, t, E) = \int dt' \int d^3r'\, G(\vec{r}, t; \vec{r}', t')\, Q(E, r', t')$$

Sost. nell'equ di prop., si trova che G deve soddisfare l'equ.

$$\frac{\partial G}{\partial t} - D\nabla^2 G + \Gamma^{sp} G = \delta(t-t')\,\delta^3(\vec{r} - \vec{r}')$$

La distribuzione delle sorgenti è

$$Q(E, r', t') = N(E)\,\delta(t-t')\,\delta^3(r-r')\,\frac{\mathcal{R}}{2\pi R_d^2 h} \quad cm^{-3} s^{-1}$$

con $N(E)$ spettro della sorgente

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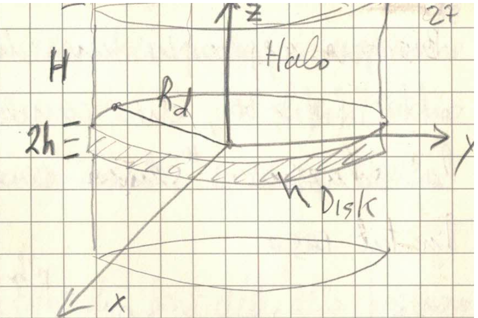


L'equ. si risolve "facilmente".

La soluzione generale senza le cond. al contorno è

$$G^{free}(\vec{r}, t; r', t') = \frac{N(E)\mathcal{R}}{2\pi R_d^2 h}\, e^{-\Gamma^{sp}(t-t')}\, \frac{e^{-[(\vec{r}-\vec{r}')^2/4D(t-t')]}}{[4\pi D(t-t')]^{3/2}}\,\theta(t-t')$$

La soluzione si estende su tutto lo spazio, anche oltre $z = \pm H$.

Per ottenere la sol. che soddisfa la cond. al contorno si usa il metodo delle "cariche immagini" (o metodo della riflessione)

Il concetto chiave è soddisfare le cond. al contorno estendendo il dominio oltre la regione di interesse e piazzando una "sorgente" mirror o un termine di forcing nella regione non fisica

Equazione di propagazione:esempio

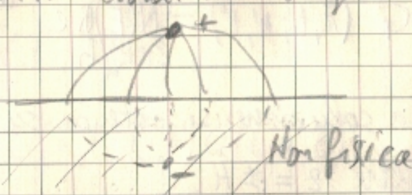
Bisogna aggiungere una distribuzione fittizia oltre $|z| > H$, che cancelli esattamente la soluzione senza cond. al contorno.

In tal caso

$$G(\vec{r},t;\vec{r}',t') = \frac{N(E)\, R\; e^{-\Gamma^{sp}\tau} \cdot e^{-[(x-x_s)^2+(y-y_s)^2]/4D\tau}}{[4\pi D\tau]^{3/2}\,(2\pi R_d^2 h)} ,$$

$$\cdot \sum_{n=-\infty}^{+\infty} (-1)^n e^{-(z-z'_n)^2/4D\tau} , \qquad \tau = t-t'$$

con $z'_n = (-1)^n z_s + 2nH$ cond. delle sorgenti immagine



Consideriamo il caso semplice in cui ci mettiamo nel centro del disco a $z = 0$ e ci limitiamo ai protoni, cioè senza nuclei e senza spallazione $\Gamma^{sp} = 0$.

In tal caso la densità di particelle alla posizione della Terra è

$$n(E) = \int_0^{\infty} d\tau \int_{-h}^{h} dz \int_0^{R_d} 2\pi r\, dr\, \frac{N(E)\, R}{2\pi R_d^2 h} \cdot \frac{e^{-\frac{r^2}{4D\tau}}}{[4D\tau]^{3/2}} \cdot$$

$$\cdot \sum_{-\infty}^{+\infty} \left[(-1)^n e^{-(2nH)^2/4D\tau} \right]$$

Integrando su z

$$n(E) = \int_0^{\infty} d\tau \int_0^{R_d} \frac{2\pi r}{\pi R_d^2} \frac{N(E)\, R}{[4D\tau]^{3/2}}\, e^{-\frac{r^2}{4D\tau}} \cdot$$

$$\cdot \sum_{-\infty}^{+\infty} \left[(-1)^n e^{-(2nH)^2/4D\tau} \right]$$

Equazione di propagazione

Integrando ancora su τ e poi su r

$$n(E) = \frac{N(E)\,R}{2\pi D(E) R_d} \sum_{-\infty}^{+\infty} \left\{ (-1)^n \left[\sqrt{1 + \left(\frac{2nH}{R_d}\right)^2} - \sqrt{\left(\frac{2nH}{R_d}\right)^2} \right] \right\}$$

Se $H \ll R_d$, la serie converge $\rightarrow \frac{H}{d}$

Quindi

$$n(E) = \frac{N(E)\,R}{2\pi R_d^2} \frac{H}{D(E)} = \frac{N(E)\,R}{2\pi R_d^2 H} \cdot \frac{H^2}{D(E)}$$

Se $D(E) \sim E^{\alpha}$ e $N(E) \sim E^{-\gamma}$

$$n(E) \sim E^{-\gamma-\alpha}$$

L'interpretazione fisica è semplice: la densità di protoni è il prodotto del rate di iniezione per unità di volume sull'intero volume della galassia e del tempo di confinamento $\tau_{esc} = H^2/D(E)$

Se c'è spallazione, introducendo il tempo scala di spallazione $\tau_{sp} = 1/\Gamma_{sp}$, la soluz. è

$$n(E) = \int_0^{\infty} d\tau \int_0^{R_d} dr \, \frac{2\pi r}{\pi R_d^2} \frac{e^{-r^2/4D\tau}}{[4\pi D\tau]^{3/2}} \cdot e^{-\tau/\tau_{sp}} \cdot \sum_{-\infty}^{+\infty} \left[(-1)^n e^{-(2nH)^2/4D\tau} \right]$$

Gli integrali su τ ed r sono analitici

$$n(E) = \frac{N(E)\,R}{2\pi D(E) R_d^2} \cdot \left[D(E)\,\tau^{sp}\right]^{1/2} \cdot \sum_{-\infty}^{+\infty} (-1)^n \left\{ e^{-(4n^2 \tau_{esc}/\tau_{sp})^{1/2}} - e^{\left[-\left(\frac{\tau_{esc}}{\tau_{sp}}\right)\left(4n^2 + R_d^2/H^2\right)\right]^{1/2}} \right\}$$

con $\tau_{esc} = H^2/D(E)$ che rappresenta il tempo di fuga dei CR dalla galassia

∴ $n(E)$ dipende dal rapporto fra tempo di fuga e di spallazione

Equazione di propagazione

Possiamo fare i limiti asintotici

i) $\tau_{esc}/\tau_{sp} \ll 1$ → Poca spallazione prima della fuga

La serie converge

$$\sum_{-\infty}^{+\infty}[\dots] = \frac{H}{(D\tau_{sp})^{1/2}}$$

NB: domina il τ più breve

quindi

$$n(E) = \frac{N(E)\,R}{2\pi R_d^2}\,\frac{H}{D(E)}$$

come nel caso senza spallazione

ii) $\tau_{esc}/\tau_{sp} \gg 1$ → La spallazione domina

In questo caso bisogna fare attenzione a definire la densità media attraversata nel volume di propagazione.

I RC dopo l'emissione sono contenuti in una regione di raggio $R \sim \sqrt{4D\tau_{sp}}$ nel disco.

Si può mostrare che, data la dipendenza $\sqrt{\tau}$, una particella può spazzare una parte suff. larga del volume da poter considerare che la particella sia esposta a densità di gas media costante pari a

$$n_{gas} = n_{disco}\frac{2h}{\sqrt{4D\tau_{sp}}} = n_{disco}\frac{h}{\sqrt{D\tau_{sp}}}$$

In queste condizioni

$$n(E) = \frac{N(E)\,R}{2\pi R_d^2 H}\left(\tau_{sp}\tau_{esc}\right)^{1/2}$$

La densità media n_{gas} può essere scritta come $n_{gas} = \frac{n_{disco}^2 h^2 \sigma c}{D}$

In queste condizioni $(\tau_{sp}\tau_{esc})^{1/2} = \left(\frac{H^2}{D}\cdot\frac{D}{n_d^2 h^2 \sigma^2 c^2}\right)^{1/2} = \left(\frac{H}{h}\frac{1}{n_d \sigma c}\right)$

è indipendente da E e lo spettro alla Terra è lo delle sorgenti

$n(E) \sim E^{-\gamma}$