Lecture 15 031218

- Il pdf delle lezioni puo’ essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/

The slides are taken from http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/bibliography/hydrodynamics_achterberg.pdf

MHD induction equation

Let assume that σ = const. in time and space.

Combining Faraday and Ohm law we eliminate E and J from equations

$$\vec{E} = \vec{j}/\sigma - \vec{v} \times \vec{B}/c$$

$$\vec{\nabla} \times (\vec{j}/\sigma - \vec{v} \times \vec{B}/c) = -\frac{1}{c}\frac{\partial B}{\partial t} \quad \Longrightarrow \quad \vec{\nabla} \times \vec{j} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c}\frac{\partial B}{\partial t})$$

From Ampere law $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} \quad \Longrightarrow \quad \frac{c}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{j}$

$$\frac{c}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c}\frac{\partial B}{\partial t})$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\nabla^2\vec{B} + \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) = -\nabla^2\vec{B} \quad \Longrightarrow \quad -\frac{c^2}{4\pi}\nabla^2\vec{B} = \sigma(\vec{v} \times \vec{B} - \frac{\partial B}{\partial t})$$

$$\boxed{\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t}}$$

The coefficient of laplacian $\lambda = \frac{4\pi\sigma}{c^2}$ Is called magnetic diffusivity

Or $\eta = c^2/4\pi\sigma$ is the magnetic diffusion coefficient

This equation, called induction equation, is very useful because only the B field appears in it

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

N.B.:

(i) → B once v is known

(ii) In MHD, v and B are primary variables:
induction eqn + eqn of motion → basic physics

(iii) $\mathbf{j} = c\nabla \times \mathbf{B} / 4\pi \quad and \quad \mathbf{E} = -\mathbf{v} \times \mathbf{B} + \mathbf{j}/\sigma$
are secondary variables

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

A B

(iv) B changes due to transport + diffusion

(v) $$\frac{A}{B} = \frac{L_0 v_0}{\eta} = R_m$$ -- * magnetic Reynold number*

eg, $\eta = 1\ \mathrm{m}^2/s$, $L_0 = 10^5$ m, $v_0 = 10^3$ m/s --> $\mathbf{R_m = 10^8}$

(vi) A >> B in most of Universe -->

B frozen to plasma -- keeps its energy

Except SINGULARITIES -- j & ∇ B large

Form at NULL POINTS, B = 0

(a) If $R_m << 1$

- The induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$$

- B is governed by a diffusion equation

  --> field variations on a scale L_0

  diffuse away on time * $t_d = \frac{L_0^{\ 2}}{\eta}$ *

  with speed $v_d = L_0 / t_d = \frac{\eta}{L_0}$

- E.g.: sunspot ($\eta = 1$ m^2/s, $L_0 = 10^6$ m), $t_d = 10^{12}$ **sec**;

  for whole Sun ($L_0 = 7 \times 10^8$ m), $t_d = 5 \times 10^{17}$ **sec**

MHD: ideal limit

$$\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} + \vec{\nabla}\times\vec{v}\times\vec{B} = \frac{\partial B}{\partial t}$$

To understand what happens when we are not in ideal limit, let consider a fluid in which the second term in the LHS is negligible

$$\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} = \frac{\partial B}{\partial t}$$

This is perfectly analogous to the heat diffusion equation, which by definition represents the dissipation: even if there is some heat concentrated in a small region, after a while it diffuses into all the space

→ the B field diffuses in space on a time scale

$$T_d = \frac{4\pi\sigma L^2}{c^2}$$

MHD: ideal limit

$$T_d = \frac{4\pi\sigma L^2}{c^2}$$

To understand whether the ideal MHD approximation fits the astrophysical situations, let calculate T_d for some concrete situation

Assume a conductivity for a hydrogen gas completely ionized of $\sigma = 6.98 \times 10^7 \frac{T^{3/2}}{ln\Lambda}\ s^{-1}$

Where T is the temperature in K and lnΛ~ 30 is an approximate factor called Coulomb logarithm (it arises from semi-classical calculation)

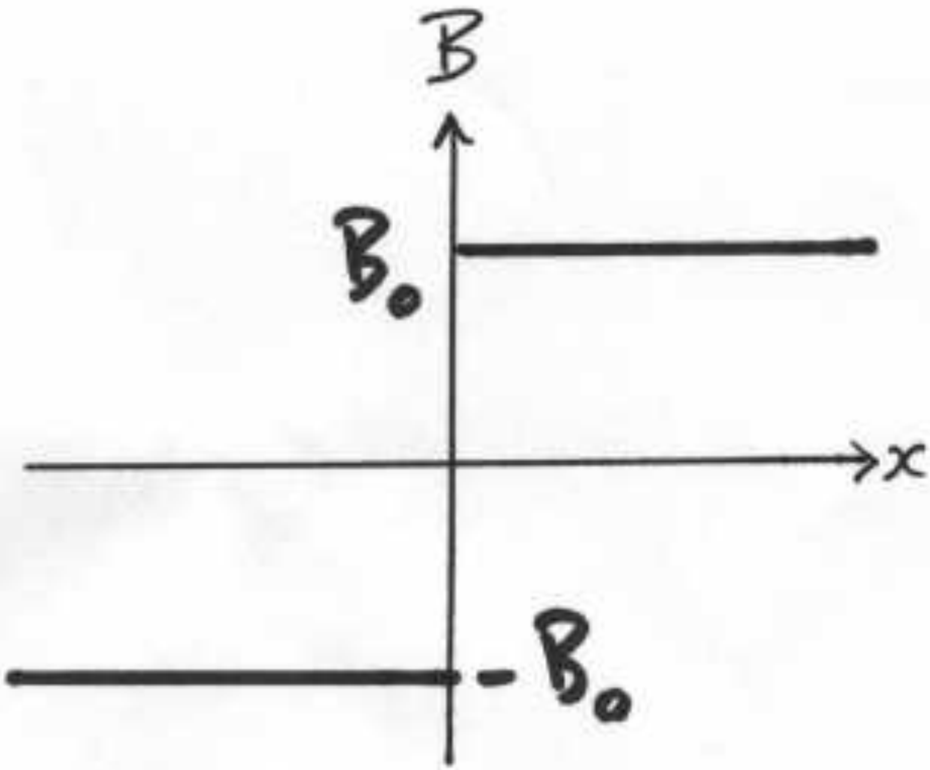
For a stellar interior for which $T\sim10^7$ K and $L\sim10^{11}$ cm, we find $T_d\sim 3\text{x}\ 10^{11}$ years, greater than universe lifetime, T_U

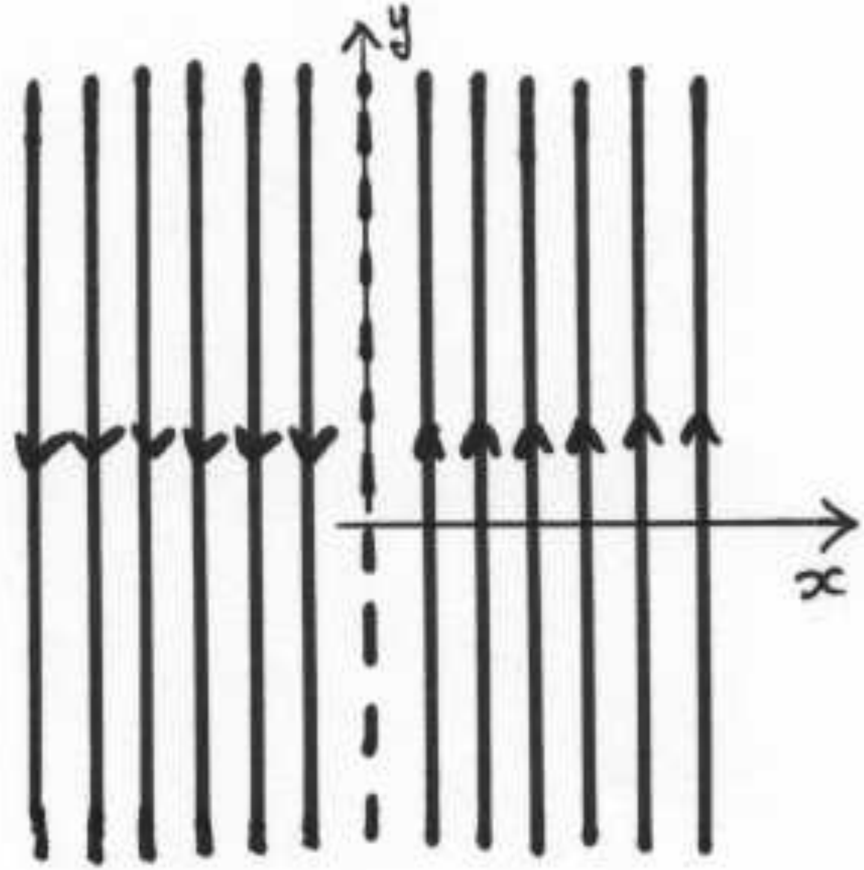
For a galaxy, L is very huge and $T_d > T_U$

But even for the smallest objects this time is long: in the inner regions of accretion disks around a black hole, with dimensions of ~ 100 Schwarzschild radii (~10^{15} cm), we find $T_d\sim10^{20}$ years

In other words, the astrophysical fluids are (almost) all such that the dissipation time of the B field is greater than Universe lifetime: the B field is never dissipated in typical astrophysical conditions

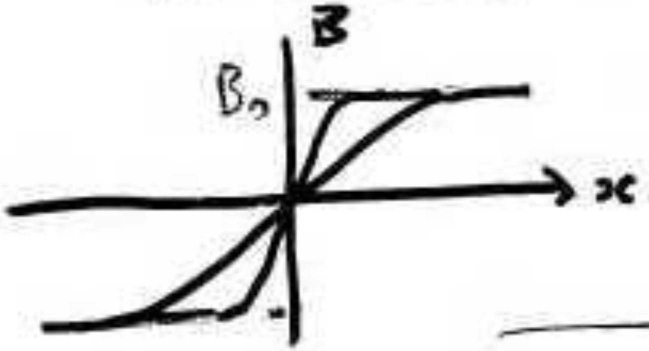
Example: given B(x,0) as in figure, determine B(x,t) in the hypothesis that diffusion dominates





To Solve $\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}$ — (1)

for $B(x,t)$



where $B(x,0) = \begin{cases} B_0, & x > 0 \\ -B_0, & x < 0 \end{cases}$ — (2)

Since initial conditions have no natural scale, seek a

self-similar solution of the form

$$B(x,t) = f(V), \text{ where } V = \frac{x}{t^{1/2}}$$

Then $\frac{\partial B}{\partial t} = \frac{df}{dV}\left(-\frac{x}{2t^{3/2}}\right) = -\frac{V}{2t}\frac{df}{dV}$

$\frac{\partial B}{\partial x} = \frac{df}{dV}\frac{1}{t^{1/2}}, \quad \frac{\partial^2 B}{\partial x^2} = \frac{d^2 f}{dV^2}\frac{1}{t}$

Thus ① becomes $-\frac{V}{2}\frac{df}{dV} = \eta \frac{d^2f}{dV^2}$

Put $\frac{df}{dV} = F$ and $V = U\sqrt{4\eta}$ $\Rightarrow$ $\frac{dF}{dU} = -2UF$

Integrate $F = \text{const}\ e^{-U^2}$

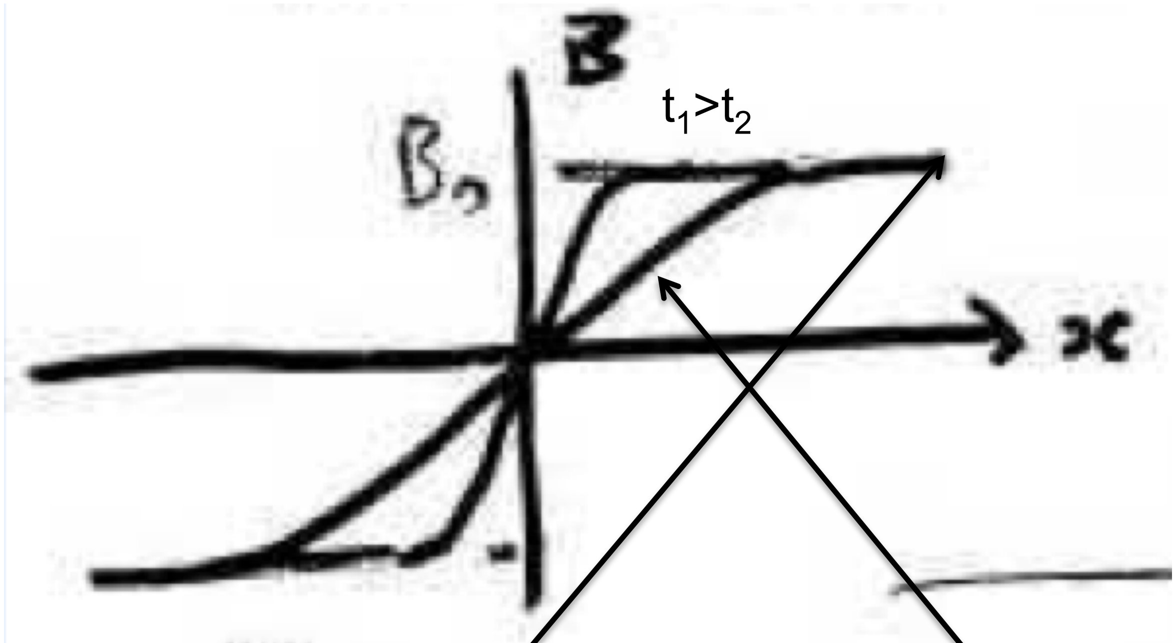
Integrate $f(U) = c\int_0^U e^{-u^2}\,du$

where $t \to 0$, $U \to \infty$ $\Rightarrow c = \frac{2B_0}{\sqrt{\pi}}$ from ②

i.e. $B = \frac{2B_0}{\sqrt{\pi}}\int_0^{x/\sqrt{4\eta t}} e^{-u^2}\,du$

$For\ x >> \sqrt{4\eta t},\quad B \approx B_0,$ *indep x*

$For\ x << \sqrt{4\eta t},\quad B \approx B_0\, x/\sqrt{\pi\eta t},$ *so slope ↓ with t*



$$For\ x >> \sqrt{4\eta t}, \quad B \approx B_0, \quad \textcolor{red}{indep\ x}$$

$$For\ x << \sqrt{4\eta t}, \quad B \approx B_0\, x / \sqrt{\pi \eta t}, \quad \textcolor{red}{so\ slope \downarrow with\ t}$$

MHD: ideal limit

The equations simplify in the so called "ideal MHD" when R>>1 or $\sigma \to \infty$

In such a case the energy equation turns back to the form of pure entropy conservation

$$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma} \Rightarrow \rho T \frac{Ds}{Dt} = 0 \Rightarrow \frac{Ds}{Dt} = 0$$

Euler eqn remains the same $\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}/c$

With the aid of $(c/4\pi)\vec{\nabla} \times \vec{B} = \vec{j}$ $\quad \rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$

Mass conservation law is the same $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

The field equation simplifies

$$\frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t} \Rightarrow \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t}$$

These are the MHD equations in the ideal limit

(b) If $R_m >> 1$

The induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

and Ohm's law -->

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

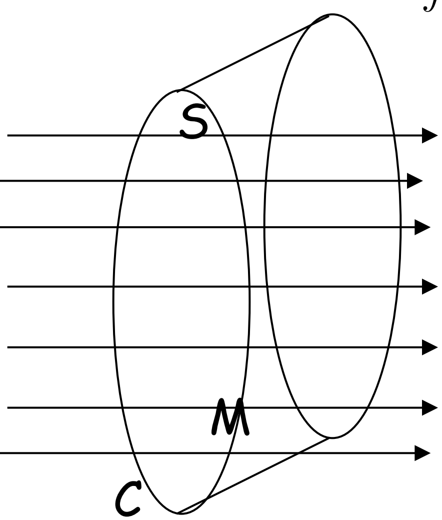
Magnetic field is "* frozen to the plasma *"

MHD: Frozen-in flux

Let assume that the B field is $B(r,t_o)$ at time t_o

The magnetic flux through a surface S enclosed by a curve C is

$$\Phi = \int_S \vec{B} \cdot d\vec{S}$$



if the curve moves, Φ changes because:
(a) the B field changes in time and (b) the field lines move into or out of S

As C moves, it creates a cylinder (a "flux tube") with a mantle surface M

Every field line leaving or entering C is associated with a flux of the SAME line through M

$$d\Phi = dt \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_M \vec{B} \cdot d\vec{S_M}$$

MHD

$$d\Phi = dt\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_M \vec{B}\cdot d\vec{S_M}$$

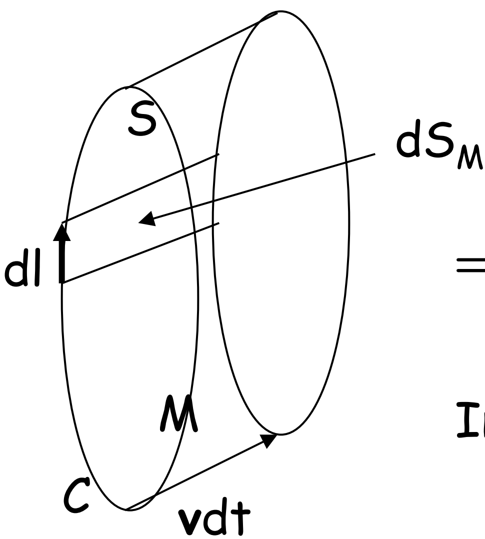
The surface element on M can be written as $d\vec{S_M} = \vec{v}\times d\vec{l}dt$

$$d\Phi = dt\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_C \vec{B}\cdot \vec{v}\times d\vec{l}dt$$

$$d\Phi/dt = \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_C \vec{B}\cdot \vec{v}\times d\vec{l}$$

$$\vec{B}\cdot\vec{v}\times d\vec{l} = -\vec{v}\times\vec{B}\cdot d\vec{l}$$

$$d\Phi/dt = \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} - \int_C \vec{v}\times\vec{B}\cdot d\vec{l}$$



$$= \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} - \int_S \vec{\nabla}\times\vec{v}\times\vec{B}\cdot d\vec{S}$$

With the Stokes' theorem for the last term

$$= \int_S [\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times\vec{v}\times\vec{B}]\cdot d\vec{S} = -\int_S \vec{\nabla}\times\vec{j}/\sigma\cdot d\vec{S} = -\int_C \vec{j}/\sigma\cdot d\vec{l}$$

In the convective limit, R_M >> 1 or S << 1, ie $\sigma \rightarrow \infty$

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times\vec{v}\times\vec{B} = 0 \Rightarrow d\Phi/dt = 0$$

The magnetic flux cannot change→ therefore field lines must be swept away with the plasma motion, ie B is tied to the particles in the element: FROZEN FIELD→ flux tube does not break when plasma shuffles around!

Frozen-in Condition

In astrophysical systems with high R_m, we can imagine the magnetic flux to be frozen in the plasma and to move with the plasma flows

Suppose we have straight magnetic field liens going through a plasma column

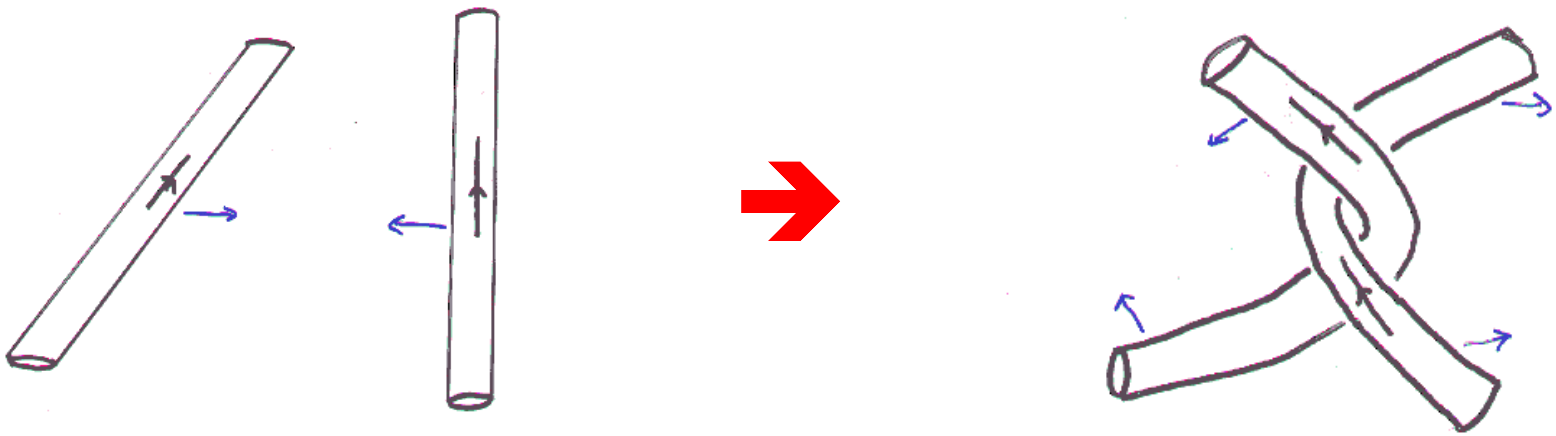
If the plasma column is bent, the magnetic field lines are bent too

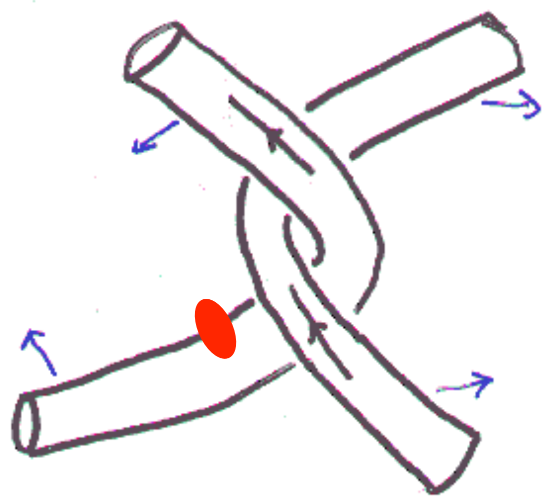
On the other hand, if one end of the plasma column is twisted (because, for istance, is in rotation), then the magnetic field lines are twisted too

As a result of frozen-in theorem, the B field in a astrophysical system can be almost regarded as a plastic material which can be bent, twisted or distorted by making the plasma move appropriately

This view of a magnetic field is radically different from that we encounter in laboratory situations, where it appears as something rather passive which we can switch on or off by sending a current through a coil. In the astrophysical setting, the magnetic field appears to acquire a life of its own

Magnetic Field Lines Can’t Break

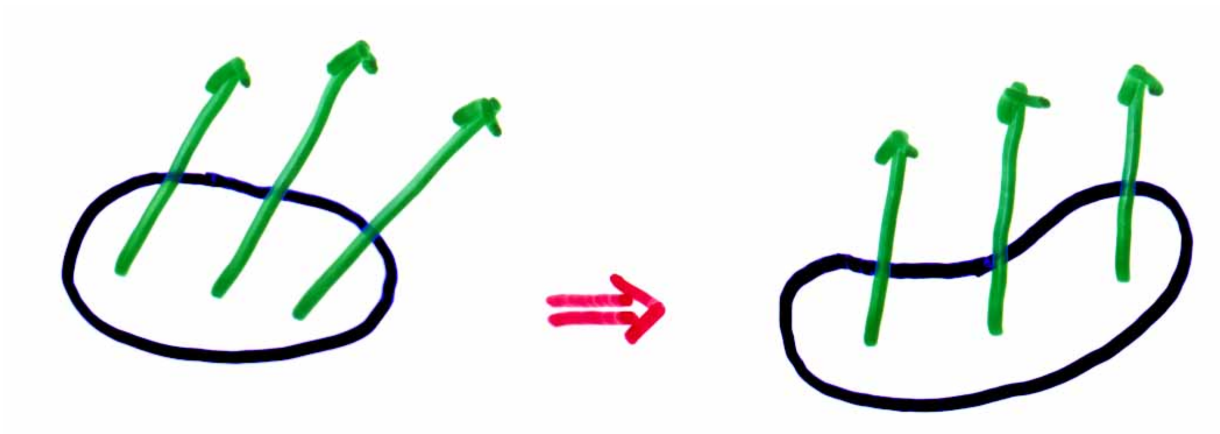




Everything Breaks Eventually

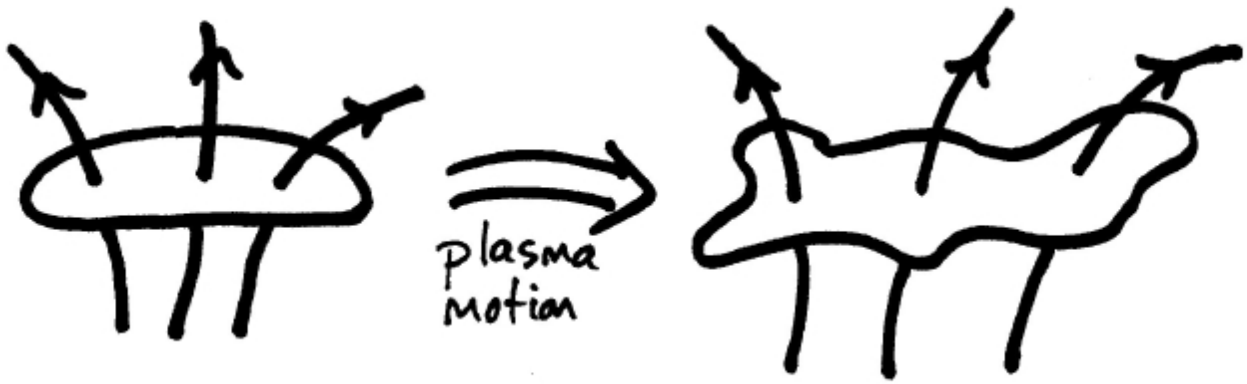
Frozen-in Condition

- In a simple form of plasma, the plasma moves so that the magnetic flux through any surface is preserved.

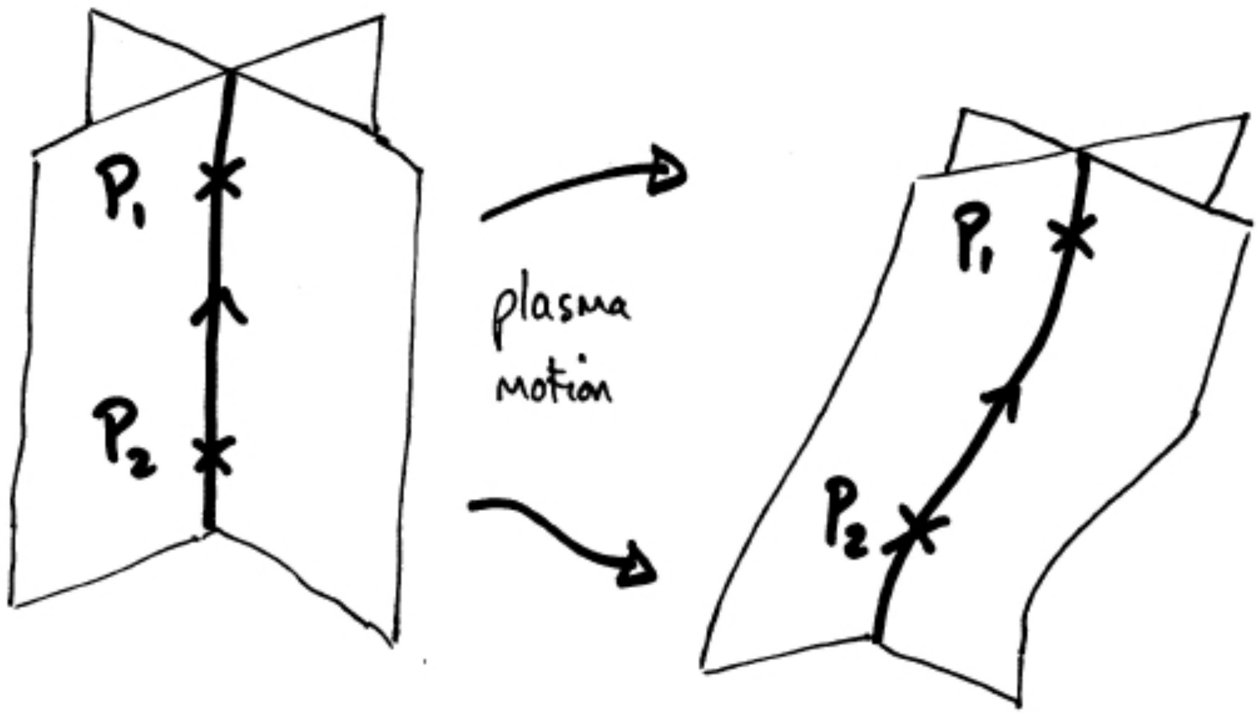


The frozen-in field condition is exactly the opposite of dissipation (ie magnetic diffusion), which tends to decrease the flux through the spire

Magnetic Flux Conservation:



Magnetic Field Line Conservation:



Frozen-in field: an example

When an astronomical object shrinks due to gravitational attraction, its magnetic field is expected to become stronger

If r is the radius of the equatorial cross-section of the body through which the the magnetic field of the order of B is passing, then the flux is of the order of Br^2

If the field is perfectly frozen, then this flux should remain an invariant during the contraction of the object

Some neutron stars are believed to have B fields of the order of 10^{12} G

Let us see if we can explain this magnetic field by assuming that the neutron star formed due to the collapse of an ordinary star of which the B field got compressed

A star like the Sun has a radius $r=10^{11}$ cm and B~ 10 G at poles

Since the radius of a typical NS is 10^6 cm, the equatorial area would decrease by a factor 10^{10} if an ordinary star were to collapse to become the NS

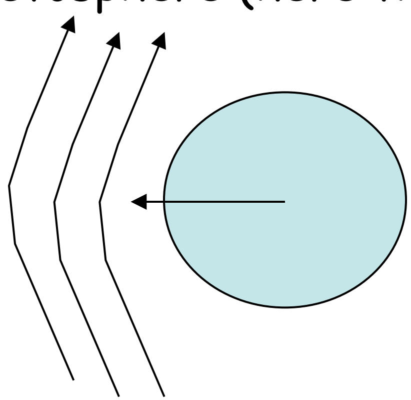
If the magnetic flux remained frozen during the collapse, the the initial field of 10 G would finally become 10^{11} G

MHD

$$d\Phi/dt = \int_S [\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{v} \times \vec{B}] \cdot d\vec{S} - \int_S \vec{\nabla} \times \vec{j}/\sigma \cdot d\vec{S} = -\int_C \vec{j}/\sigma \cdot d\vec{l}$$

A reversal of the concept, the frozen-out field, exists

If a field free plasma bubble moves toward a region filled with a magnetic field, then it pushes the field away until the pressures equals. The field cannot enter into the bubble because the magnetic flux inside the bubble would change. This happens with the solar wind frozen-out of the magnetosphere (here the field inside in the convective limit...$(R_M)_{mgtsph} \gg 1$



Magnetic force:

$$\mathbf{j} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \frac{\mathbf{B}}{\mu}$$

$$= (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{\mu} - \nabla \left(\frac{B^2}{2\mu} \right)$$

Magnetic field lines have a

Tension B^2/μ ----> force when lines curved

Pressure $B^2/(2\mu)$----> force from high to low B^2

Magnetic force(s)

The internal forces acting on a charged fluid are the mechanical pressure, already seen and the Lorentz force, which is new because due to the particle charge and the B field

$$\vec{f} = (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Using again the identity $(\vec{\nabla} \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla} B^2$

$$\vec{f} = \frac{1}{4\pi}[(\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla} B^2]$$

The first term can be rewritten

We note that it represents B times the derivative along the field line

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B\frac{\partial \vec{B}}{\partial s}$$

Where *s* is a curvilinear coordinate along the field line

Magnetic force(s) $\vec{f} = \frac{1}{4\pi}[(\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2]$

$(\vec{B}\cdot\vec{\nabla})\vec{B} = B\frac{\partial \vec{B}}{\partial s}$ Where *s* is a curvilinear coordinate along the field line

In general, B varies both in intensity and direction $\vec{B} = B\hat{b}$

With b is the local tangent versor to B

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = B\frac{\partial (B\hat{b})}{\partial s} = B^2\frac{\partial \hat{b}}{\partial s} + B\hat{b}\frac{\partial B}{\partial s}$$ but $\frac{\partial B^2}{\partial s} = 2B\frac{\partial B}{\partial s}$

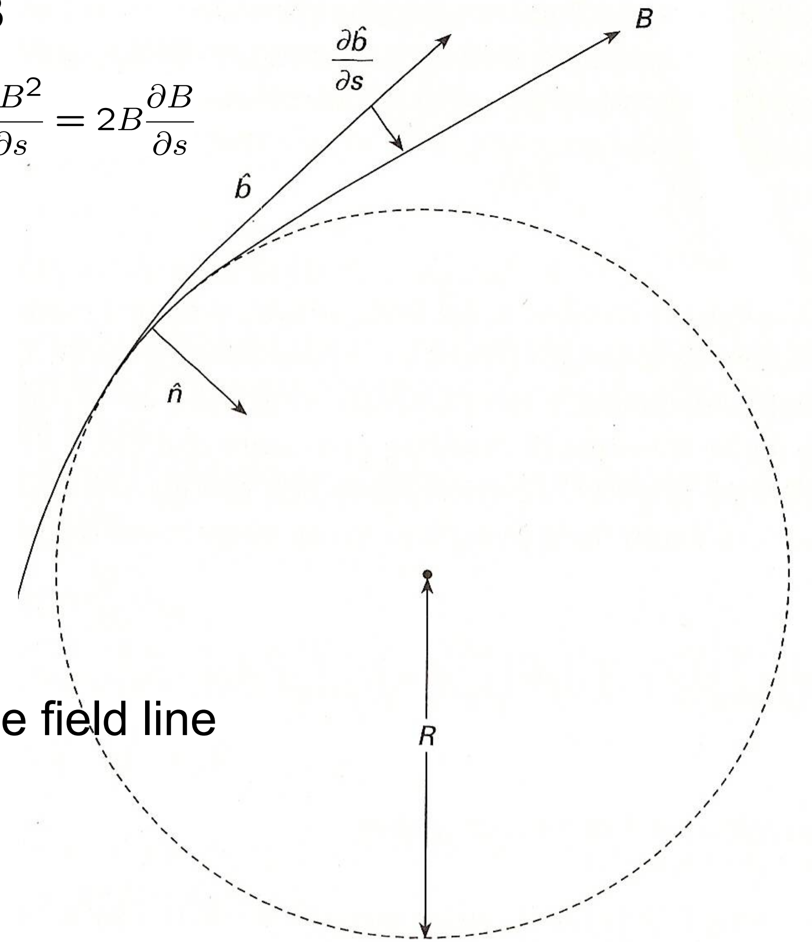
$$\Rightarrow \quad (\vec{B}\cdot\vec{\nabla})\vec{B} = B^2\frac{\partial \hat{b}}{\partial s} + \hat{b}\frac{\partial B^2}{2\partial s}$$

The first term is due to the change of the field direction. The versor derivative is, by definition, perp to the versor (ie field line) and measures the curvature of the field $\frac{\partial \hat{b}}{\partial s} = \frac{\hat{n}}{R}$

The 2nd term may rewritten as $\hat{b}\frac{\partial B^2}{2\partial s} = \hat{b}\vec{\nabla}_{||}(\frac{B^2}{2})$

Where $grad_{||}$ is the gradient component along the field line

$$\Rightarrow \quad (\vec{B}\cdot\vec{\nabla})\vec{B} = B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{||}\frac{B^2}{2}$$



Magnetic force(s)

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2}$$

Lorentz force is therefore $\vec{f} = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2} - \frac{1}{2}\vec{\nabla}B^2]$

Writing grad as $\vec{\nabla} = \hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel}$ We get

$$\vec{f} = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2} - (\hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel})\frac{B^2}{2}] = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} - \hat{n}\nabla_{\perp}\frac{B^2}{2}]$$

Parallel component of the gradient cancels with the corresponding pressure term, leaving only the perp component: both terms give forces normal to the field line (as it is reasonable to expect since the Lorentz force is ~ vxB

Magnetic force(s)

$$\vec{f} = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel}\frac{B^2}{2} - (\hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel})\frac{B^2}{2}] = \frac{1}{4\pi}[B^2\frac{\hat{n}}{R} - \hat{n}\nabla_{\perp}\frac{B^2}{2}]$$

The first term represents a force directed along the istantaneous center of curvature: it tends therefore to shorten the field line

This term is very similar to the elastic force exerted on a guitar string when is tweaked: it exerts a force that tends to bring the string back to its minimum length

That's why it's called magnetic tension

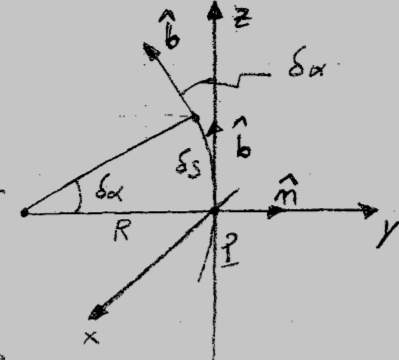
The 2nd term is the gradient of a scalar quantity→ $B^2/2$ can be interpreted as a magnetic pressure, opposite to the gradient: the more the field is increasing along perp dir, the more is the force that tends to decrease the field...in terms of field lines this means when you try to compress field lines, there is a force which opposes to the compression

Curvature can be eliminated since

$$\vec{\nabla}_\perp B = -\frac{B}{R}\hat{m}$$

In P, $\vec{B}(0,0,B)$ e $\boxed{\frac{\partial B_x}{\partial z}=0}$

Due to osculator circle in P



$\vec{\nabla}\times\vec{B}=0$

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=0 \qquad \boxed{-\frac{\partial B_z}{\partial x}+\frac{\partial B_x}{\partial z}=0} \qquad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=0$$

$$\boxed{\frac{\partial B_z}{\partial x}=0}$$

$$\rightarrow \vec{\nabla}B = \frac{\partial}{\partial x}\left[\left(B_x^2+B_y^2+B_z^2\right)^{1/2}\right] = \frac{2B_x\partial_x B_x + 2B_y\partial_x B_y + 2B_z\partial_x B_z}{2B} = \left(\frac{B_z}{B}\right)\frac{\partial B_z}{\partial x} = 0$$

($\frac{B_z}{B}=1$)

$$\frac{\partial}{\partial y}\left[\left(\cdots\right)^{1/2}\right] = \frac{B_x\partial_y B_x + B_y\partial_y B_y + B_z\partial_y B_z}{B} = \frac{B_z}{B}\frac{\partial B_z}{\partial y} = \frac{\partial B}{\partial y}$$

$$\frac{\partial}{\partial z}\left[\left(\cdots\right)^{1/2}\right] = \frac{\partial B}{\partial s} = (\hat{b}\cdot\vec{\nabla})B \equiv \nabla_\parallel B$$

Gradient lies in the yz-plane

But $\vec{\nabla}=\vec{\nabla}_\parallel+\vec{\nabla}_\perp \Rightarrow \nabla_\perp B = \frac{\partial B}{\partial y} \Rightarrow \vec{\nabla}B = \nabla_\parallel B\,\hat{b} + \nabla_\perp B\,\hat{n}$

The trasverse comp (B_y) changes when moving along field line

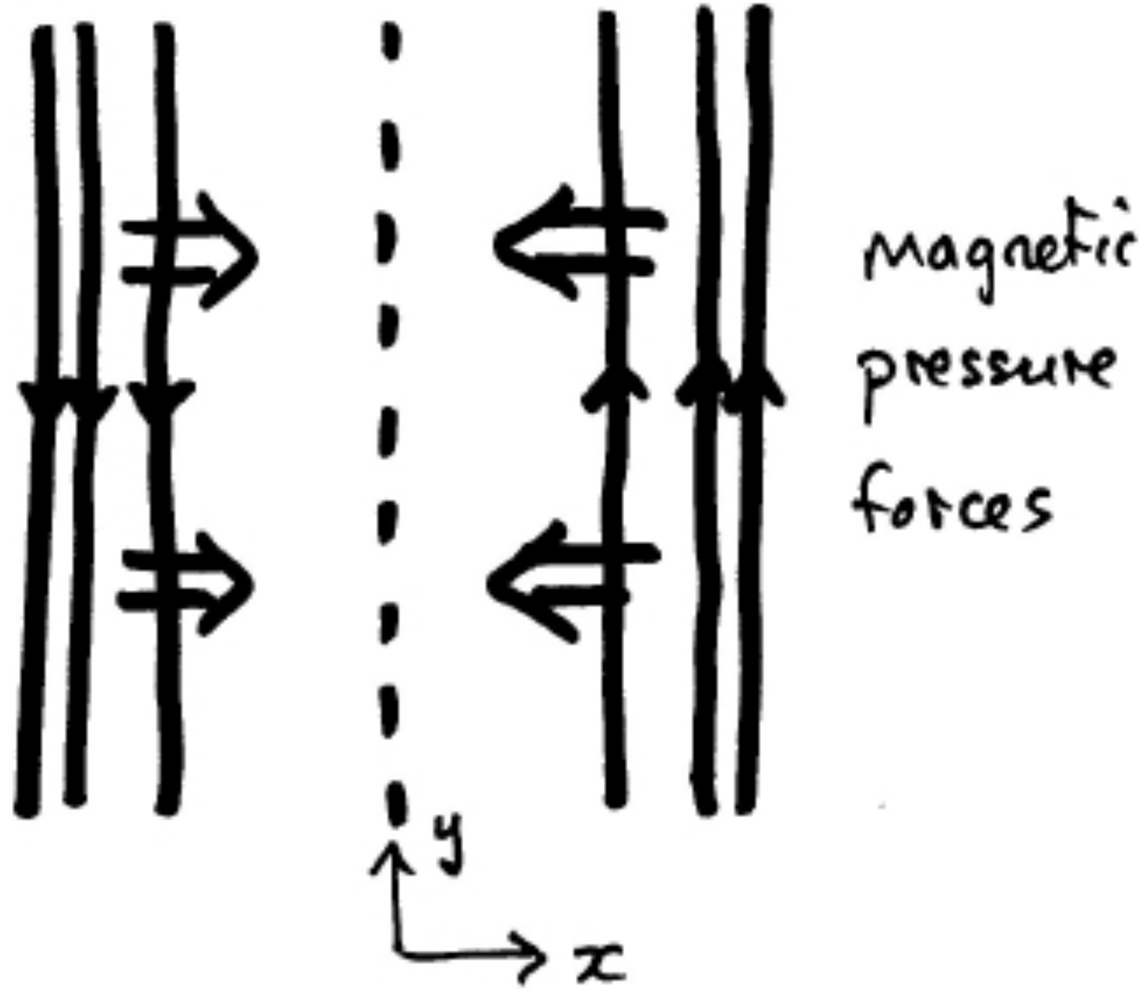
$$dB_y \approx -B\sin\delta\alpha \approx -B\,\delta\alpha \approx -B\frac{\delta s}{R} \Rightarrow \frac{\partial B_y}{\partial s} = \frac{\partial B_y}{\partial z} \approx -\frac{B}{R}$$

But $\frac{\partial B_y}{\partial z}=\frac{\partial B_z}{\partial y}$ from $\vec{\nabla}\times\vec{B}=0 \Rightarrow \frac{\partial B_z}{\partial y}\approx -\frac{B}{R}$

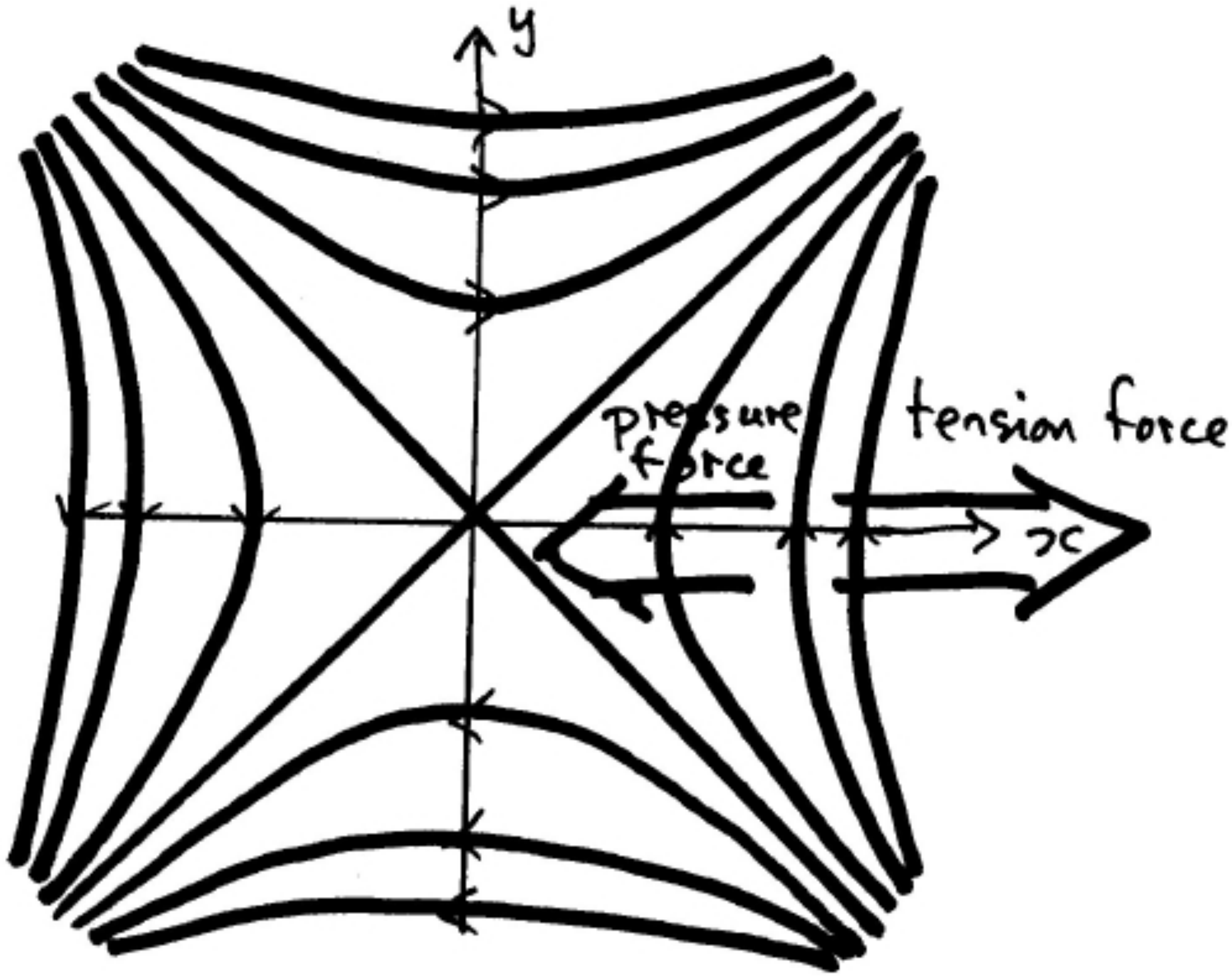
$\Rightarrow \nabla_\perp B \approx -\frac{B}{R}$ or in vector form $\frac{\vec{\nabla}_\perp B}{B} = -\frac{\hat{n}}{R}$ (with $\vec{\nabla}_\perp = \hat{m}\nabla_\perp$)

Ex.

$$\mathbf{B} = x\,\hat{\mathbf{y}}$$



Ex. $\mathbf{B} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$



Conservative form

We have already seen that the equation of hydrodynamics can be written in a manifestly conservative form

Einstein convention adopted

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\,\boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi = 0$$

or in components

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i(\rho v_i v_j + p\delta ij) = 0$$

The tensor $\mathbf{R_{ik}} = \boldsymbol{\rho V_i V_k + p\delta_{ik}}$ is the Reynolds stress tensor for an ideal fluid and represents the mechanical momentum flux $\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial R_{ik}}{\partial x_k}$

This is possible for MHD too

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + (1/4\pi)(\vec{\nabla}\times\vec{B})\times\vec{B}$$

Make use of the identity $(\vec{\nabla}\times\vec{B})\times\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2$

The double vector prod can be written as the divergence of a tensor

$$[(\vec{\nabla}\times\vec{B})\times\vec{B}]_i = B_j\nabla_j B_i - \frac{1}{2}\nabla_i B^2$$

Now $\nabla_j(B_jB_i) = B_i\nabla_j B_j + B_j\nabla_j B_i$ but $\nabla_j B_j = \vec{\nabla}\cdot\vec{B} = 0$

$\Rightarrow$ $\nabla_j(B_jB_i) = B_j\nabla_j B_i$ $\Rightarrow$ $[(\vec{\nabla}\times\vec{B})\times\vec{B}]_i = \nabla_j(B_jB_i) - \frac{1}{2}\nabla_i B^2 = \nabla_j(B_jB_i) - \nabla_j\frac{1}{2}B^2\delta_{ij}$

$$= \nabla_j[(B_jB_i) - \frac{1}{2}B^2\delta_{ij}]$$

$\Rightarrow$ $$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + (1/4\pi)\nabla_j[(B_jB_i) - \frac{1}{2}B^2\delta_{ij}]$$

Conservative form

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i(\rho v_i v_j + p\delta ij) = 0 \qquad \mathbf{R_{ik} = \rho V_i V_k + p\delta_{ik}} \qquad \frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial R_{ik}}{\partial x_k}$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \frac{1}{4\pi}\nabla_j[(B_j B_i) - \frac{1}{2}B^2\delta_{ij}]$$

The Euler eqn can be put therefore in the same form as for hydrodynamics case defining

$$M_{ij} = -\frac{1}{4\pi}[(B_j B_i) - \frac{1}{2}B^2\delta_{ij}]$$

And $T_{ij} = R_{ij} + M_{ij} = \rho v_i v_j + p\delta ij - \frac{1}{4\pi}[(B_j B_i) - \frac{1}{2}B^2\delta_{ij}]$ ⇒ $\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial T_{ik}}{\partial x_k}$

M_{ij} describes flux per time unit of the i-th component of the B field momentum through a surface oriented along the j-th direction and it's called Maxwell stress tensor

The relative importance in concrete astrophysical situations, as pulsar winds and black hole jets, of the mechanical and magnetic momentum flux is, at the moment, uncertain and of great interest

Conservative form

It is well known that elm field transports energy too

The energy density of the field is $\epsilon_B = B^2/8\pi$

The energy flux is described by Poynting vector $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}$

We have to eliminate the E field

Consider the Ohm's law $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}/c)$

In the ideal limit $\sigma \rightarrow \infty$, to avoid infinite currents we must have

$$\vec{E} + \vec{v} \times \vec{B}/c = 0$$

$\Longrightarrow$ $\vec{E} = -\vec{v} \times \vec{B}/c$ $\Longrightarrow$ $\vec{S} = -\frac{1}{4\pi}(\vec{v} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}\vec{B} \times (\vec{v} \times \vec{B})$

Conservative form

$\epsilon_B = B^2/8\pi$

$$\vec{S} = -\frac{1}{4\pi}(\vec{v}\times\vec{B})\times\vec{B} = \frac{1}{4\pi}\vec{B}\times(\vec{v}\times\vec{B})$$

We have to add these terms into the energy equation of the hydro case

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\text{eff}}$$

h=ε + P/ρ

$$\mathcal{H}_{\text{eff}} \equiv \mathcal{H} + \rho\,\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2+\rho e+\rho\Phi+\frac{1}{8\pi}B^2)+\vec{\nabla}\cdot[\vec{v}(\frac{1}{2}\rho v^2+\rho h+\rho\Phi)+\frac{1}{4\pi}\vec{B}\times(\vec{v}\times\vec{B})] = H_{eff}$$

With no irreversible losses and no time dependent gravitational potentail H_{eff}=0

Summary in the ideal limit

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

Mass conservation law

M density M flux

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i[\rho v_i v_j + p\delta ij - \frac{1}{4\pi}(B_j B_i - \frac{1}{2}B^2 \delta_{ij})] = 0$$

Momentum conservation law

p density p flux

Energy conservation law

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2 + \rho e + \rho \Phi + \frac{1}{8\pi}B^2) + \vec{\nabla} \cdot [\vec{v}(\frac{1}{2}\rho v^2 + \rho e + P + \rho \Phi) + \frac{1}{4\pi}\vec{B} \times (\vec{v} \times \vec{B})] = 0$$

E density E flux

Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.
-> Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.
-> Then all variables can be expressed by one, say the magnetic field.

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys:

$$|\delta \mathbf{B}| \ll B_0$$

In the uniform plasma with straight field lines, the field provides the only ***symmetry axis*** which may be chosen as *z*-axis of the coordinate system such that: $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_{||}$.

Small perturbations

We will treat the small perturbations of a magnetohydrodynamical equilibrium

Let choose an equilibrium configuration of an omogeneous, infinite medium at rest for sake of simplicity (but the method can be applied to any equilibrium configuration, even non omogeneous and time dependent)

Let therefore consider a configuration in which $p=p_o$, $\rho=\rho_o$, $\mathbf{v}=0$ and $\mathbf{B}=B_o\mathbf{z}$

Consider now a situation in which the quantities are perturbed from equilibrium, that is $p=p_o+\delta p$, $\rho=\rho_o+\delta\rho$, $v=\delta v$ and $B=B_oz+\delta B$

Our goal is find perturbations corresponding to acoustic waves, therefore let assume that perturbations are iso-entropic, so that we have a relation between pressure and density:

$$\delta p=c^2_s\delta\rho$$

With $c_s=\gamma\ p_o/\rho_o$ iso-entropic sound speed

Small perturbations

The motion equations, under the assumption that perturbations are small, can be linearized

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 & \text{(i)} \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\dfrac{\vec{\nabla}p}{\rho} + \dfrac{1}{4\pi\rho}(\vec{\nabla}\times\vec{B})\times\vec{B} & \text{(ii)} \\ \dfrac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) & \text{(iii)} \end{cases}$$

$$\rho = \rho_0 + \delta\rho \qquad p = p_0 + \delta p \qquad \vec{v} = \delta\vec{v} \qquad \vec{B} = B_0\hat{z} + \delta\vec{B}$$

$$\delta\rho \ll \rho_0 \qquad \delta p \ll p_0 \qquad \delta\vec{v} \ll 1 \qquad \delta\vec{B} \ll B_0\hat{z}$$

$$\delta p = c_s^2\,\delta\rho \qquad c_s^2 = \frac{\gamma p_0}{\rho_0}$$

$$\text{(i)}\quad \frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \vec{\nabla}\cdot\left[(\rho_0 + \delta\rho)\delta\vec{v}\right] = \frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v} + \vec{\nabla}\cdot(\delta\rho\,\delta\vec{v}) \approx \frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_0}\right) \approx -\vec{\nabla}\cdot\vec{v}$$

$$\text{(ii)}\quad \frac{\partial\delta\vec{v}}{\partial t} + (\delta\vec{v}\cdot\vec{\nabla})\delta\vec{v} = -\frac{\vec{\nabla}(p_0 + \delta p)}{(\rho_0 + \delta\rho)} + \frac{1}{4\pi(\rho_0 + \delta\rho)}\left[\vec{\nabla}\times(B_0\hat{z} + \delta\vec{B})\right]\times(B_0\hat{z} + \delta\vec{B})$$

$$\frac{\partial\delta\vec{v}}{\partial t} \approx -\frac{\vec{\nabla}\delta p}{\rho_0 + \delta\rho} + \frac{1}{4\pi(\rho_0 + \delta\rho)}\Big[\underbrace{(\vec{\nabla}\times B_0\hat{z})}_{=0}\times B_0\hat{z} + \underbrace{(\vec{\nabla}\times B_0\hat{z})}_{=0}\times\delta\vec{B} + (\vec{\nabla}\times\delta\vec{B})\times B_0\hat{z} + \underbrace{(\vec{\nabla}\times\delta\vec{B})\times\delta\vec{B}}_{\ll 1}\Big] \approx$$

$$\approx -\frac{\vec{\nabla}\delta p}{\rho_0} + \frac{1}{4\pi\rho_0}(\vec{\nabla}\times\delta\vec{B})\times B_0\hat{z} = -c_s^2\,\vec{\nabla}\left(\frac{\delta\rho}{\rho_0}\right) + \frac{1}{4\pi\rho_0}(\vec{\nabla}\times\delta\vec{B})\times B_0\hat{z}$$

$$\text{(iii)}\quad \frac{\partial\delta\vec{B}}{\partial t} = \vec{\nabla}\times\left[\delta\vec{v}\times(B_0\hat{z} + \delta\vec{B})\right] = \vec{\nabla}\times(\delta\vec{v}\times B_0\hat{z}) + \vec{\nabla}\times(\delta\vec{v}\times\delta\vec{B}) \approx \vec{\nabla}\times(\delta\vec{v}\times B_0\hat{z})$$

(iv) Ds/Dt=0 in automatically satisfied since entropy is assumed constant

The magnetic field equation can be furtherly simplified noting that Bo has null all the spatial derivatives. Using the vectorial identity:

$$\vec{\nabla}\times(\delta\vec{v}\times\vec{B}) = (\vec{B}\cdot\nabla)\delta\vec{v}-(\delta\vec{v}\cdot\nabla)\vec{B}-(\vec{B}\nabla)\cdot\delta\vec{v}+\delta\vec{v}(\nabla\cdot\vec{B})$$

The terms grad(B)=0 and (v•grad)B= 0

$$= (\vec{B}\cdot\nabla)\delta\vec{v} - (\vec{B}\nabla)\cdot\delta\vec{v} \quad = B_o\frac{\partial}{\partial z}\delta\vec{v} - B_o\hat{z}\nabla\cdot\delta\vec{v}$$

$$\frac{\partial}{\partial t}\delta\vec{B} = B_o\frac{\partial}{\partial z}\delta\vec{v} - B_o\hat{z}\nabla\cdot\delta\vec{v}$$

Small perturbations

$$\frac{\partial}{\partial t}(\frac{\delta\rho}{\rho_o}) + \nabla\cdot\delta\vec{v} = 0$$

$$\frac{\partial\vec{v}}{\partial t} = -c_s^2\nabla(\frac{\delta\rho}{\rho_o}) + \frac{1}{4\pi\rho_o}(\nabla\times\delta\vec{B})\times B_o\hat{z}$$

$$\frac{\partial\delta\vec{B}}{\partial t} - \vec{\nabla}\times(\delta\vec{v}\times\vec{B_o}\hat{z}) = 0$$ or $$\frac{\partial}{\partial t}\delta\vec{B} = B_o\frac{\partial}{\partial z}\delta\vec{v} - B_o\hat{z}\nabla\cdot\delta\vec{v}$$

This is a system of partial derivative equations

Since the coefficients are constant we can develop the unknowns in Fourier series and find a solution for each Fourier amplitude

Derive wrt t the momentum equation

$$\frac{\partial^2\vec{v}}{\partial t^2} = -c_s^2\nabla(\frac{\partial}{\partial t}[\frac{\delta\rho}{\rho_o})] + \frac{1}{4\pi\rho_o}(\nabla\times\frac{\partial}{\partial t}\delta\vec{B})\times B_o\hat{z}$$

Use 1 and 3 to eliminate time deriv in RHS

$$\frac{\partial^2\vec{v}}{\partial t^2} = c_s^2\nabla(\nabla\cdot\delta\vec{v}) + \frac{1}{4\pi\rho_o}(\nabla\times(\vec{\nabla}\times(\delta\vec{v}\times B_o\hat{z})))\times B_o\hat{z}$$

Introducing the Alven speed $\vec{v}_A \equiv \frac{\vec{B}_o}{\sqrt{4\pi\rho_o}}$ We get

$$\frac{\partial^2\vec{v}}{\partial t^2} = c_s^2\nabla(\nabla\cdot\delta\vec{v}) + (\nabla\times(\vec{\nabla}\times(\delta\vec{v}\times\vec{v}_A)))\times\vec{v}_A$$

This is the wave equation for MHD

Small perturbations

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_s^2 \nabla(\nabla \cdot \delta\vec{v}) + (\nabla \times (\vec{\nabla} \times (\delta\vec{v} \times \vec{v}_A))) \times \vec{v}_A$$

$$\vec{v}_A \equiv \frac{\vec{B}_o}{\sqrt{4\pi\rho_o}}$$

In absence of B field, it reduces to the acoustic wave equation:
Infact taking 1-dim case, ie δ**v** directed along the x-axis (δv,0,0), we get

$$\nabla \cdot \delta\vec{v} = \frac{\partial v}{\partial x} \qquad \nabla = \frac{\partial}{\partial x} \qquad \Rightarrow \qquad \frac{\partial^2 v}{\partial t^2} = c_s^2 \frac{\partial^2 v}{\partial x^2}$$

When B field is present, the situation is much more complicated:
i) There are two speeds involved in the wave propagation, c_s and v_A
ii) There are waves types which are not acoustic, since a speed completely dependent on the unperturbed B field appears

Small perturbations

Therefore let look for solutions of type

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi} \qquad \delta\vec{v} = \vec{V}e^{i\Phi} \qquad \frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \qquad \Phi = \vec{k}\cdot\vec{r} - \omega t$$

With k and ω completely arbitrary at this moment

Inserting these solutions into the equations we get

$$\omega r = \vec{k}\cdot\vec{V}$$

$$\omega\vec{V} = c_s^2 r\vec{k} - \frac{(\vec{k}\times\vec{b})\times\hat{z}B_o^2}{4\pi\rho_o}$$

$$\omega\vec{b} = -k_z\vec{V} + \hat{z}\vec{k}\cdot\vec{V}$$

In this way the coupled equation system becomes a simple omogeneous linear system in the unknowns r,***V*** and ***b***

The system admits only the trivial solution r=***V***=***b*** =0, unless the determinant of the system is zero ➔ the condition for non zero solutions to exist is that det=0

Small perturbations

It is convenient to decompose the wave vector ***k*** in its components || and perp to ***B**$_o$* (directed along z dir) $\vec{k} = k_{\parallel}\hat{z} + k_{\perp}\hat{y}$

After a little bit of algebra, we find the determinant of the system

$$\det\begin{pmatrix} \omega^2 - v_A^2 k_{\parallel}^2 & 0 & 0 \\ 0 & \omega^2 - v_A^2 k_{\parallel}^2 - (c_s^2 + v_A^2) k_{\perp}^2 & -c_s^2 k_{\parallel} k_{\perp} \\ 0 & -c_s^2 k_{\parallel} k_{\perp} & \omega^2 - c_s^2 k_{\parallel}^2 \end{pmatrix} = 0$$

$$(\omega^2 - v_A^2 k_{\parallel}^2)(\omega^4 - \omega^2 (c_s^2 + v_A^2) k^2 + c_s^2 v_A^2 k_{\parallel}^2 k^2) = 0$$

This establishes a relation between the wave frequency ω and the wave vector k, which is called dispersion relation

$v_f = \omega/k$ is not the real wave speed (but only the phase speed), which is the group speed $v_g = \partial\omega/\partial k$

There is a dispersion (ie v depends on ω) every time phase and group velocity differ

Small perturbations

Occasionally it may happens that, for a given k, ω has an imaginary part (ie is complex)

Remembering that all the physical quantities depend on time as $e^{i\omega t}$, this means that the phenomenon either is strongly dumped or is strongly amplified

When even only one solution is amplified, one speaks of instabilities of the 0th order solutions: perturbations, even if initially small, tend to increase without bound with time

This is not the case for previous relation dispersion

Small perturbations

$$(\omega^2 - v_A^2 k_{\parallel}^2)(\omega^4 - \omega^2 (c_s^2 + v_A^2) k^2 + c_s^2 v_A^2 k_{\parallel}^2 k^2) = 0$$

This equation has three distinct solutions in ω^2 (ie six independent solutions in ω) but the pure k^2 dependence in the dispersion relation tells us that couniugated solutions +- ω represent the same wave propagating in the +B and −-B directions

The first is $\omega^2 = v_A^2 k_{||}$ and it is called Alfven's wave

Its peculiarity is that it is transmitted exclusively along the B field direction with the characteristic speed v_A, as shown by the presence of the || k component only in the spatial part

The 2nd and 3rd are $\omega^2 = \frac{k^2}{2}(c_s^2 + v_A^2 \pm \sqrt{\frac{(c_s^2 - v_A^2)^2 + 4c_s^2 v_A^2 k_\perp^2}{k^2}})$

Called magneto-sonic waves, fast or slow, depending on the sign in front of square root

The fact that the dispersion relation is even in k, that depends exclusively on k^2 tells us that the two propagation directions are completely equivalent

Alfvén waves

Inspection of the determinant shows that the fluctuation in the *y*-direction decouples from the other two components and has the linear dispersion

$$\omega_A = \pm k_{\|} v_A$$

This ***transverse wave*** travels parallel to the field. It is called ***shear Alfvén wave***. It has no density fluctuation and a constant group velocity, $\mathbf{v}_{gr,A} = \mathbf{v}_A$, which is always oriented along the background field, along which the wave energy is transported.

*The transverse velocity and **magnetic field components are (anti)-correlated** according to:* $\delta v_y/v_A = \pm\, \delta B_y/B_0$, *for **parallel (anti-parallel) wave propagation**. The wave electric field points in the x-direction:* $\delta E_x = \delta B_y/v_A$

Magnetosonic waves

The remaing four matrix elements couple the fluctuation components, $\delta v_{||}$ and $\delta v_{\perp}$. The corresponding determinant reads:

$$\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k^2 k_{\|}^2 = 0$$

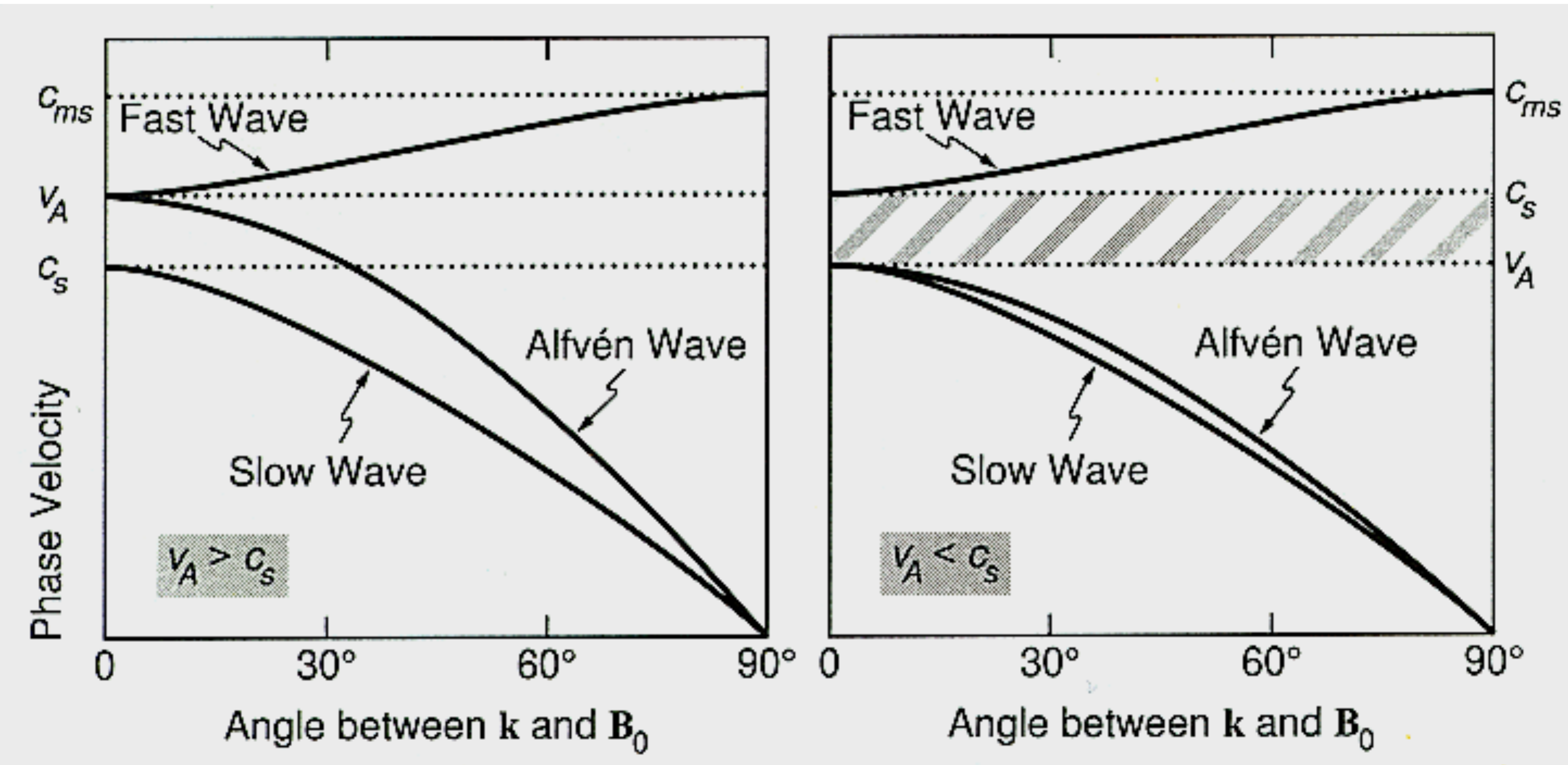
This bi-quadratic equation has the roots:

$$\omega_{ms}^2 = \frac{k^2}{2}\left\{c_{ms}^2 \pm \left[\left(v_A^2 - c_s^2\right)^2 + 4 v_A^2 c_s^2 \frac{k_\perp^2}{k^2}\right]^{1/2}\right\}$$

which are the phase velocities of the compressive ***fast and slow magnetosonic waves***. They depend on the propagation angle θ, with $k_\perp^2/k^2 = sin^2\theta$. For $\theta = 90^0$ we have: $\omega = kc_{ms}$, and $\theta = 0^0$:

$$\omega^2 = \frac{1}{2}k^2\left[c_s^2 + v_A^2 \pm \left(c_s^2 - v_A^2\right)\right]$$

Dependence of phase velocity on propagation angle



Magnetosonic wave dynamics

In order to understand what happens physically with the dynamic variables, δv_x, δB_x, $\delta B_{||}$, $\delta v_{||}$, δp, and δn, inspect again the equation of motion written in components:

$$\omega \delta \mathbf{v} = \frac{\mathbf{k}}{m_i n_0}\left(\delta p + \frac{1}{\mu_0}\mathbf{B}_0 \cdot \delta \mathbf{B}\right) - \frac{\mathbf{k}\cdot\mathbf{B}_0}{\mu_0 m_i n_0}\delta \mathbf{B}$$

Parallel direction: $$\omega v_{\|} = \frac{k_{\|}\delta p}{m_i n_0}$$

Parallel pressure variations cause parallel flow.

Oblique direction: $$\omega\left(k_{\|}v_{\|} + k_{\perp}v_x\right) = \frac{k^2 \delta p_{tot}}{m_i n_0}$$

Total pressure variations ($p_{tot}=p+B^2/2\mu_0$) accelerate (or decelerate) flow, for in-phase (or out-of-phase) variations of δp and δB, *leading to the* ***fast and slow mode waves.***

Small perturbations

$\omega^2 = v_A^2 k_{\|}$ Alfven's wave

$$\omega^2 = \frac{k^2}{2}(c_s^2 + v_A^2 \pm \sqrt{\frac{(c_s^2 - v_A^2)^2 + 4c_s^2 v_A^2 k_\perp^2}{k^2}})$$

magneto-sonic waves, fast or slow

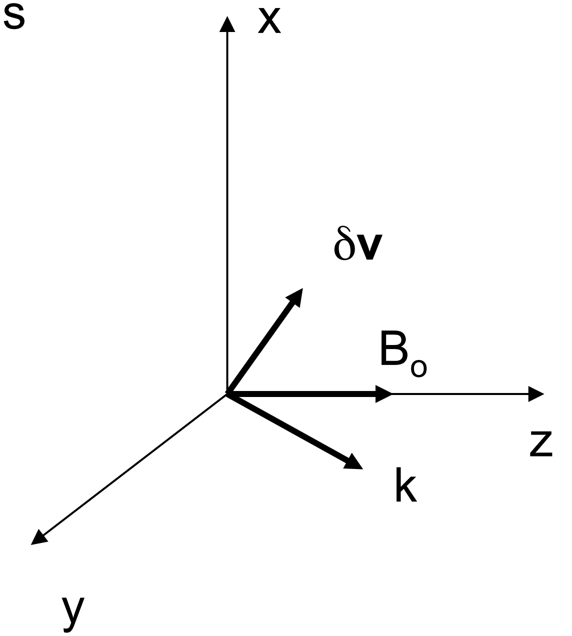
Since there are three independent solution there must be three distinct modes of oscillation

In principle, these modes could be identified finding the solutions, linearly independent and orthogonal of the system, but there is a simpler, more physical way to follow

Small perturbations

Let choose a suitable reference frame: z axis oriented along the B field, k in the yz plane (this simplifies the representation with no loss of generality

The speed perturbation has three components



With the reference choice k_n lies in the xy plane and $k_{||}$ is along z-axis

Small perturbations

Let consider the perturbed momentum equation $\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla (\frac{\delta \rho}{\rho_o}) + \frac{1}{4\pi \rho_o} (\nabla \times \delta \vec{B}) \times B_o \hat{z}$

Get the z component, that is along the unperturbed field direction and put the solutions in

$$\frac{\delta \rho}{\rho_o} = r e^{i\Phi} \qquad \delta \vec{v} = \vec{V} e^{i\Phi} \qquad \frac{\delta \vec{B}}{B_o} = \vec{b} e^{i\Phi} \qquad \Phi = \vec{k} \cdot \vec{r} - \omega t$$

We find that $\omega \delta v_z = k_z \frac{\delta p}{\rho_o}$ (the double vector prod is by construction perp to z)

So, every perturbation of the speed along the field direction is exclusively due to pressure gradient and is not connected to any perturbation of the B field and can be considered as an acoustic wave

Let consider now the y comp of the equation $\omega \delta B_z = k_\perp \delta p + \frac{B_o}{4\pi} (k_\perp \delta B_z - k_\parallel \delta B_y)$

The div equation divB=0 for the field becomes $k_\parallel \delta B_z = -k_\perp \delta B_y$

Small perturbations

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi}$$

$$\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla(\frac{\delta\rho}{\rho_o}) + \frac{1}{4\pi\rho_o}(\nabla \times \delta\vec{B}) \times B_o\hat{z}$$

$$\frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \qquad \delta\vec{v} = \vec{V}e^{i\Phi}$$

$$\Phi = \vec{k}\cdot\vec{r} - \omega t$$

Z comp

$$\omega\delta v_z = k_z \frac{\delta p}{\rho_o}$$

Y comp

$$\omega\delta B_z = k_\perp \delta p + \frac{B_o}{4\pi}(k_\perp \delta B_z - k_\parallel \delta B_y)$$

$$k_\parallel \delta B_z = -k_\perp \delta B_y$$

Insert this condition in y comp eqn

$$\omega\rho_o k_\perp \delta v_y = k_\perp^2 \delta p + k^2 \frac{B_o \delta B_z}{4\pi} = k_\perp^2 \delta p + k^2 \frac{\delta B_z^2}{8\pi}$$

Use now the z comp eqn and finally we get

$$\vec{k}\cdot\delta\vec{v} = \frac{k^2}{\omega\rho_o}\delta(p + B^2/8\pi)$$

From this we see this wave has as restoring force the total pressure of the fluid, that is mechanical + magnetic pressure

For this reason they are called magneto-sonic waves

The reason why there are slow and fast modes depends on the fact that p and $B^2/8\pi$ may have signs equal or opposite: when with same sign, the restoring force is greater and the waves are faster, when with opposite signs the two restoring forces δp and δB^2 tend to cancel and the wave will propagate more slowly

Small perturbations

$$\frac{\delta\rho}{\rho_o} = re^{i\Phi}$$

$$\frac{\delta\vec{B}}{B_o} = \vec{b}e^{i\Phi} \qquad \delta\vec{v} = \vec{V}e^{i\Phi}$$

$$\Phi = \vec{k}\cdot\vec{r} - \omega t$$

$$\frac{\partial\vec{v}}{\partial t} = -c_s^2\nabla(\frac{\delta\rho}{\rho_o}) + \frac{1}{4\pi\rho_o}(\nabla\times\delta\vec{B})\times B_o\hat{z}$$

To examine the 3rd wave type, consider the 3rd component of the motion equation

This is directed along x direction, perp B_o and k

$$\omega\rho_o\delta v_x = -\delta B_x\frac{k_\parallel B_o}{4\pi}$$

From this we see that the pressure is not involved at all in the Alfven wave, neither the mechanical or the magnetic pressure: the restoring force is entirely due to the magnetic tension and the wave is transverse

(i) Sound Waves ($B_0 = 0$)

- Uniform medium of pressure p_0, density ρ_0
- Disturbance v=$\mathbf{v_1}$, $p = p_0 + p_1$, $\rho = \rho_0 + \rho_1$
- Linearise eqns motion, continuity, energy

$$(p / \rho^{\gamma} = c)$$

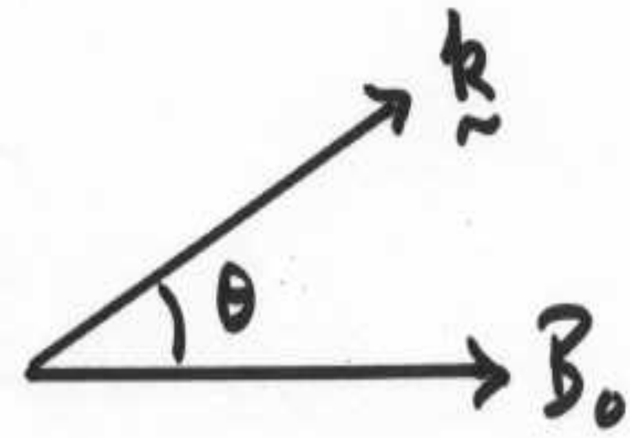
- Fourier analyse $\mathbf{v_1}, p_1, \rho_1 \approx \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$

$\rightarrow$ $$\boxed{\omega^2 = k^2 c_s^2}$$ **Dispersion Relation**

$\rightarrow$ **Waves propagate with speed** $\boxed{\omega / k = c_s}$

(ii) Magnetic Waves ($p_0 = 0$)

Repeat, but uniform (B_0)
- include **j** x **B** force
- assume wave propagates at angle to B_0



Either $$\omega^2 = k^2 {v_A}^2 \cos^2\theta$$ ***Alfvén Waves***

Incompressible - due to magnetic tension

Or $$\omega^2 = k^2 {v_A}^2$$ **Compressional Alfvén Waves**

Compressible - due to magnetic pressure

- propagate at speed v_A

(iii) MHD Waves (p_0 and B_0 nonzero)

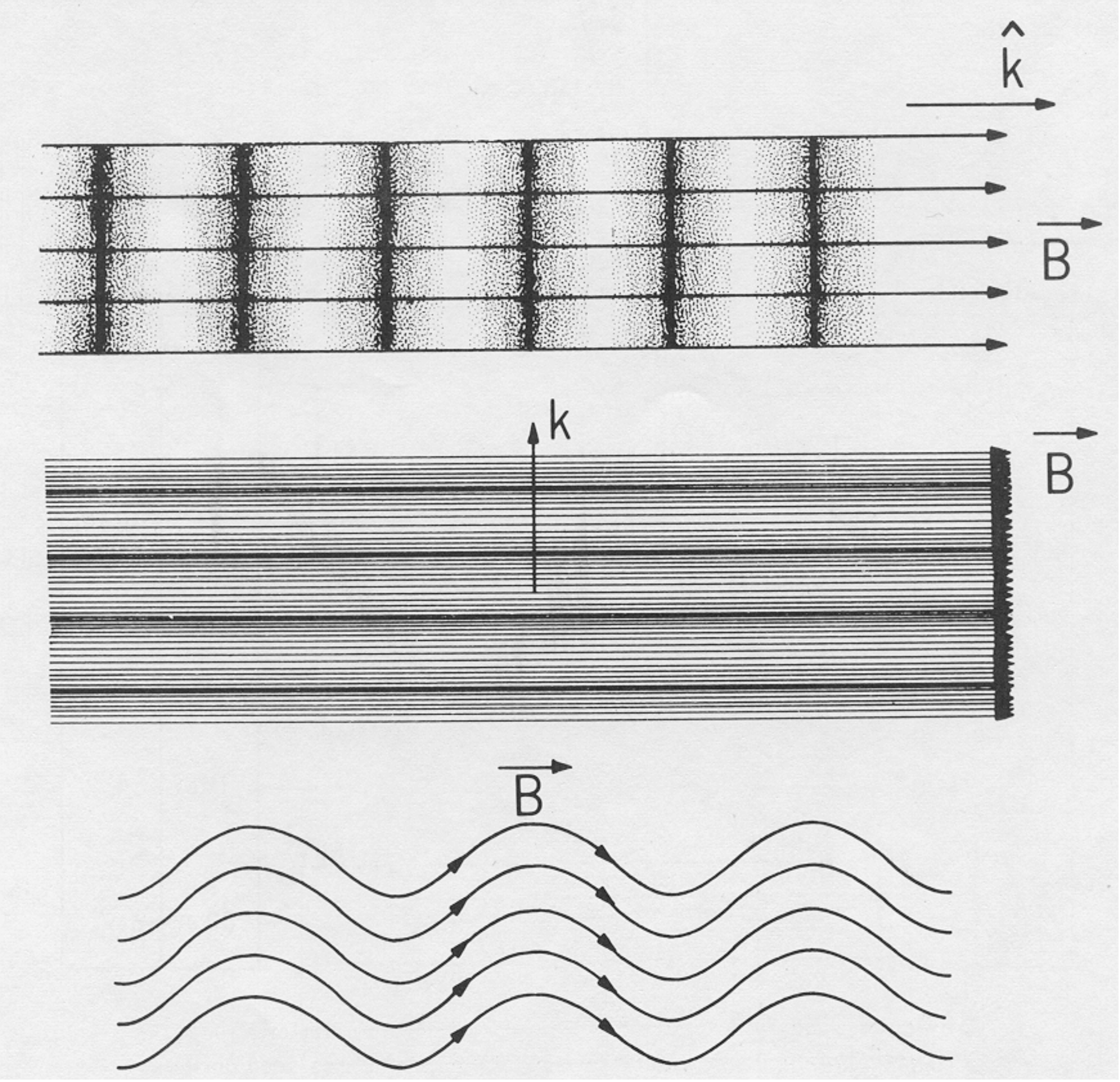
- Alfvén Wave is unaffected

- Compressional Alfvén Wave and Sound Wave are coupled:

  Slow Magnetoacoustic Wave (Slow-Mode)
  + **Fast Magnetoacoustic Wave** (Fast-Mode)

  Propagate slower/faster than Alfvén Wave

Magnetohydrodynamic waves



• Magnetosonic waves

compressible

- parallel slow and fast

- perpendicular fast

$c_{ms} = (c_s^2 + v_A^2)^{-1/2}$

• Alfvén wave

incompressible

parallel and oblique

$v_A = B/(4\pi\rho)^{1/2}$

Shock waves

The description of discontinuities in MHD is more complex of pure hydrodynamics, due to the presence of magnetic fields.
The MHD shock waves form

However, in astrophysics exists the following circumstance that simplifies considerably the problem

Except that around the pulsars, astrophysical fluids are never dominated by the magnetic field, in the sense that the plasma-β par is such that

$$\frac{B^2}{8\pi p} = \frac{\gamma v_A^2}{2c_s^2} < 0.1 - 0.3$$ (take the defs of v_A and c_s)

This implies that the dynamical importance of B field can be approximated as small

Thanks to this we can also neglect the fact that the shock waves must superalfvenic, $V_s > v_A$

In effect, since $v_A = (B^2/4\pi\rho)^{1/2}$ and from previous relation we have $B^2/8\pi < \rho c_s^2$, the condition $V_s > v_A$ is equivalent to $V_s > c_s$, which is absolutely necessary for shock waves

Equation of Motion

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho\, \mathbf{g}$$

(1) (2) (3) (4)

(i) $$\frac{(2)}{(3)} = \beta = \frac{p}{B^2/(2\mu)}$$ * **Plasma beta** *

When $\beta << 1$, $\mathbf{j} \times \mathbf{B}$ dominates

(ii) $$(1) \approx (3) \rightarrow v \approx v_A = \frac{B}{\sqrt{\mu\rho}}$$ * **Alfvén speed** *

Shock waves

The jump conditions at the discontinuity surface are the same as for hydrodynamics:

Conservation of:

i) mass flux

ii) momentum flux (parallel and normal flux)

iii) energy flux

Putting $[X] = X_2 - X_1$ We get

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0$$

$$[\rho v_n(v^2/2 + w) + \frac{1}{4\pi}(v_n B^2 - B_n(\vec{v} \cdot \vec{B}))] = 0$$

There additional conditions due to electromagnetic fields:

From Maxwell eqns, at the discontinuity, we hace B_n and E_t are continous →

$[B_n] = 0 \qquad [E_t] = 0$ Since we are in the ideal limit $E = -v \times B/c$ we get

$$[B_n v_t - B_t v_n] = 0$$

Shock waves

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0 \quad [B_n] = 0$$

$$[\rho v_n (v^2/2 + w) + \frac{1}{4\pi}(v_n B^2 - B_n(\vec{v} \cdot \vec{B}))] = 0 \quad [B_n v_t - B_t v_n] = 0$$

The analysis is simple in two limiting cases

The first is one in which the B field before the shock is normal to the shock surface, B_{t1}=0

This case is called "parallel" shock since the B field is parallel to shock normal direction

In such a case it is easy to show that B_{t2} =0 behind the shock

Since both B_n and B_t are continous with B_{n1} =B_{n2} and B_{t1}=B_{t2}=0, it follows that the parallel shocks reduce to the pure hydrodynamic case, as if the B field is not present

Shock waves

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0$$

$$[\rho v_n (v^2/2 + w) + \frac{1}{4\pi}(v_n B^2 - B_n(\vec{v} \cdot \vec{B}))] = 0$$

$$[B_n] = 0$$

$$[B_n v_t - B_t v_n] = 0$$

The analysis is simple in two limiting cases

The 2nd case is when the B field is parallel to the shock surface, ie perpendicular to the normal, B_{n1} =0

In such a case, from 3rd equation we see that v_t is continous → therefore we can choose a reference frame in which v_t=0 and the shock is a normal shock

From $[B_n v_t - B_t v_n] = 0$ And $[\rho v_n] = 0$ We get

$$B_{t1}/B_{t2} = \rho_1/\rho_2$$

While the others reduce to

$$[\rho v] = 0 \qquad [p + \rho v^2 + \frac{B_t^2}{8\pi}] = 0 \qquad [v^2/2 + w + \frac{B_t^2}{4\pi\rho}] = 0$$

Which are the hydrodynamics eqns with the additional magnetic terms is momentum and energy flux

If the plasma is made of electrons and protons (or ions), there is the question about the particle temperature after the shock, that is if we have $T_e=T_i$

We have seen that what transforms the the ordered kinetic energy in internal kinetic energy are not the collisions, but are time-varying induced electromagnetic fields

The electrons are subjected to same forces as ions, but, due to their much lower mass, the accelerations are much higher

In there conditions, it is perfectly possible that electron irradiate so that they do not retain, ie dissipate, the internal kinetic energy transferred by the shock wave → it is possible that the electrons come out form shock front with $T_e<T_i$ or even $T_e \ll T_i$

This means that the shock dissipates a fraction of its energy, but since before the shock, where the kinetic energy is mainly ordered, the electrons have only a fraction $m_e/m_p \ll 1$ of the total energy, even if almost all this energy would be dissipated, we have not important dynamic consequences on the jump conditions

We can say that, without radiative losses within the shock thickness and with no heat transfer from protons to electrons, the electron temperature behind the shock is a factor m_e/m_p lower than for protons, which is the temperature we get from RH conditions

Consider a subsonic disturbance moving through a conventional neutral fluid. As is well-known, *sound waves* propagating ahead of the disturbance give advance warning of its arrival, and, thereby, allow the response of the fluid to be both smooth and adiabatic. Now, consider a supersonic disturbance. In this case, sound waves are unable to propagate ahead of the disturbance, and so there is no advance warning of its arrival, and, consequently, the fluid response is sharp and non-adiabatic. This type of response is generally known as a *shock*. Let us investigate shocks in MHD fluids.

Since information in such fluids is carried via three different waves--namely, *fast* or compressional-Alfvén waves, *intermediate* or shear-Alfvén waves, and *slow* or magnetosonic waves we might expect MHD fluids to support three different types of shock, corresponding to disturbances traveling faster than each of the aforementioned waves. This is indeed the case.

(iv) Shock Waves

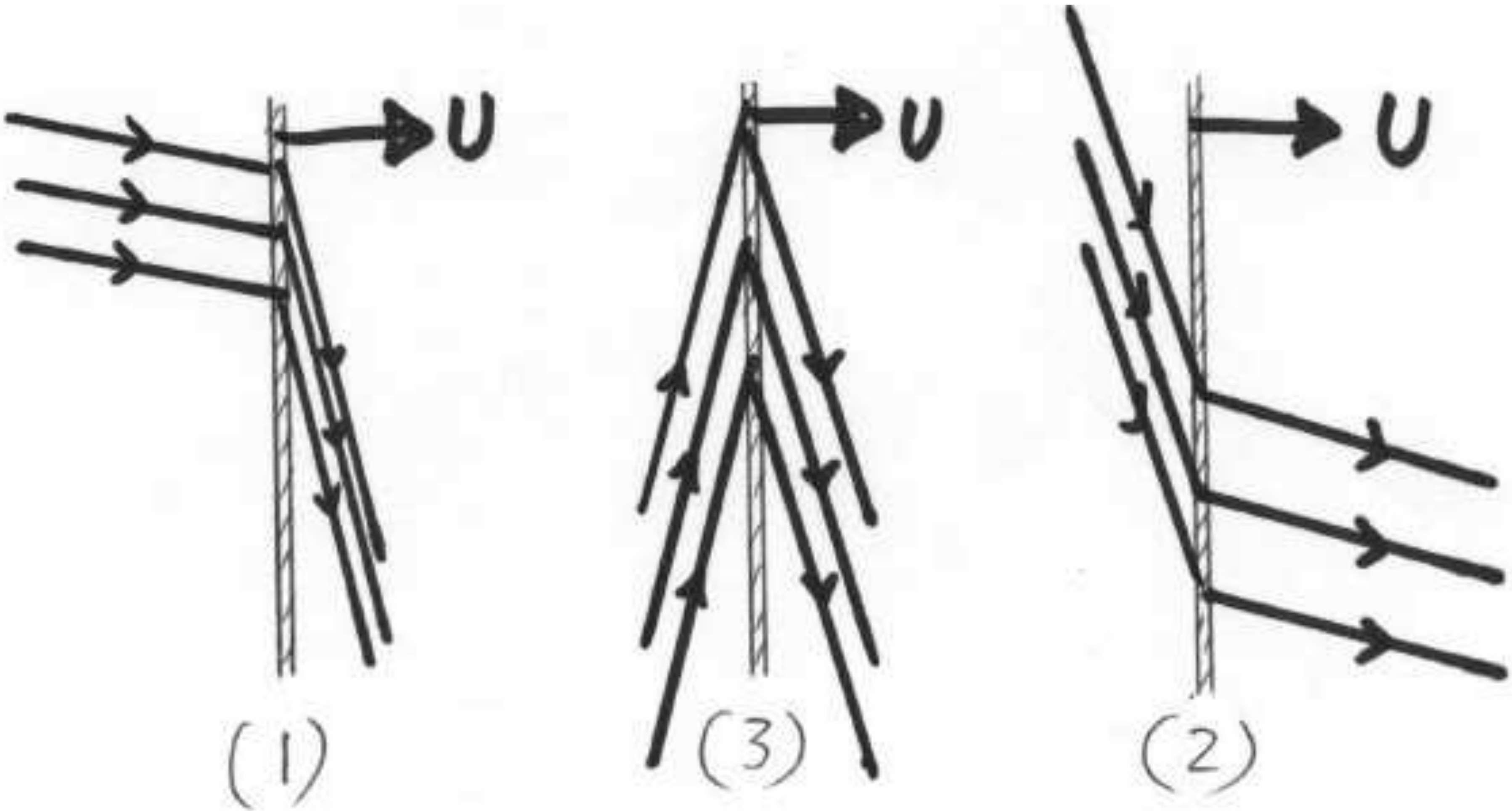
- Nonlinear sound wave can steepen to a shock wave
  -- propagates at speed > c_s

- In MHD 3 modes:

(1) **Slow-mode shock**
- propagates faster than slow-mode speed
- turns B towards normal

(2) **Fast-mode shock**
- propagates faster than fast-mode speed
- turns B away from normal

(3) **Finite-amplitude Alfvén Wave**
- no change in p - reverses tangential magnetic field



Slow-mode **Alfvén** **Fast-mode**

– Conservation of momentum $\left[\rho v_n \vec{v}_t - \frac{B_n}{\mu_0}\vec{B}_t\right] = 0$. The subscript t refers to components that are transverse to the shock (i.e. parallel to the shock surface).

– Conservation of energy $\left[\rho v_n\left(\tfrac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\right) + v_n\frac{B^2}{\mu_0} - \vec{v}\cdot\vec{B}\frac{B_n}{\mu_0}\right] = 0$

There we have used $p\rho^{-\gamma} = const.$

The first two terms are the flux of kinetic energy (flow energy and internal energy) while the last two terms come from the electromagnetic energy flux $\vec{E}\times\vec{B}/\mu_0$

– Gauss Law $\nabla\cdot\vec{B} = 0$ gives $\left[B_n\right] = 0$

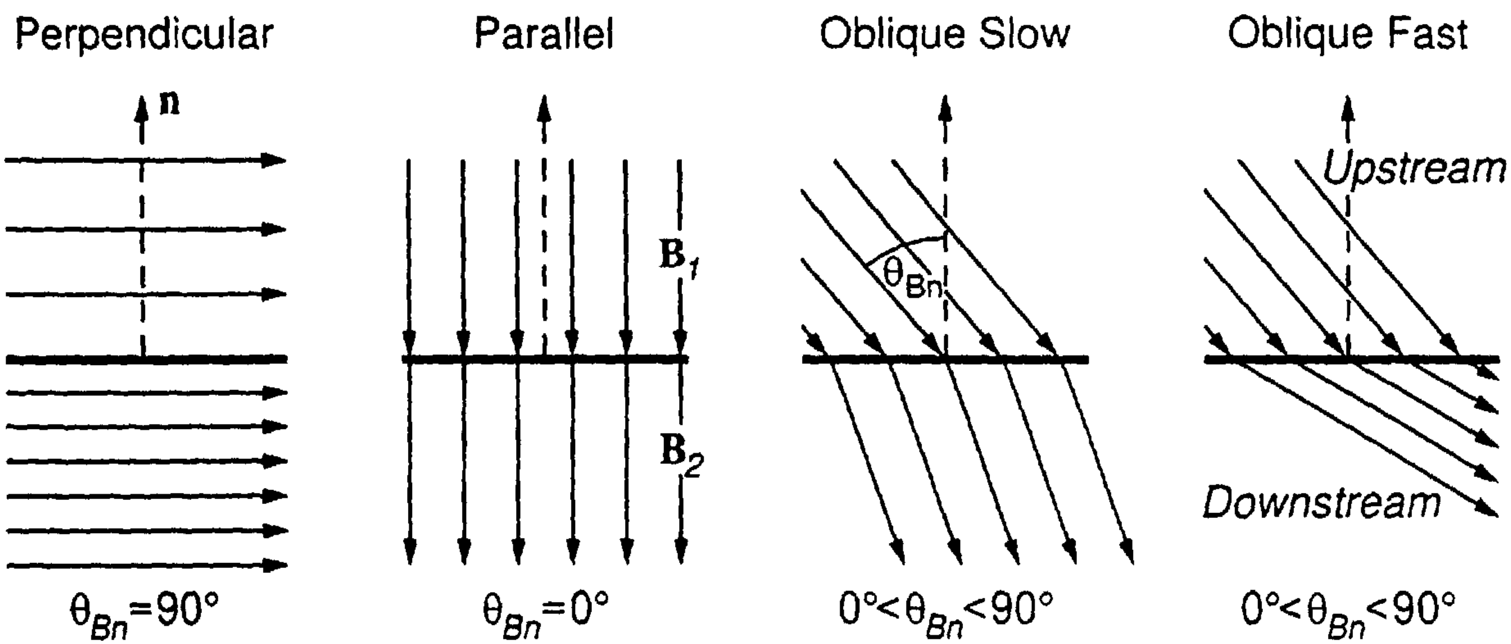
– Faraday's Law $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$ gives $\left[v_n\vec{B}_t - B_n\vec{v}_t\right] = 0$

- The jump conditions are a set of 6 equations. If we want to find the downstream quantities given the upstream quantities then there are 6 unknowns ($\rho, v_n, v_t, p, B_n, B_t$).
- The solutions to these equations are not necessarily shocks. These are conservation laws and a multitude of other discontinuities can also be described by these equations.

Types of Discontinuities in Ideal MHD		
Contact Discontinuity	$v_n = 0$, $B_n \neq 0$	Density jumps arbitrary, all others continuous. No plasma flow. Both sides flow together at v_t.
Tangential Discontinuity	$v_n = 0$, $B_n = 0$	Complete separation. Plasma pressure and field change arbitrarily, but pressure balance
Rotational Discontinuity	$v_n \neq 0$, $B_n \neq 0$ $v_n = B_n / (\mu_0 \rho)^{\frac{1}{2}}$	Large amplitude intermediate wave, field and flow change direction but not magnitude.

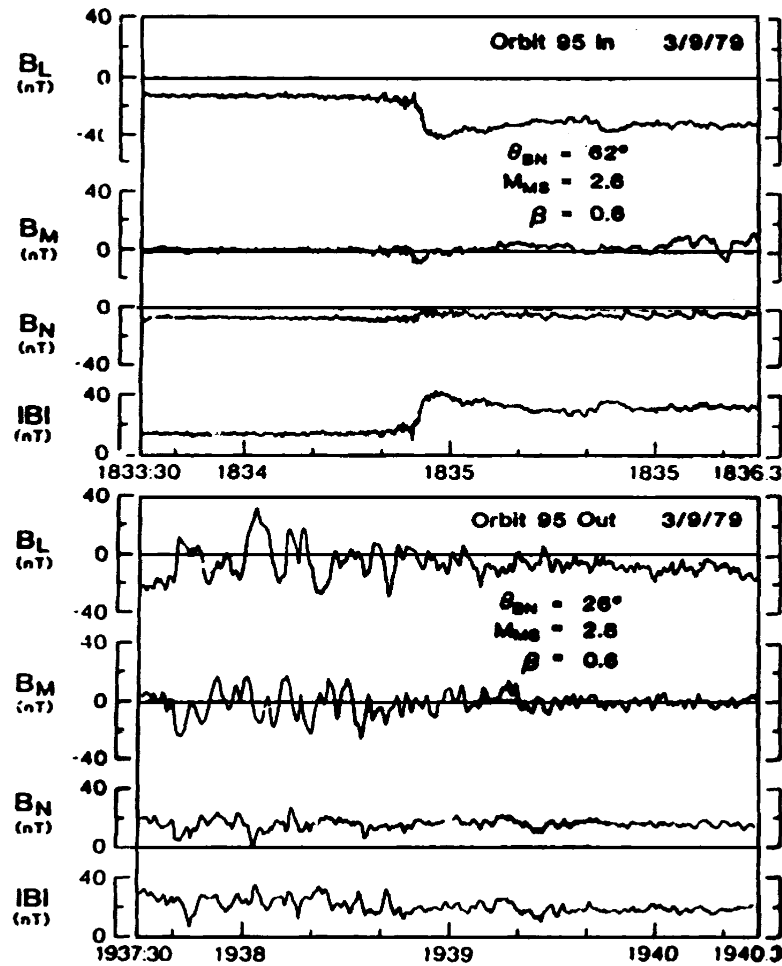
Types of Shocks in Ideal MHD		
Shock Waves	$v_n \neq 0$	Flow crosses surface of discontinuity accompanied by compression.
Parallel Shock ($\vec{B}$ along shock normal)	$B_t = 0$	B unchanged by shock.
Perpendicular Shock	$B_n = 0$	P and B increase at shock
Oblique Shocks	$B_t \neq 0, B_n \neq 0$	
Fast Shock		P, and B increase, B bends away from normal
Slow Shock		P increases, B decreases, B bends toward normal.
Intermediate Shock		B rotates 180^0 in shock plane. [p]=0 non-compressive, propagates at $u_{A,}$ density jump in anisotropic case.

•Configuration of magnetic field lines for fast and slow shocks. The lines are closer together for a fast shock, indicating that the field strength increases.

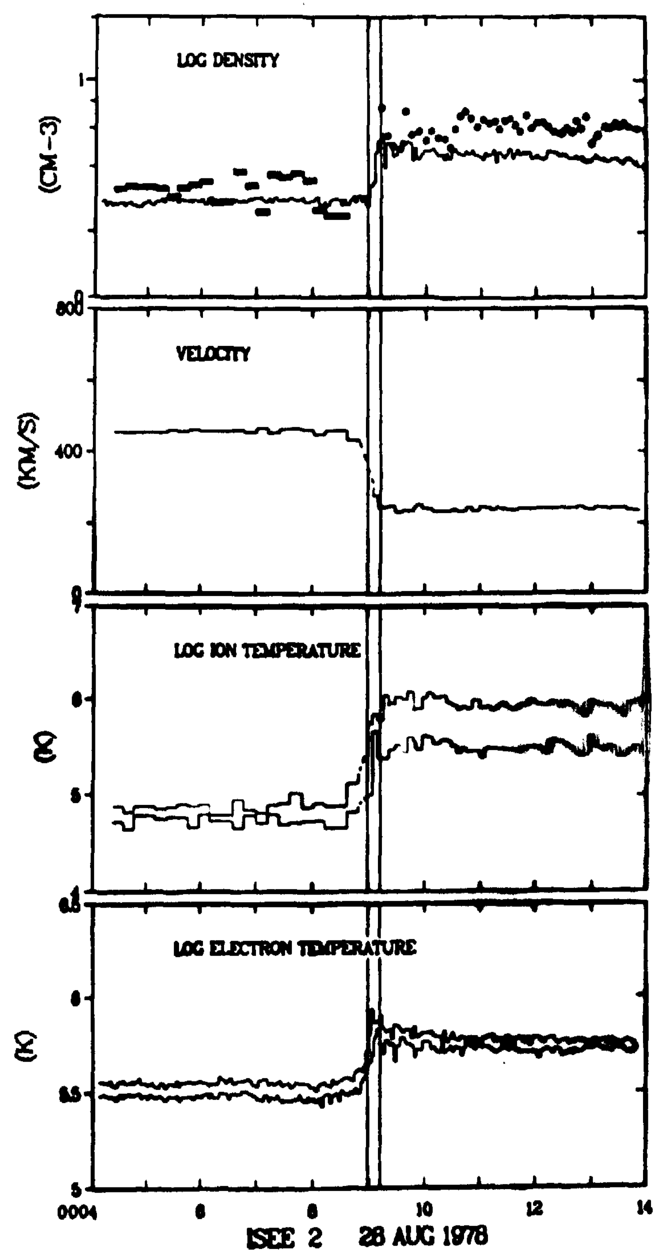


• Quasi-perpendicular and quasi-parallel shocks.

– Call the angle between $\vec{B}$ and the normal θ_{Bn} .
– Quasi-perpendicular shocks have $\theta_{Bn} > 45^0$ and quasi-parallel have $\theta_{Bn} < 45^0$.
– Perpendicular shocks are sharper and more laminar.
– Parallel shocks are highly turbulent.
– The reason for this is that perpendicular shocks constrain the waves to the shock plane while parallel shocks allow waves to leak out along the magnetic field
– In these examples of the Earth's bow shock – N is in the normal direction, L is northward and M is azimuthal.



- Examples of the change in plasma parameters across the bow shock
  - The solar wind is super-magnetosonic so the purpose of the shock is to slow the solar wind down so the flow can go around the obstacle.
  - The density and temperature increase.
  - The magnetic field (not shown) also increases.
  - The maximum compression at a strong shock is 4 but 2 is more typical.



- For compressive fast-mode and slow-mode oblique shocks the upstream and downstream magnetic field directions and the shock normal all lie in the same plane. (***Coplanarity Theorem***)

$$\hat{n}\cdot(\vec{B}_d \times \vec{B}_u) = 0$$

- The transverse component of the momentum equation can be written as $\left[\rho v_n \vec{v}_t - \frac{B_n}{\mu_0}\vec{B}_t\right] = 0$ and Faraday's Law gives $\left[v_n \vec{B}_t - B_n \vec{v}_t\right] = 0$
- Therefore both $\left[v_n \vec{B}_t\right]$ and $\left[\vec{B}_t\right]$ are parallel to $\left[\vec{v}_t\right]$ and thus are parallel to each other.
- Thus $\left[\vec{B}_t\right] \times \left[v_n \vec{B}_t\right] = 0$. Expanding $v_{un}\vec{B}_{ut} \times \vec{B}_{ut} + v_{dn}\vec{B}_{dt} \times \vec{B}_{dt} - v_{dn}\vec{B}_{ut} \times \vec{B}_{dt} - v_{un}\vec{B}_{dt} \times \vec{B}_{ut} = 0$ $(v_{n,u} - v_{n,d})(\vec{B}_{t,u} \times \vec{B}_{t,d}) = 0$
- If $v_{n,u} \neq v_{n,d}$ $\vec{B}_{t,u}$ and $\vec{B}_{t,d}$ must be parallel.
- The plane containing one of these vectors and the normal contains both the upstream and downstream fields.
- Since $(\vec{B}_u - \vec{B}_d)\cdot\hat{n} = 0$ this means both $\vec{B}_d \times \vec{B}_u$ and $\vec{B}_u - \vec{B}_d$ are perpendicular to the normal and $\hat{n} = (\vec{B}_u - \vec{B}_d)\times(\vec{B}_u \times \vec{B}_d) / \left|(\vec{B}_u - \vec{B}_d)\times(\vec{B}_u \times \vec{B}_d)\right|$

Magnetic buoyancy

We have treated always ideal fluids, in which the turbulence does not play a role

But many astrophysical settings are intrinsecally turbulent: classical examples are the heat convective transport regions inside the stars and the accretion disks

In such conditions, the fluid element is in average dynamical equilibrium, with the gravity force balanced by the pressure gradient

But occasionally, due to turbulence, bubbles with high magnetic field content are formed

These bubbles have a different dynamics with respect to non magnetized fluid: the magnetic bubbles are less dense than the average fluid, therefore a bouyancy force will make to float them in the fluid until they are expelled from it

Magnetic buoyancy

Let consider a stratified atmosphere along the z direction

Here with atmosphere, i mean a region in which the gravitational attraction is mainly due an "external" source.

In a star, the atmosphere includes the more external layers, which contain only a small fraction of the stellar mass, subjected to the " external" force of the star (exactly as the Earth's atmosphere is subjected to Earth's gravitational force)

In a accretion disk, the gravitational force is provided by the compact object (NS, BH) around which the disk rotates

Magnetic buoyancy

Let assume that the fluid element is in dynamical equilibrium

Assume also that the B field in the average fluid element is negligible, so that that we can treat it with hydrodynamical Euler equation

since at equilibrium we must have Dv/Dt =0, ie $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = 0$

This implies no explicit time dependence and v=0, so that

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p - \rho\nabla\Phi \quad \Longrightarrow \quad 0 = -\vec{\nabla}p - \rho\nabla\Phi$$

Assuming (for sake of simplicity) a planar atmosphere with all quantities constant along x and y we get the Stevino's law

$$\frac{dp}{dz} = -\rho\frac{d\Phi}{dz}$$

Magnetic buoyancy

For a fluid element containing an important contribution from B field, which may be assumed to point mainly in a fixed direction normal to gravity gradient , say in the x direction, we must apply the complete Euler equation

$$\rho\frac{\partial \vec{v}}{\partial t}+\rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p+(1/4\pi)(\vec{\nabla}\times\vec{B})\times\vec{B}-\rho\nabla\Phi$$

$$\rho_m\frac{Dv_z}{Dt} = -\frac{dp}{dz} + (1/4\pi)[(\vec{\nabla}\times\vec{B})\times\vec{B}]_z - \rho_m\frac{d\Phi}{dz}$$

Where the index m denotes the possibility that here the density might be different from that of non magnetized fluid

The Lorentz force can be decomposed as $(\vec{\nabla}\times\vec{B})\times\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2$

In our planar system, the first term can be neglected since $\vec{B}\cdot\vec{\nabla} \approx B_x\frac{\partial}{\partial x}$

While the main variation is directed alond z direction

This simplifies a lot the problem...the general case is much more difficult but we will reach the same conclusions

Magnetic buoyancy

$$\rho_m \frac{Dv_z}{Dt} = -\frac{dp}{dz} - \frac{1}{8\pi}\frac{d}{dz}B^2 - \rho_m \frac{d\Phi}{dz} \quad = -\frac{d}{dz}(p + \frac{1}{8\pi}B^2) - \rho_m \frac{d\Phi}{dz}$$

Initially the magnetic bubble will be at a distance $z_o<0$ form the star surface, but the convective transport will drive it toward the surface

When this happens, we may assume that the bubble is always in pressure equilibrium (if not, it would expand or shrink quickly until equilibrium is reached) and that no heat is exchanged with the external fluid since motions are much faster than any entropy exchange rate

In such a situation, the bubble density can not remain constant

In the case of non magnetized fluid, the thermodynamical evolution of the bubble would be fixed by the politropic law $p \sim \rho^\gamma$ with $\gamma \sim 4/3$, proper for an ideal fluid

Magnetic buoyancy

But since there is B field, we must take into account the evolution of the magnetic pressure

From frozen-in theorem we know that the flux is constant through the bubble surface, $BR^2=\Phi/4\pi=\text{constant}$

The magnetic pressure is $p_m \sim B^2$ so that $p_m \sim R^{-4}$ but $R \sim V^{1/3}$
$\rightarrow p_m \sim V^{-4/3}$

Taking the specifiv volume $V=1/\rho_m$, we get
$P_m \sim \rho^{4/3}$

The magnetic field behaves as as politropic fluid with index 4/3

Magnetic buoyancy

The total pressure within the bubble is $p_{tot} = p + p_{mag} = p + B^2/8\pi$

If the plasma β-parameter $\beta = p/p_{mag} << 1$, that is B field pressure dominates, the magneto-fuid can be approximated as fluid with $p \sim \rho^{\gamma m}$ with $\gamma_m < \gamma$

It follows that, if, on a infinitesimal path, the pressure decreases of δp, within the non magnetized medium, density decreases of

$$\frac{\delta\rho}{\rho} = \frac{\delta p}{\gamma p}$$

While in the bubble the same pressure change is realized with a density change

$$\frac{\delta\rho_m}{\rho_m} = \frac{\delta p}{\gamma_m p}$$

Since $\gamma_m < \gamma$ we find that $\frac{\delta\rho_m}{\rho_m} = \frac{\gamma\delta\rho}{\gamma_m\rho} > \frac{\delta\rho}{\rho}$

So while the bubble floats toward the surfaces, density reduction inside is higher than the surrounding medium, hence even starting from the same initial density, the bubble becomes lighter while floating and therefore is driven to the surface by Archimede's law or buoyancy

Magnetic buoyancy

$$\rho_m \frac{Dv_z}{Dt} = -\frac{d}{dz}(p_b + \frac{1}{8\pi}B^2) - \rho_m \frac{d\Phi}{dz}$$

The key point is the total pressure equilibrium between bubble and surrounding

$$-\rho g = \frac{dp}{dz} = \frac{d}{dz}(p_b + \frac{1}{8\pi}B^2)$$

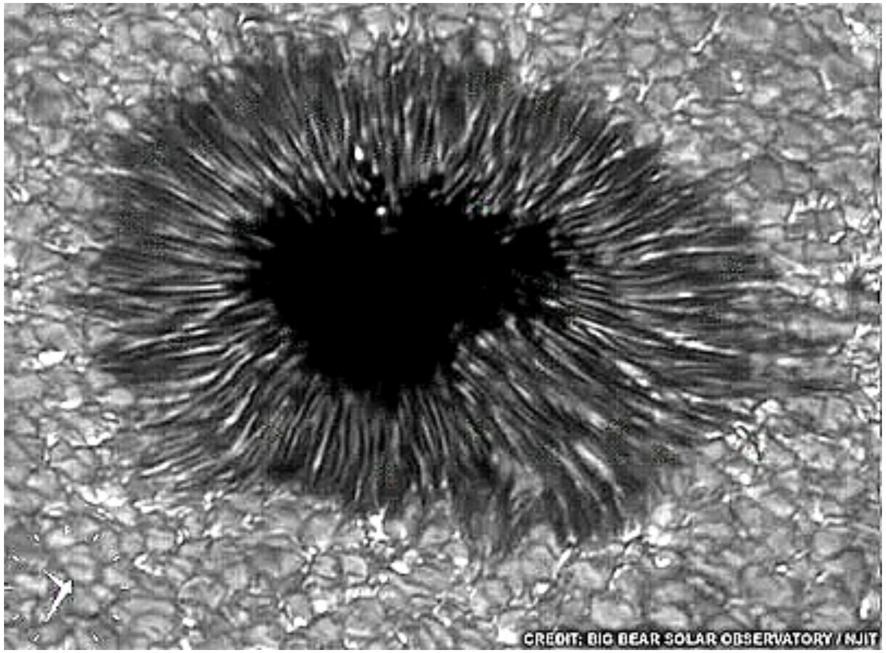
g=dΦ/dz is the local gravity acceleration, p_b mech pressure in the bubble

So that Euler eqn becomes $\rho_m \frac{Dv_z}{Dt} = \rho g - \rho_m g$

Since we have shown that $\rho_m < \rho$ at any quote z, the bubbles is driven to surface by Archimede's law

When it reaches the surface (of star or disk), the bubble there is not longer in pressure equilibrium, the bubble freely expands and expels the magnetized medium in the surrounding (~vacuum) medium, so that the magnetic field is expelled from the region (star or disk)

An example: magnetic bipolar regions

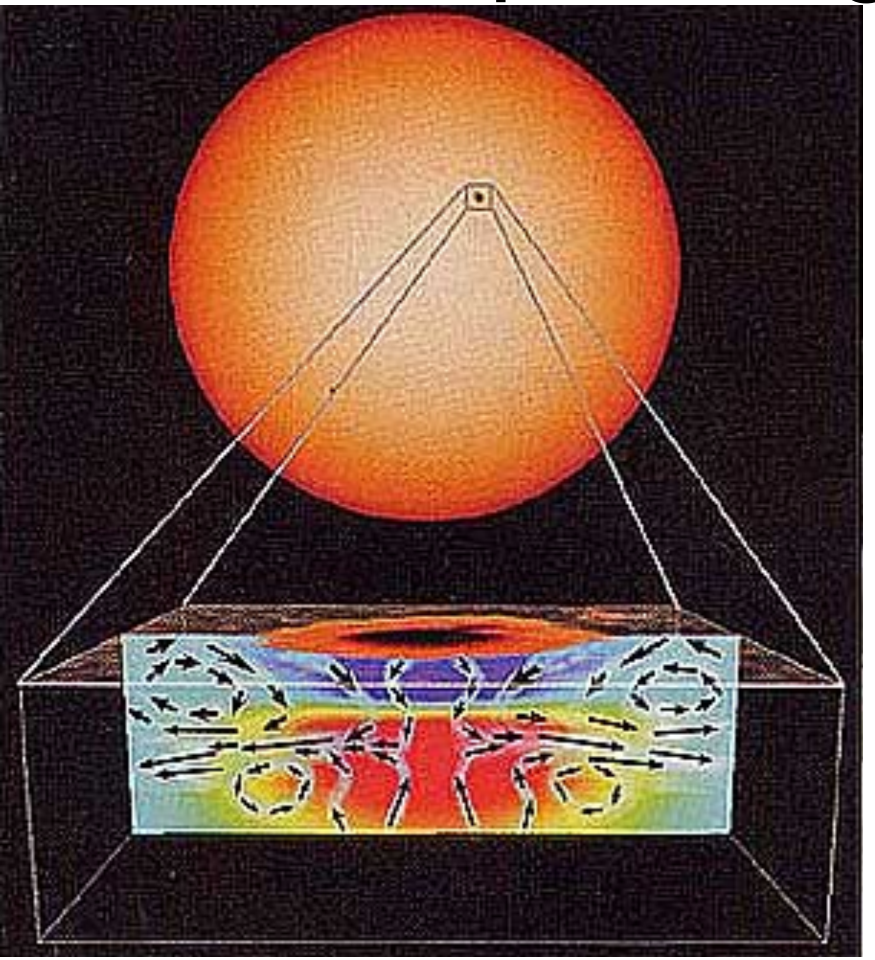


A **sunspot** is a region on the Sun's surface (photosphere) that is marked by a lower temperature than its surroundings and has intense magnetic activity (as seen from Zeeman splitting of emission lines), which inhibits convection, forming areas of low surface temperature.

Although they are blindingly bright at temperatures of roughly 4000-4500 K, the contrast with the surrounding material at about 5800 K leaves them clearly visible as dark spots. If they were isolated from the surrounding photosphere they would be brighter than an electric arc.

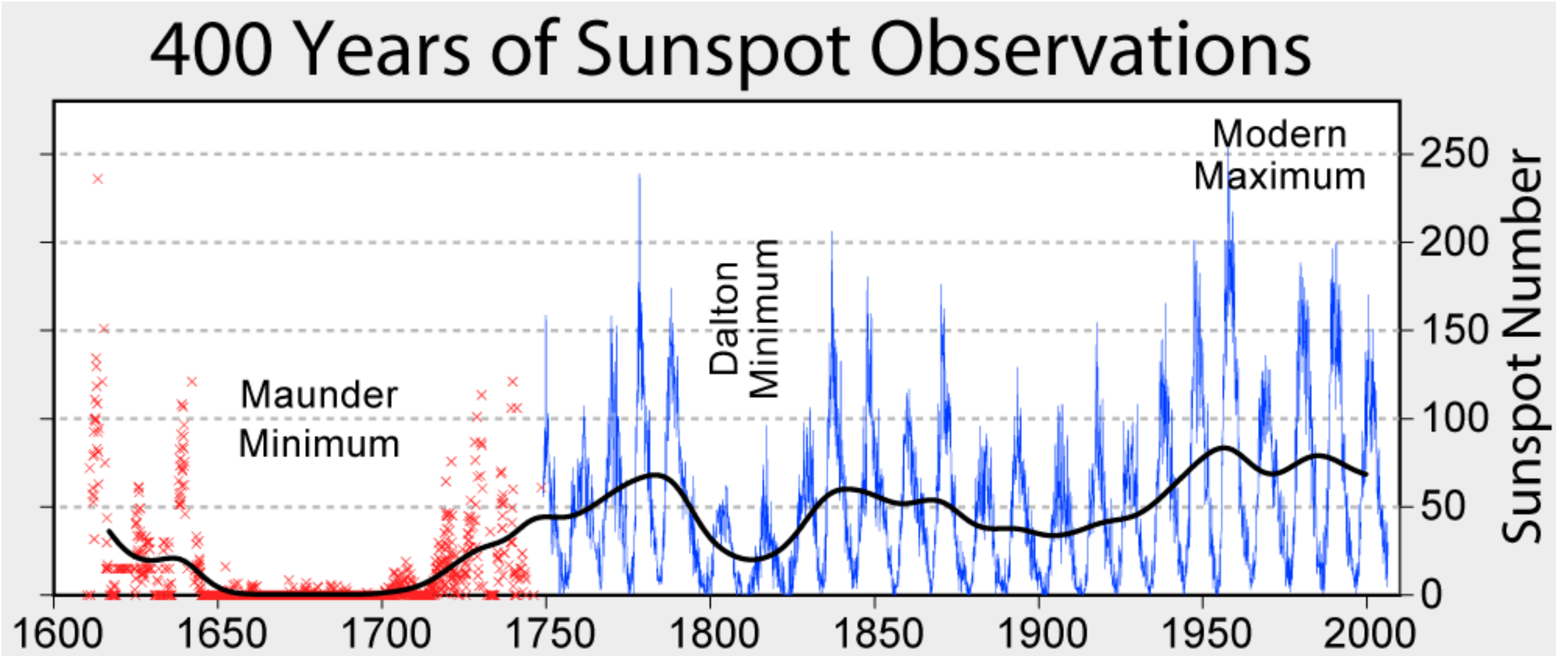
Similar phenomena observed on stars other than the Sun are commonly called ***starspots***

An example: magnetic bipolar regions



Although the details of sunspot generation are still somewhat a matter of research, it is quite clear that sunspots are the visible counterparts of magnetic flux tubes in the convective zone of the sun that get "wound up" by differential rotation. If the stress on the flux tubes reaches a certain limit, they curl up quite like a rubber band and puncture the sun's surface. At the puncture points convection is inhibited, the energy flux from the sun's interior decreases, and with it the surface temperature.

They tend to cluster around a band extending from the solar equator. They are caused by churning up of hydrogen gas by the strong solar bipolar magnetic field. In effect, they are manifestations or products of solar magnetic "storms"



Sunspot numbers rise and fall with an irregular cycle with a length of approximately 11 years. In addition to this, there are variations over longer periods. The recent trend is upward from 1900 to the 1960s, then somewhat downward. The Sun was last similarly active over 8,000 years ago.

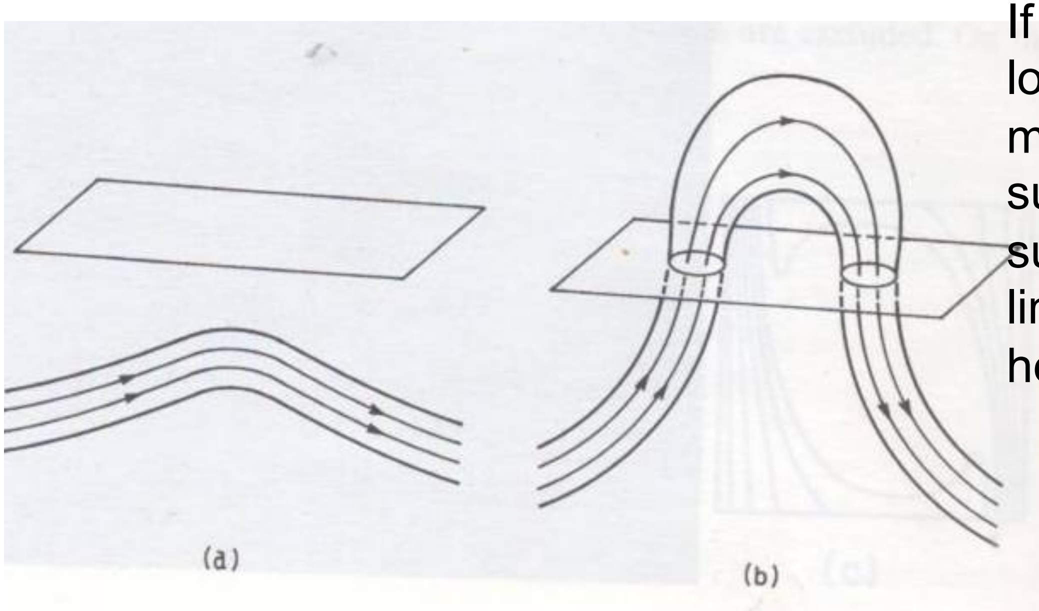
The number of sunspots has been found to correlate with the intensity of solar radiation over the period - since 1979 - when satellite measurements of radiation are available. Since sunspots are dark it might be expected that more sunspots lead to less solar radiation. However, the surrounding areas are brighter and the overall effect is that more sunspots means a brighter sun. The variation is very small (of the order of 0.1%).

During the Maunder Minimum in the 17th Century there were hardly any sunspots at all. This coincides with a period of cooling known as the Little Ice Age.

magnetic bipolar regions

One features of sunspots is that often two large sunspots appear side by side and that the two sunspots in such a pair almost always hace opposite magnetic polarities

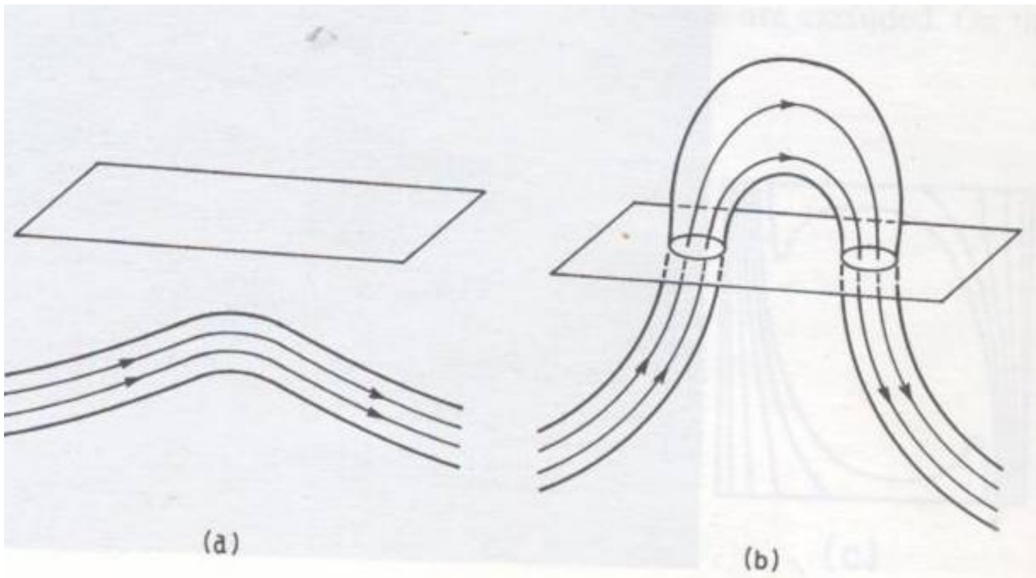
The most obvious explanation for this is that a strand of magnetic field has come through the solar surface



If the two sunspots are merely the two locations where this strand of magnetic field intersects the solar surface, then we readily see that one sunspot must have magnetic field lines coming out and the other must heve field lines going in

This can be explained by magnetic buoyancy

magnetic bipolar regions



Let us consider a nearly horizontal cylindrical region within which some B field is concentrated, ie a magnetic flux tube

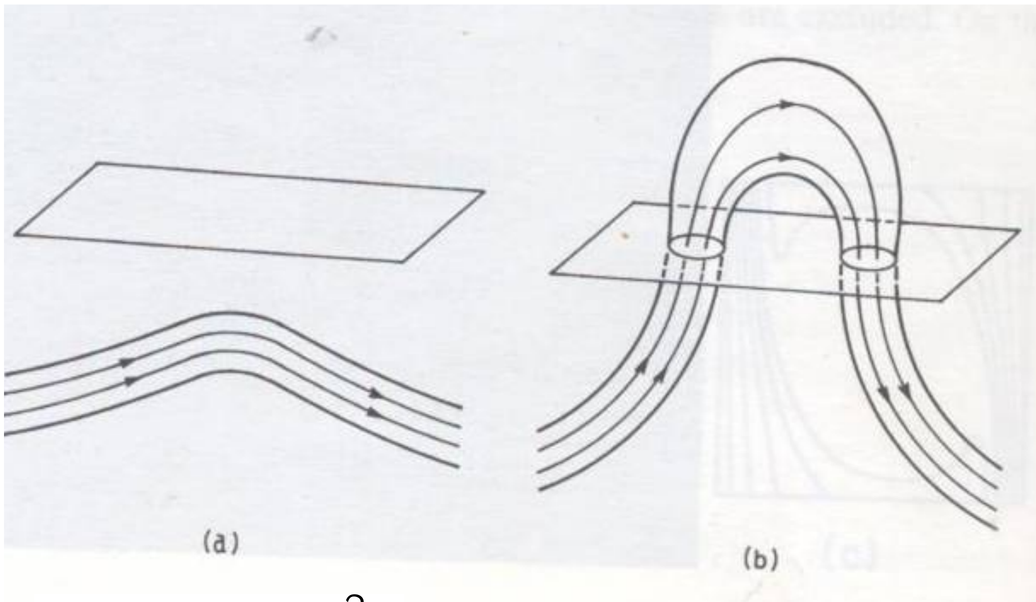
We have seen that there is pressure equilibrium between the pressure within the tube and outside and that the B filed contributes to the tube pressure

$$p_e = p_i + \frac{1}{8\pi}B^2$$

From which follows that $p_i<p_e$

This usually implies, though not always, that the internal density ρ_i is also less than the external density ρ_e

magnetic bipolar regions



$$p_e = p_i + \frac{1}{8\pi}B^2$$ $p_i < p_e$

From gas equation $p = R\rho T$

$$R\rho_e T_e = R\rho_i T_i + \frac{1}{8\pi}B^2$$

$$\frac{\rho_i}{\rho_e} = \frac{T_e}{T_i}(1 - \frac{B^2}{8\pi p_e})$$

If $\rho_i < \rho_e$, the fluid in the interior of the flux tube is lighter and must buoyant

This happens when $$\frac{T_i}{T_e} > (1 - \frac{B^2}{8\pi p_e})$$

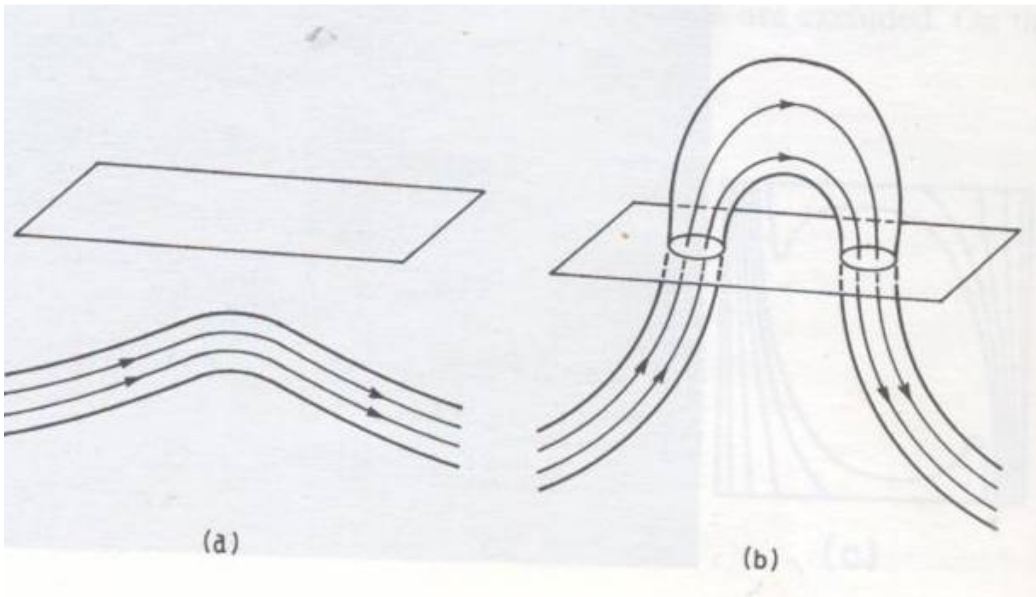
Usually T_i ~ 4500 K and T_e ~ 5800 K → T_i/T_e =0.78

The condition for buoyancy is therefore that $$\frac{B^2}{8\pi p_e} > 0.2$$

The inverse of this ratio is called plasma-β parameter

Which is usually satisfied in the Sun

magnetic bipolar regions



$$p_e = p_i + \frac{1}{8\pi} B^2$$

$p_i < p_e$

From gas equation $p = R\rho T$

$$R\rho_e T_e = R\rho_i T_i + \frac{1}{8\pi} B^2$$

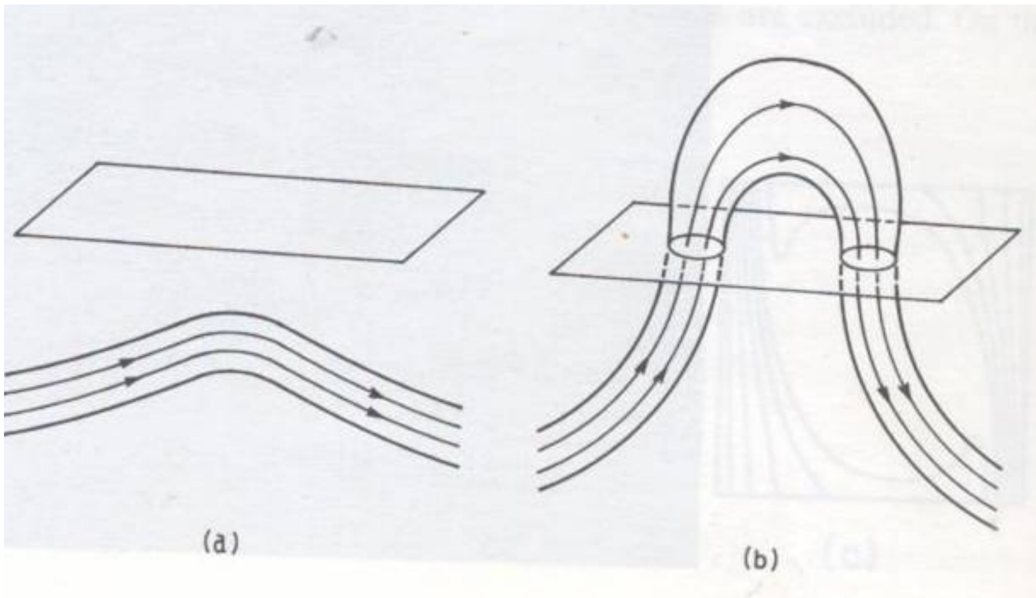
If $T_e = T_i$ the condition is satisfied and we get

$$\frac{\rho_e - \rho_i}{\rho_e} = \frac{B^2}{8\pi}$$

In the limit of high R_m, the magnetic field is frozen in this lighter fluid

As a result, the flux tube as an entity becomes buoyant and rises against the gravitational field

magnetic bipolar regions



$$p_e = p_i + \frac{1}{8\pi} B^2$$

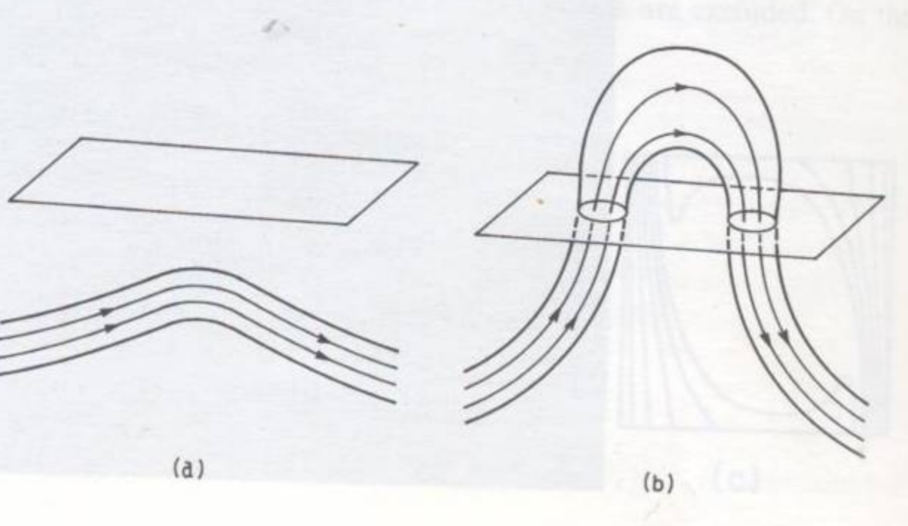
$p_i < p_e$

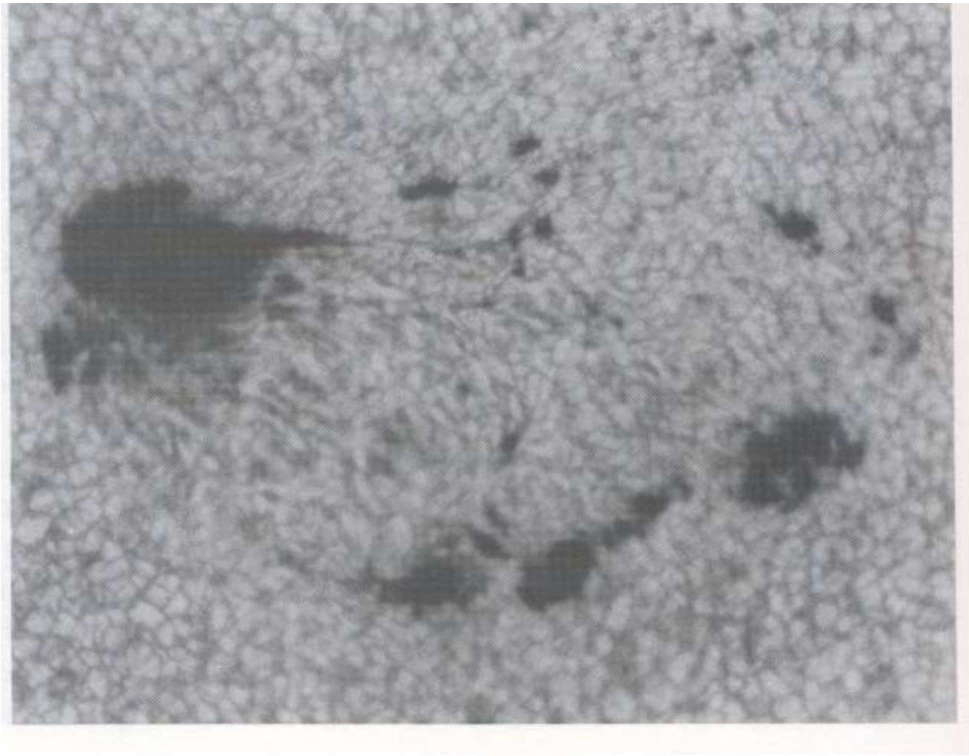
As we have seen, the condition $p_i < p_e$ does not always implies that the interior is lighter

Then it is possible the one part of the flus tube becomes buoyant and not the other parts

If, for istance, the middle part becomes lighter, then it is expected to rise, eventaully piercing through the surface and creating the configuration (b)

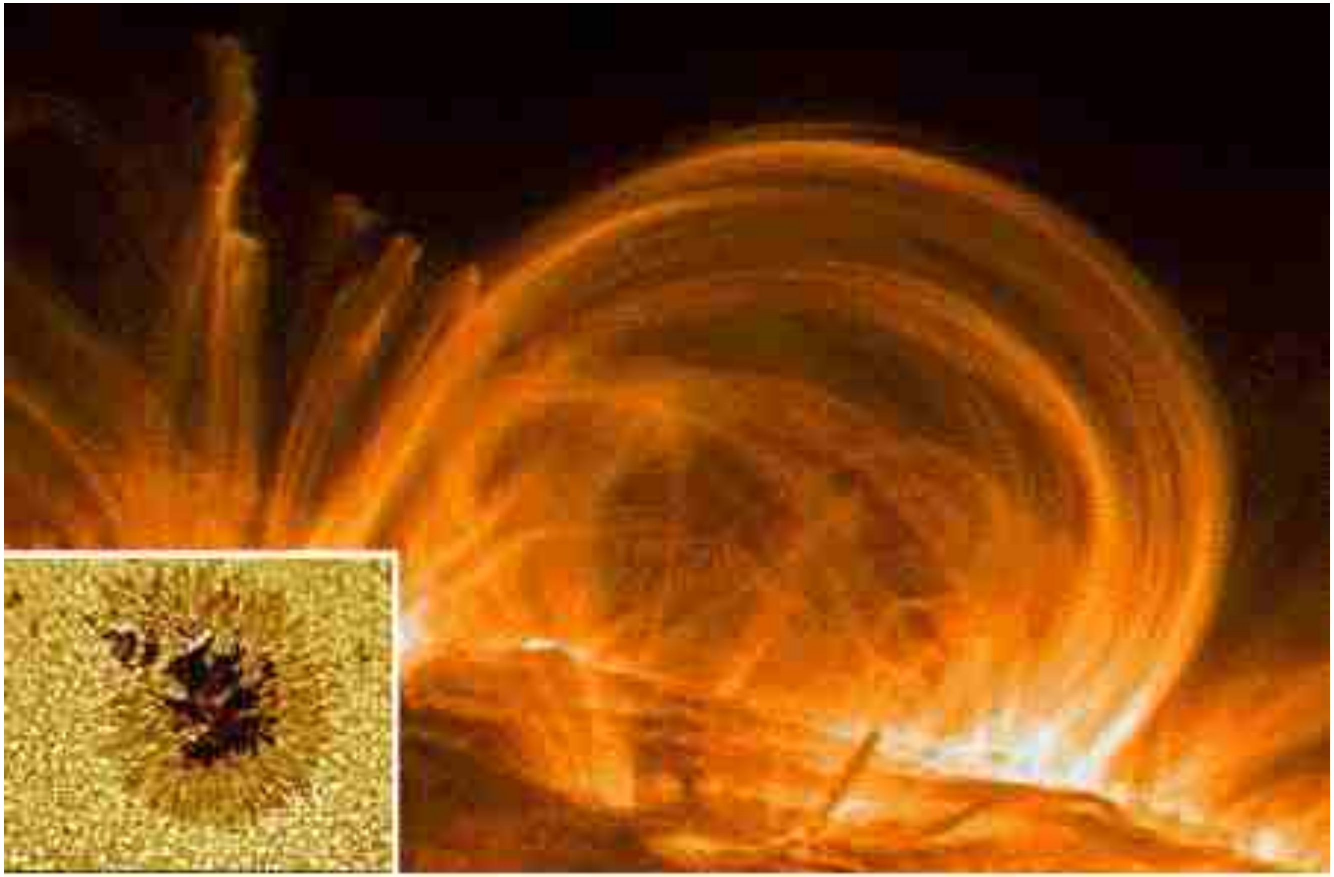
magnetic bipolar regions





The photograph shows a freshly emerged bipolar region showing two large sunspots along with some smaller spots

The granules lying between the two large sunspots seem somewhat distorted and elongated: clear signature that something recently come up through the solar surface between the two large spots



B field lines are made visible by hot plasma moving along them