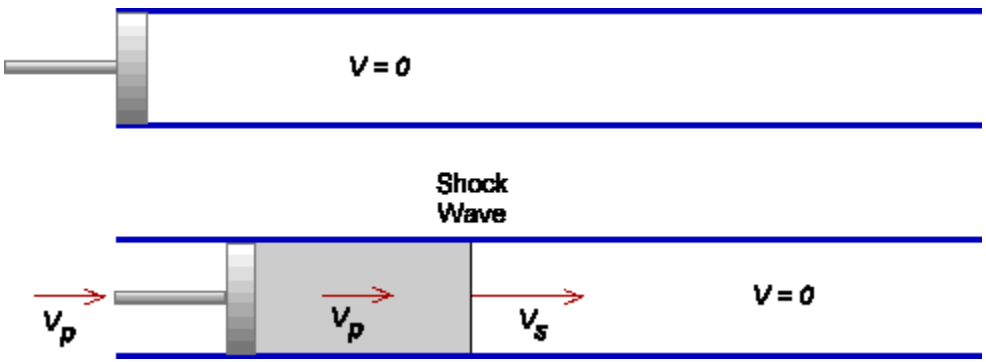
Lecture 14 281118

- Il pdf delle lezioni puo’ essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/

The slides are taken from http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/bibliography/hydrodynamics_achterberg.pdf

The supersonic piston

A common situation in high energy astrophysics is one in which an object is driven supersonically into a gas, or equivalently, a supersonic flow past a stationary object (ie what is important is the relative speed between fluid and object), as for istance in the case of supernovae explosions



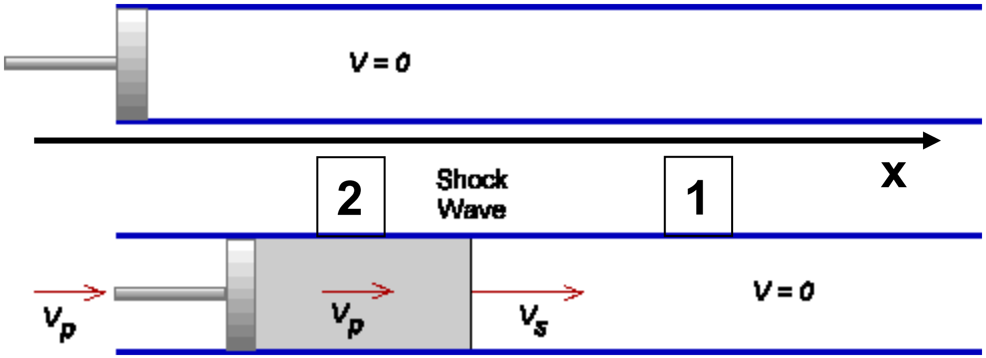
A good illustrative example is a piston driven supersonically into a tube filled with stationary gas

Basically, a hollow tube is filled with a uniform gas at rest and fitted with a piston at one end. At time $t = 0$, the piston is suddenly put into motion with a constant speed, V_p $(>c_s)$.

The motion of the piston creates a shock wave, ahead of the piston, that moves in the same direction as the piston, but at faster, constant speed, V_s (faster than the speed of sound).

The problem is to calculate the shock wave speed

The supersonic piston



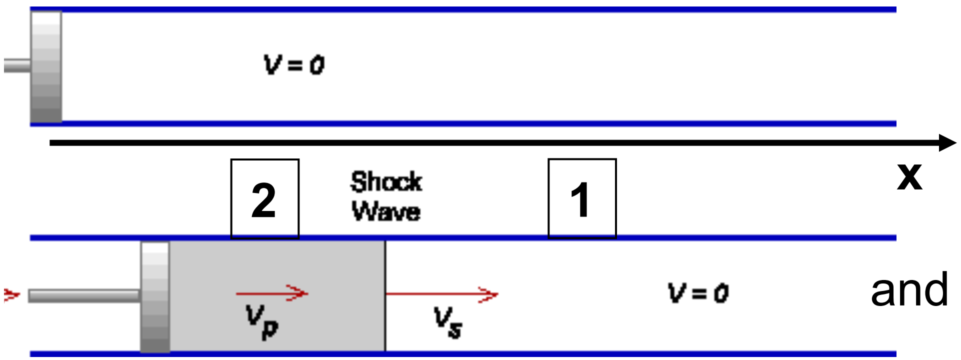
Let work out the problem in the shock wave reference frame, which is moving in the observer frame at speed $+V_s$

In the obs frame, the pre-shock fluid is at rest $V_1=0$, while for the post-shock fluid is reasonable to assume that is moving at the same speed of the piston $V_2=V_p>c_s$ because the particles of the fluid are swept up when the piston reach them, very much like snow is swept up by a snowplow

The galileo's tranformation is $\mathbf{V_{sh}=V_{obs} - V_s}$

In the shock frame, the pre-shock fluid moves with speed $V_1'=-V_s$ and the post shock fluid moves at (unknown) speed V_2'

The supersonic piston



From mass flux conservation

$$V_1'^2 = J^2 v_1^2$$

and $J^2 = \dfrac{p_2 - p_1}{v_1 - v_2} = \dfrac{p_2 - p_1}{v_1(1 - v_2/v_1)}$ (cfr. pag 233)

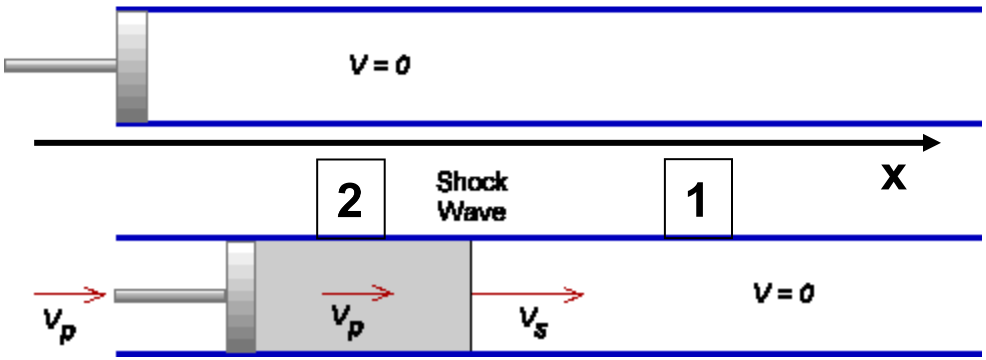
but $\dfrac{v_1}{v_2} = \dfrac{(\frac{\gamma+1}{\gamma-1})p_2 + p_1}{(\frac{\gamma+1}{\gamma-1})p_1 + p_2}$ (cfr. pag 234) $\Rightarrow$ $J^2 = \dfrac{(\gamma-1)p_1 + (\gamma+1)p_2}{2v_1}$

$$\Rightarrow \quad V_1^2 = \frac{v_1}{2}(\gamma-1)p_1 + (\gamma+1)p_2 = \frac{p_1 v_1}{2}[(\gamma-1) + (\gamma+1)(p_2/p_1)]$$

But $c_1^2 = \gamma p_1 v_1$ $\Rightarrow$ $V_1^2 = \dfrac{c_1^2}{2\gamma}[(\gamma-1) + (\gamma+1)(p_2/p_1)]$

Then we have to find the pressure ratio

The supersonic piston



We dont know V_2', but we know the speed difference

$$V_1' - V_2' = V_p$$

From Galileo's transformation

From mass flux conservation we have $\rho_1 V_1' = \rho_2 V_2' \equiv J \Rightarrow (\rho_1/\rho_2)V_1' = V_2'$

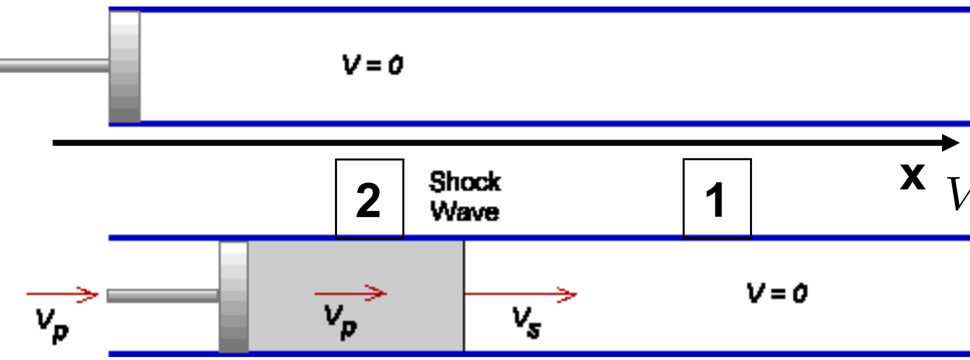
$$V_1' - V_2' = V_p = V_1' - (\rho_1/\rho_2)V_1'$$

$$V_1' - V_2' = V_p = V_1'[1 - (\rho_1/\rho_2)] = \rho_1 V_1'[1/\rho_1 - 1/\rho_2] = \rho_1 V_1'[v_1 - v_2] = J[v_1 - v_2]$$

$$\Rightarrow \frac{V_1' - V_2'}{v_1 - v_2} = J \qquad \text{but} \qquad J^2 = \frac{p_2 - p_1}{v_1 - v_2}$$ (See pag 234)

$$\frac{(V_1' - V_2')^2}{(v_1 - v_2)^2} = \frac{p_2 - p_1}{v_1 - v_2} \Rightarrow V_1' - V_2' = (p_2 - p_1)(v_1 - v_2)^{1/2}$$

The supersonic piston



$$V_1' - V_2' = V_p$$

$$V_1' - V_2' = (p_2 - p_1)(v_1 - v_2)^{1/2} = V_p \quad \text{(a)}$$

Now eliminate specific volume v_2 in (a) by using

$$\frac{v_1}{v_2} = \frac{(\frac{\gamma+1}{\gamma-1})p_2 + p_1}{(\frac{\gamma+1}{\gamma-1})p_1 + p_2}$$

(cfr. pag 234)

Then square (a) and solve for for p_2/p_1

$$(p_2/p_1)^2 - (p_2/p_1)[2+(\gamma+1)\frac{V_p^2}{2p_1v_1'}] + [1 - \frac{(\gamma-1)V_p^2}{2p_1v_1'}] = 0$$

NB: v is the specific volme

The sound speed in the pre-shock region is $\gamma \mathbf{v_1'p_1} = \mathbf{c_1^2}$ ➔ we can solve for p_2/p_1

$$(p_2/p_1) = 1 + \frac{\gamma(\gamma+1)V_p^2}{4c_1^2} + \frac{\gamma V_p}{c_1}[1 + \frac{(\gamma+1)^2V_p^2}{16c_1^2}]^{1/2}$$

The supersonic piston

$$(p_2/p_1) = 1+\frac{\gamma(\gamma+1)V_p^2}{4c_1^2}+\frac{\gamma V_p}{c_1}[1+\frac{(\gamma+1)^2V_p^2}{16c_1^2}]^{1/2} \qquad \text{(a)}$$

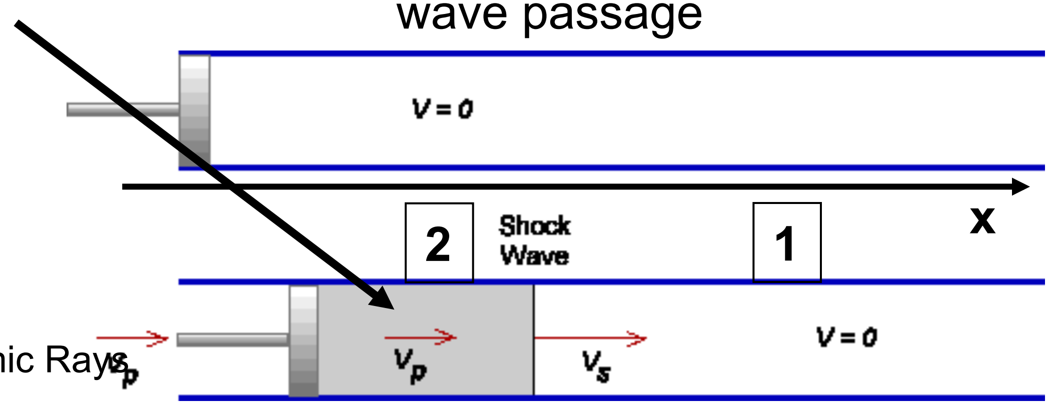
$$V_1^2 = \frac{c_1^2}{2\gamma}[(\gamma-1)+(\gamma+1)(p_2/p_1)] \qquad \text{(b)}$$

Inserting (a) into (b) after some simple but tedious algebra we have

$$V_1' \equiv |V_s| = \frac{(\gamma+1)}{4}V_p + [c_1^2 + \frac{(\gamma+1)^2V_p^2}{16}]^{1/2}$$

The shock wave travels at higher speed with respect to the piston

➔ there is a layer of shocked material in between the shock wave and the piston, traveling at the piston speed, compressed and heated by the shock wave passage



The supersonic piston

$$V_1' \equiv |V_s| = \frac{(\gamma+1)}{4}V_p + [c_1^2 + \frac{(\gamma+1)^2 V_p^2}{16}]^{1/2}$$

In the limit of strong shocks, the expression reduces to $|V_s| \approx \frac{(\gamma+1)}{2}V_p$

→ the ratio between the shock position and the piston position is $|V_s|/V_p \approx \frac{(\gamma+1)}{2}$

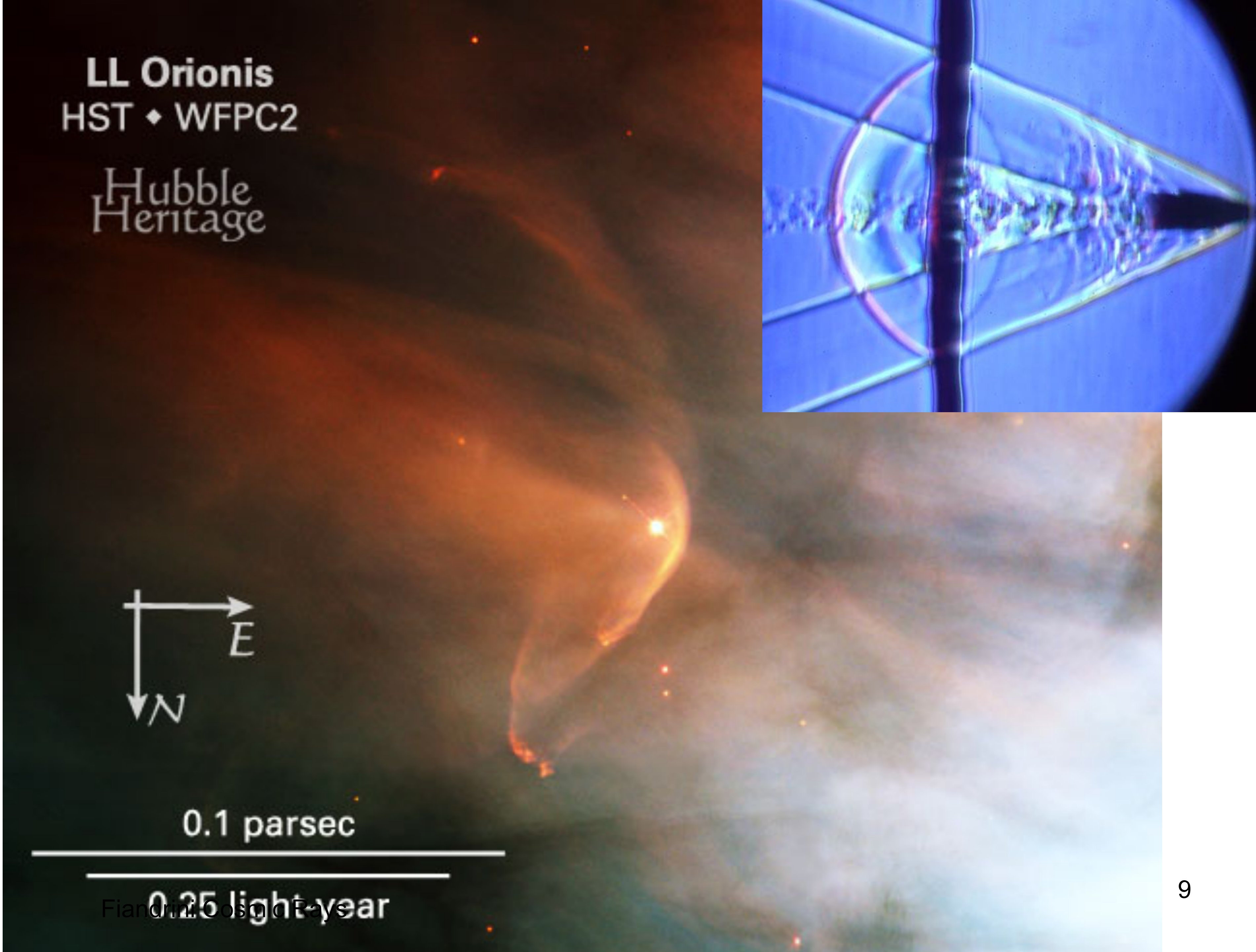
E.g. for a monoatomic gas γ=5/3 → $\mathbf{V_s/V_p=4/3}$

All the gas originally in the tube between x=0 and the shock position is squeezed into a smaller distance $(V_s-V_p)t$

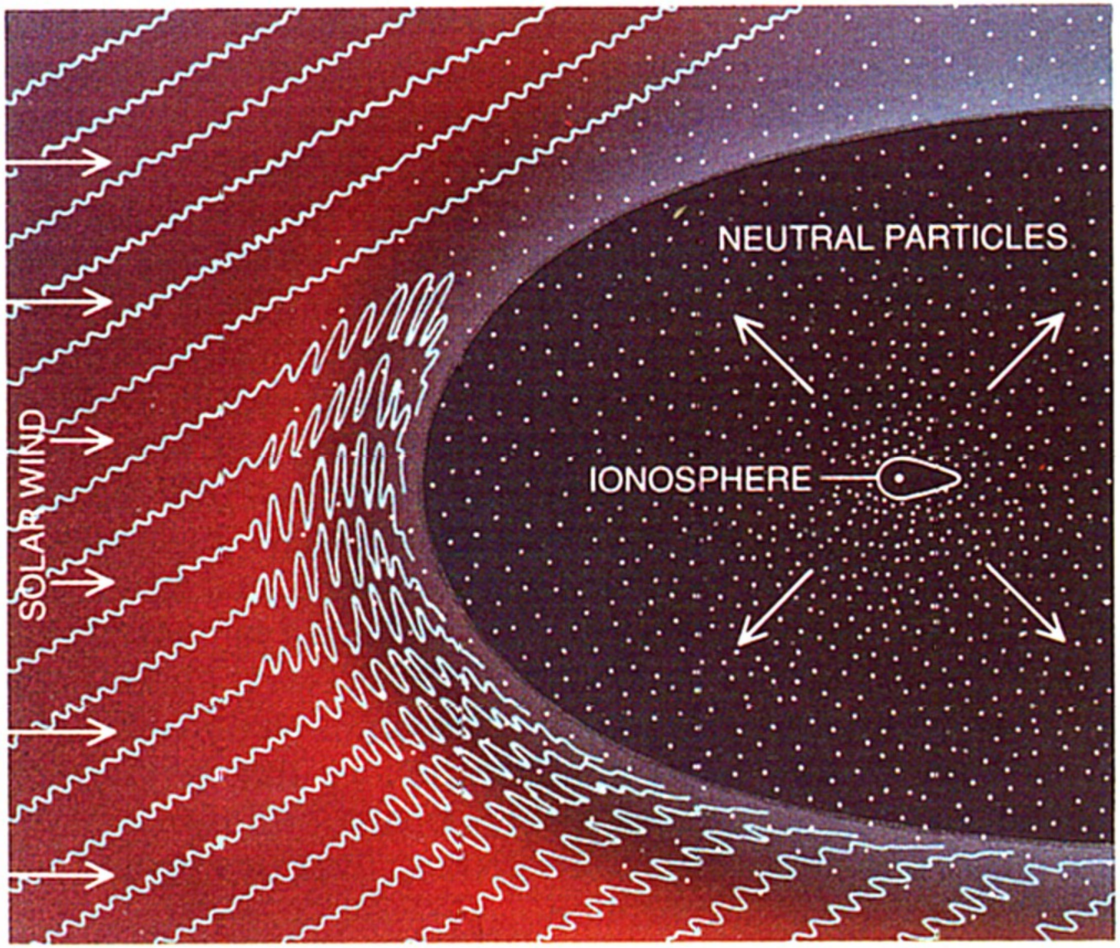
So behind the shock wave and ahead of the obstacle (the piston in this example) there is a layer of material compressed and heated

It is also seen that there is a stand-off distance of a shock front from a blunt object placed in the flow, as for istance in the case of solar wind past the Earth's magnetic dipole

This is what is expected to occur when a supernova ejects a sphere of hot gas into the ISM


LL Orionis
HST ◆ WFPC2
Hubble Heritage
E
N
0.1 parsec
0.25 light year

Examples of Astrophysical shocks



Cometary bow-shocks

Blast shock waves

A blast shock wave is a shock wave formed by a hot gas bubble expanding supersonically in the ambient medium

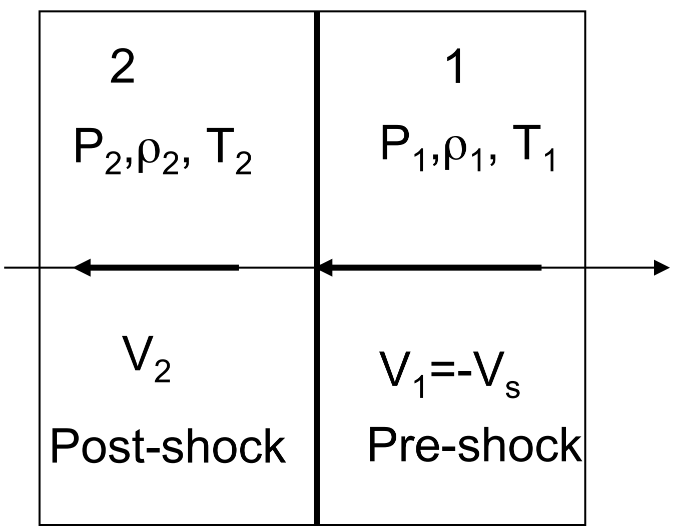
Let us assume that the expansion occurs in a uniform stationary polytropic medium with density ρ_o and pressure p_o

how does the shock wave evolve in time?

First, since the surrounding medium is uniform, the expansion will have spherical symmetry

Blast shock waves

We worked out the physics of the (strong) shock in the shock reference frame, where it is stationary



Shock rest frame

$$\frac{\rho_2}{\rho_1} \approx \frac{(\gamma+1)}{(\gamma-1)}$$

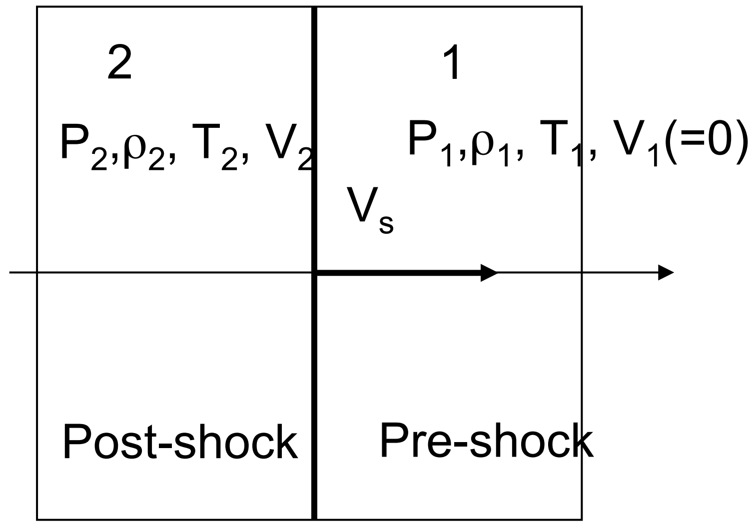
$$\frac{p_2}{p_1} \approx \frac{2\gamma M_s^2}{(\gamma+1)}$$

$$\frac{V_2}{V_1} \approx \frac{\gamma-1}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_s^2$$

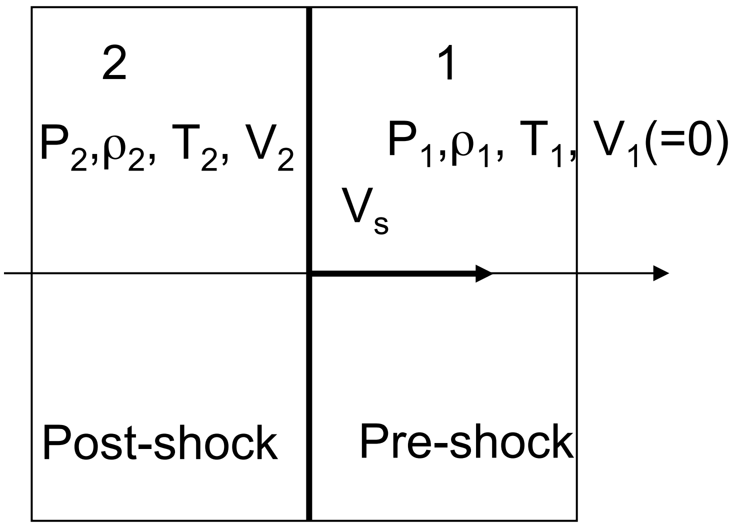
Blast shock waves

Now we have an (supersonic) expansion in a uniform, stationary (V_1=0) polytropic medium with density $\rho_1 = \rho_o$ and pressure $p_1 = p_o$ and a supersonic shock wave propagating to the right (+x dir) with speed V_s ahead the expanding gas, as in the case of the supersonic piston

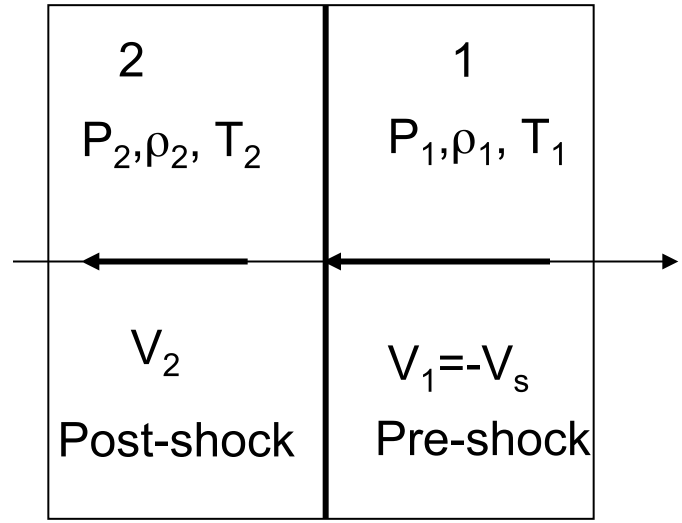


As in the supersonic piston example, the situation in a reference frame where the shock is traveling with speed V_s can be obtained, for non-relativistic shocks, from a simple galileian transformation

Blast shock waves



Observer frame



Shock rest frame

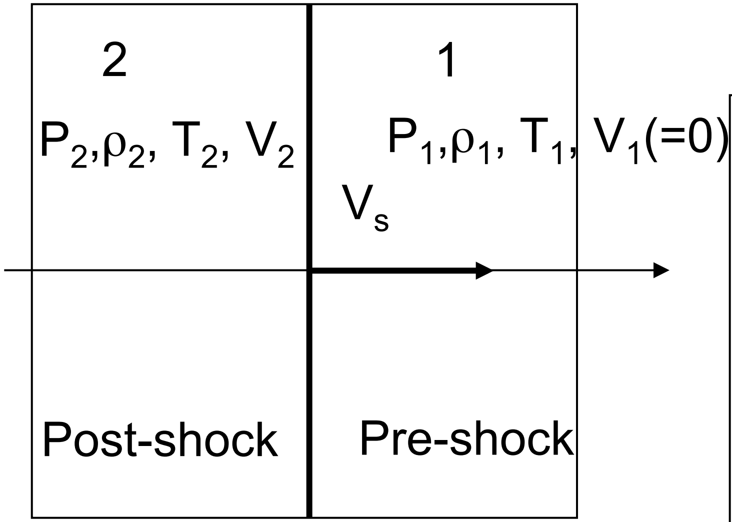
The RH conditions can be still applied, provided one interprets the speeds V_1 and V_2 as relative speeds with respect to the shock, ie apply a galileian velocity tranformation

$$\vec{V} \rightarrow \vec{V}_{rel} = \vec{V} - \vec{V}_s$$

For a shock propagating with velocity $\mathbf{V}_s$ into a medium at rest, $\mathbf{V}$=0, one has <u>$\mathbf{V_1 = -V_s}$</u> and θ_s=0 (as for the supersonic piston) ➔ in this case any shock is a normal shock, with $\mathbf{V}_t$=0, even when the shock surface itself is not a plane!

➔ spherical shocks are normal shocks

Blast shock waves



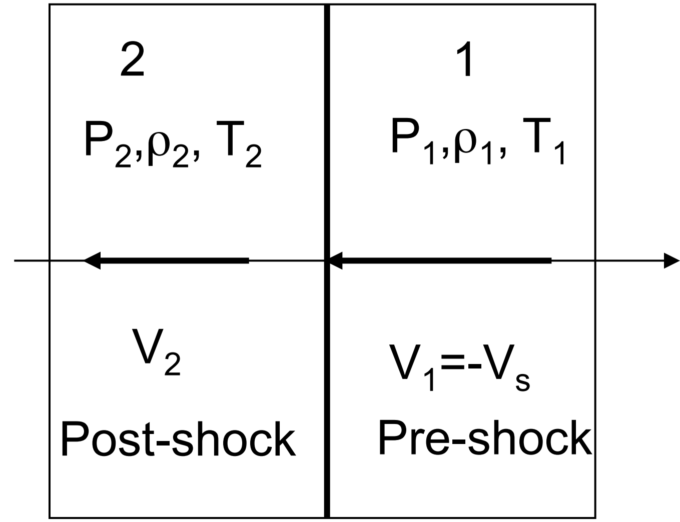
$$\vec{V}_{sh} = \vec{V}_{obs} - \vec{V}_s$$

$$\frac{\rho_2}{\rho_1} \approx \frac{(\gamma+1)}{(\gamma-1)}$$

$$\frac{p_2}{p_1} \approx \frac{2\gamma M_s^2}{(\gamma+1)}$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_s^2$$

$$\frac{V_2}{V_1} \approx \frac{\gamma-1}{\gamma+1}$$

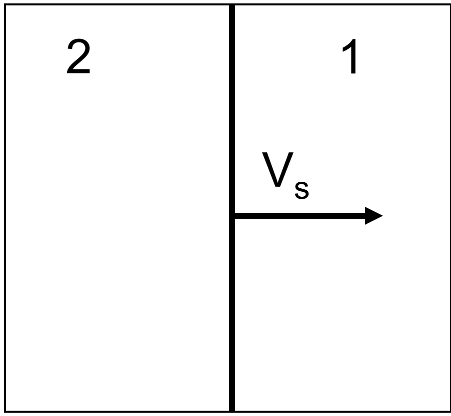


All the conditions may be directly applied, except the speed for which we need to make the substitution $V_2^{sh} = V_2^{obs} - V_s$ and $V_1 = -V_s$

$$\frac{V_2}{V_1} \rightarrow \frac{V_2 - V_s}{-V_s} \approx \frac{\gamma-1}{\gamma+1} \quad \Rightarrow \quad V_2 \approx \frac{2V_s}{\gamma+1}$$

Is the post shock speed in the obs frame

Blast shock waves



Let assume also that the explosion occurs in a uniform stationary polytropic medium with density $\rho_1 = \rho_o$ and pressure $p_1 = p_o$

The strong shock satisfies the relation $M_s^2 = (\frac{V_s}{c_s})^2 = \frac{\rho_o V_s^2}{\gamma p_o} \gg 1$

The RH relations then give

$$\frac{\rho_2}{\rho_1} \approx \frac{(\gamma+1)}{(\gamma-1)}$$

$$\boxed{\frac{p_2}{p_1} \approx \frac{2\gamma M_s^2}{(\gamma+1)}}$$

$$V_2 \approx \frac{2V_s}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_s^2$$

We can have the pressure p_2 immediately behind the shock

$$p_2 \approx \frac{2\gamma M_s^2}{(\gamma+1)} p_o = \frac{2\rho_o V_s^2}{(\gamma+1)}$$

Inverting this relation, one can calculate the shock speed as a function of the post-shock pressure and the pre-shock density

$$V_s = (\frac{\gamma+1}{2})^{1/2} (\frac{P_2}{\rho_o})^{1/2}$$

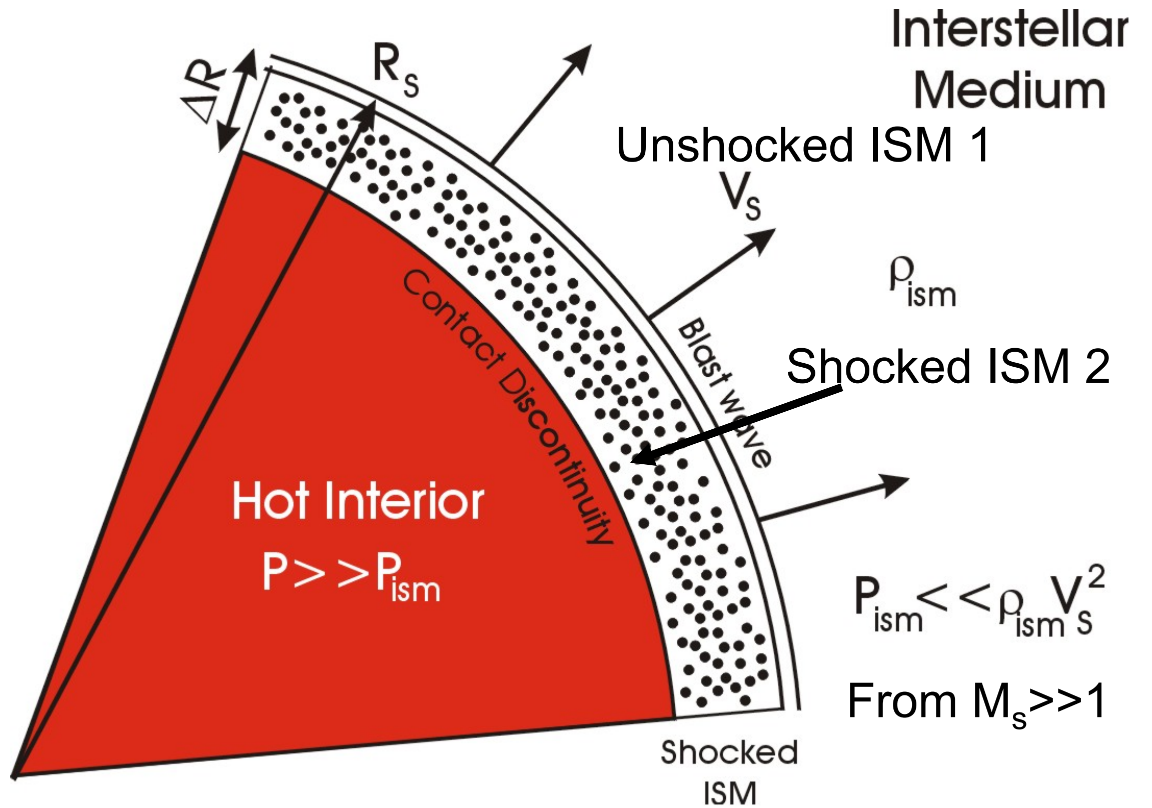
This result can be applied for the formation of high pressure bubbles in a stationary medium

As for istance SuperNova remnants (SNRs) and stellar wind bubbles in the interstellar medium

(...and to nuclear explosions too, unfortunately)

Blast shock waves

Consider a spherical bubble containing a low density, very hot gas with internal pressure p_i and density ρ_i embedded in a cold, dense stationary medium with low pressure p_o and a high density ρ_o (...we can have low pressure with high density at low T because p=nkT)



Because of the high pressure difference, the bubble will start to expand rapidly

Blast shock waves

If the difference between internal and exte
pressure is sufficiently large, the expansi
speed will be supersonic with respect t
sound speed of the surrounding mediur

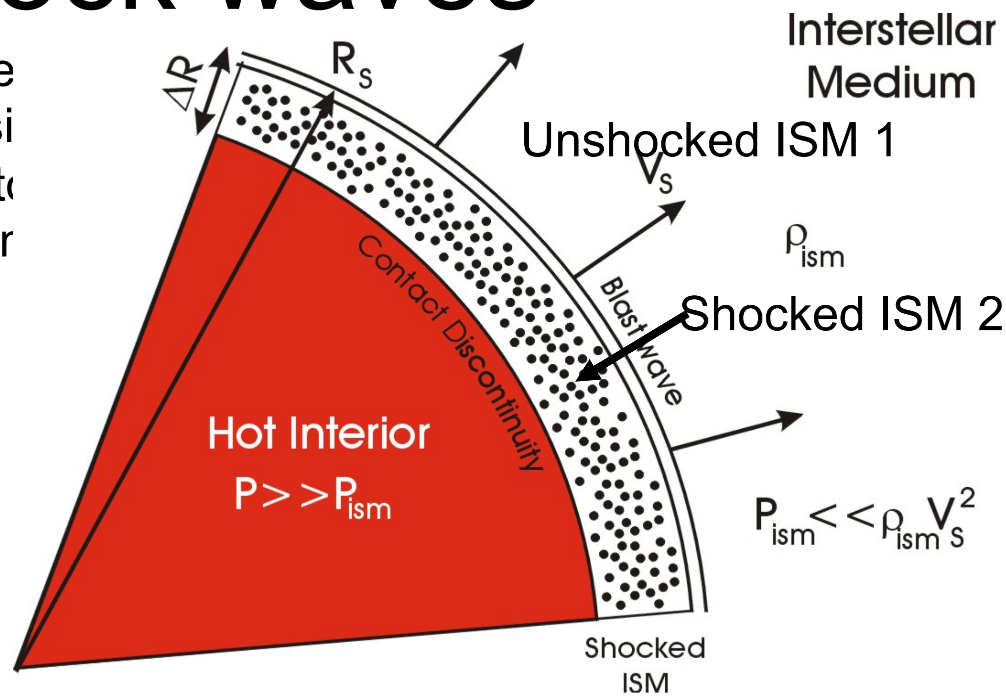
$$M_s^2 = (\frac{V_s}{c_s})^2 = \frac{\rho_o V_s^2}{\gamma p_o} \gg 1$$

$$V_s = (\frac{\gamma+1}{2})^{1/2}(\frac{P_2}{\rho_o})^{1/2}$$

Substituting Vs in M^2_s we get

$M^2_s = [(\gamma + 1)/\gamma](p_2/p_0)$

Imply that $M_s>1$ if $p_2/p_o > 2\gamma/(\gamma+1)$



The key point is that at the interface between hot interior and the shocked material a contact discontinuity forms where there MUST be pressure equilibrium $p_2 = p_i$, because the relative speed between the two media is zero (like in the case of supersonic piston) so that M>>1 if $p_i >> p_0$

For istance, the typical observed expansion speed of a supernova remnant is ~10000 km/s, while the sound speed in the ISM ranges 10-100 km/s

Because of supersonic speed, a shock will form at the outer edge of the bubble (which acts as a supersonic piston) .

This shock is usually called blast wave

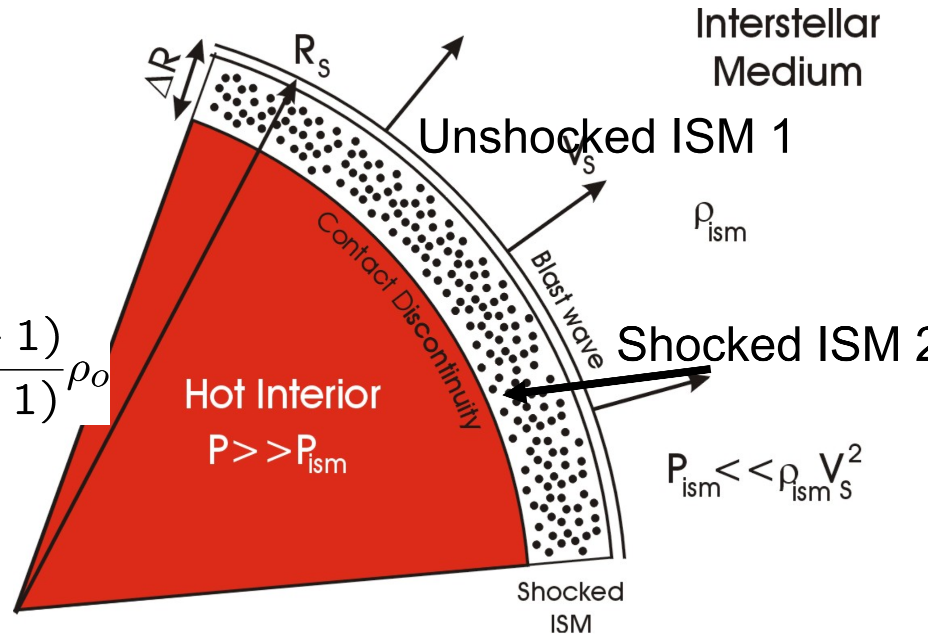
Blast shock waves

The mass that has been swept up by the expanding bubble will collect in a dense shell at its outer rim

If M_s>>1, the typical density of the shocked material in the shell is

$$\rho_{sh} \approx \frac{(\gamma+1)}{(\gamma-1)}\rho_o$$

This allows us to calculate the thickness of the shell



Neglecting, for now, the mass of the bubble (because $\rho_i<<\rho_o$), a bubble with radius R has swept up a mass from surrounding ISM

$$M_{sw} = \frac{4}{3}\pi\rho_o R^3$$

This mass is now in a shell with thickness ΔR with density ρ_{sh} → if ΔR<<R

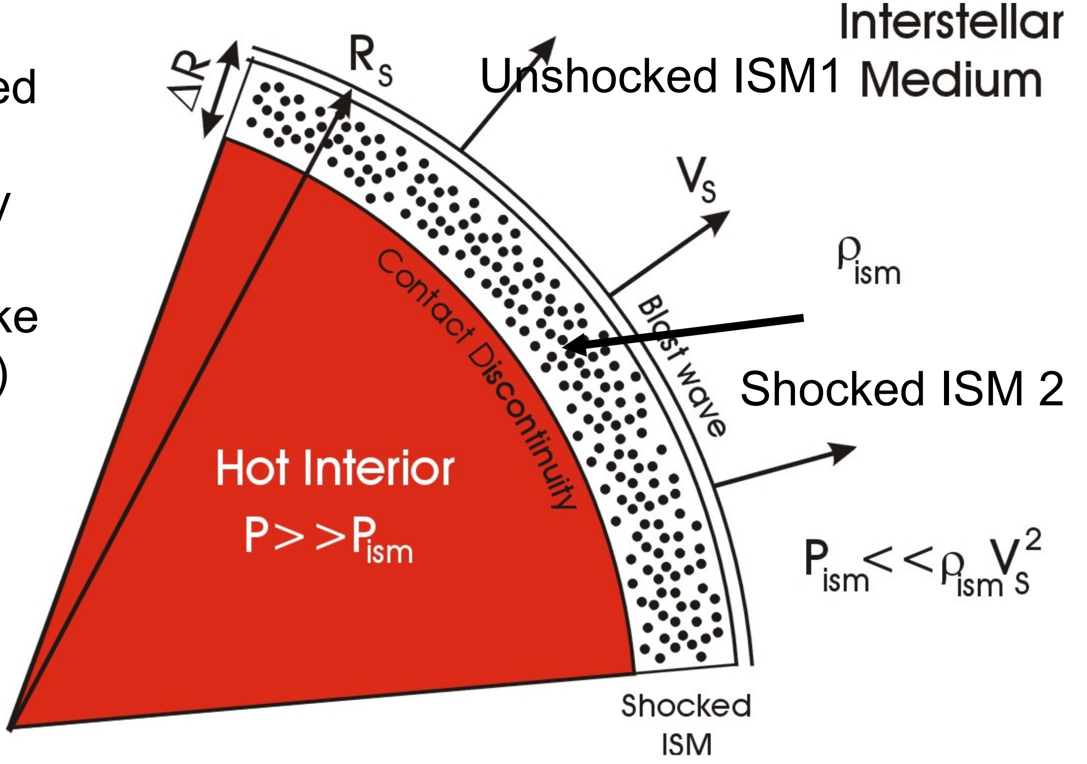
$$M_{sw} \approx 4\pi\rho_{sh}R^2\Delta R$$

For γ =5/3, ΔR=0.083 R
→ thin shell approximation is fine

Combining we get $$\Delta R \approx \frac{(\gamma-1)}{3(\gamma+1)}R$$

Blast shock waves

The swept up material is separated from the hot material inside the bubble by a contact discontinuity because the relative speed between the two media is zero (like in the case of supersonic piston)



The expansion speed is

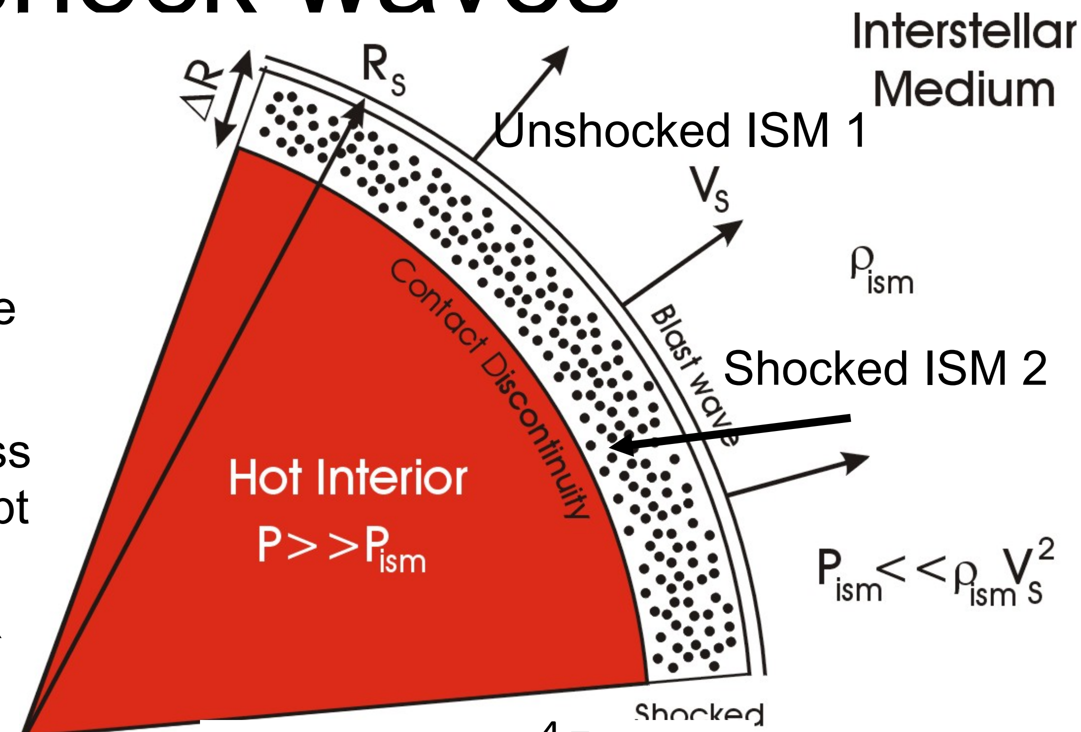
$$\frac{dR}{dt} \approx V_s = (\frac{\gamma+1}{2})^{1/2}(\frac{p_2}{\rho_o})^{1/2}$$

Blast shock waves

$$\frac{dR}{dt} \approx V_s = (\frac{\gamma+1}{2})^{1/2}(\frac{P_2}{\rho_o})^{1/2}$$

We can get the time evolution of the shock radius

Let us assume that most of the mass of the expanding bubble is the swept up mass and which resides in the shocked shell of thickness $\Delta R << R$



→ the mass contained in the bubble is then $M(t) \approx M_{sw} = \frac{4\pi}{3}\rho_o R^3(t)$

The total energy of the bubble consists of the kinetic energy of the expanding massive shell and the internal energy of the hot tenous gas inside the bubble

mistake: here ρ_o is ρ_i

$$E(t) = \frac{1}{2}M(t)(\frac{dR(t)}{dt})^2 + (\frac{4\pi}{3}\rho_o R^3)(\frac{p_i(t)}{\rho_o(\gamma-1)})$$

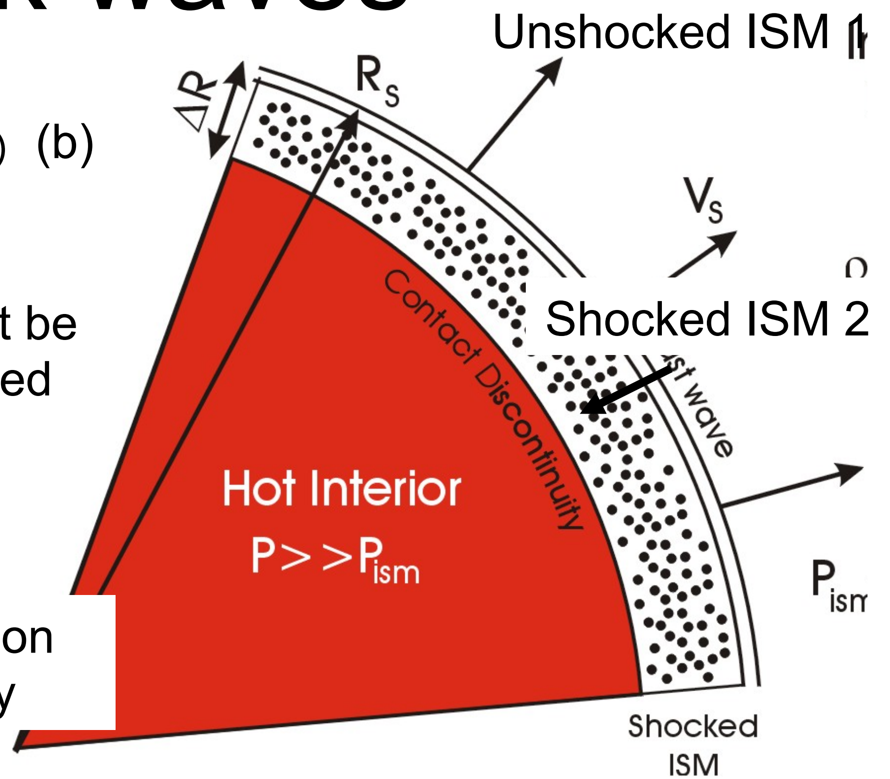
Assuming uniform pressure p_i inside the bubble

Internal energy U = uV = (ρe)V, e = specific energy

Blast shock waves

mistake: here ρ_o is ρ_i

$$E(t) = \frac{1}{2}M(t)(\frac{dR(t)}{dt})^2 + (\frac{4\pi}{3}\rho_o R^3)(\frac{p_i(t)}{\rho_o(\gamma-1)}) \quad \text{(b)}$$



The interior pressure in the hot bubble must be roughly equal to the pressure in the shocked material in the shell

$$p_i \approx p_2 \approx \frac{2}{\gamma+1}\rho_o(\frac{dR}{dt})^2 \quad \text{(a)}$$

Which is simply the pressure-balance condition which must hold at the contact discontinuity

Inserting (a) into (b) we get $$E(t) = \frac{2\pi}{3}\rho_o R^3(\frac{dR}{dt})^2[1+\frac{4}{\gamma^2-1}]$$

→ the ratio of thermal energy and kinetic energy is a constant $E_{th}/E_{kin} = \frac{4}{\gamma^2-1}$

For γ=5/3, E_{th}/E_{kin} =9/4 → E_T=E_{th}+E_{kin}= (1+9/4)E_{kin}=(13/4)E_{kin}

➔ E_{kin} = (4/13)E_T ~ 31% E_T...most part of the initial energy is thermal energy

Blast shock waves

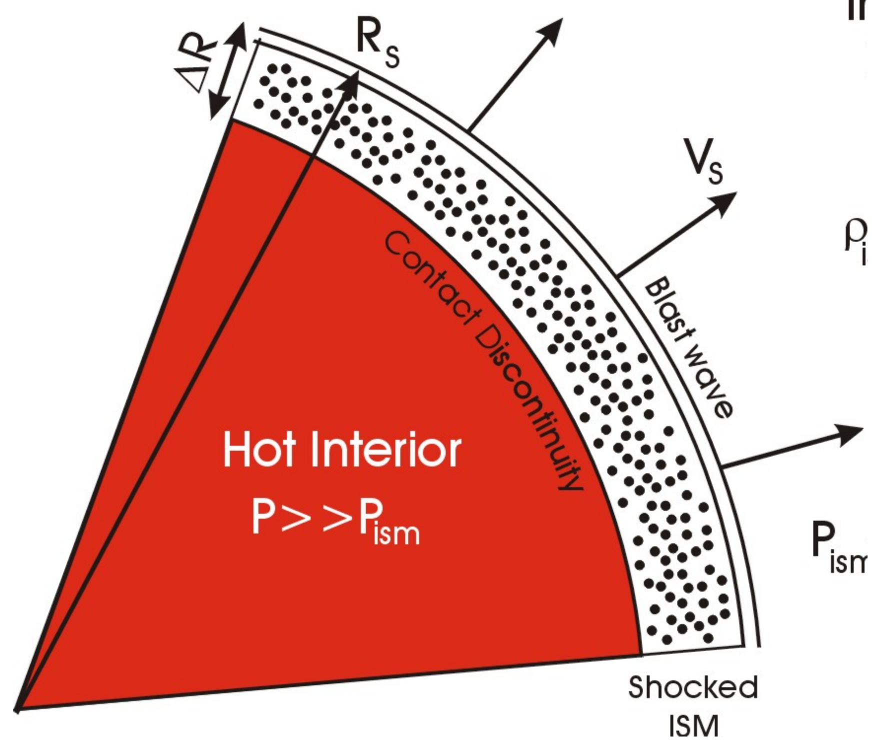
$$E(t) = \frac{2\pi}{3}\rho_o R^3 (\frac{dR}{dt})^2 [1 + \frac{4}{\gamma^2 - 1}]$$

→ the total energy can be then written as

$$E(t) = C_\gamma M(t) (\frac{dR}{dt})^2$$

With $$C_\gamma = \frac{\gamma^2 + 3}{2(\gamma^2 - 1)}$$

For an ideal monoatomic gas C_γ=1.625

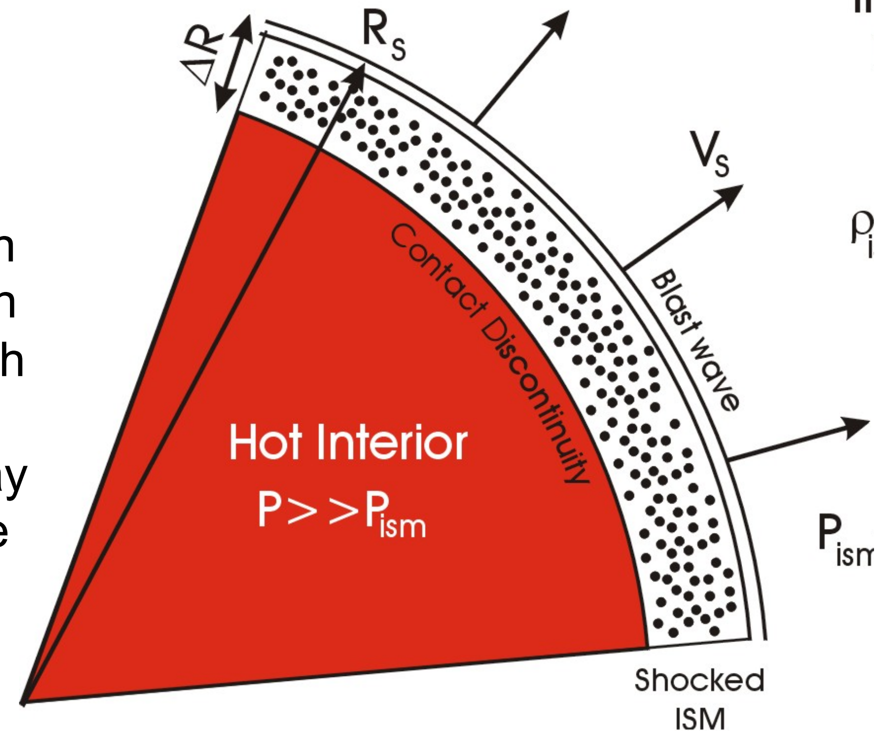


C_γ is approximate because of the various approximations made in the derivation. However, more exact treatments come at the same result with a somewhat smaller value

Blast shock waves

$$E(t) = C_\gamma M(t)(\frac{dR}{dt})^2$$

The energy may depend on time, because in general there may be some energy source in the bubble which fuels the hot bubble through emission of power L, as for istance a star driving a strong stellar wind, or the SNRs may radiate away part or all their energy at some point of its evolution



Two cases:

i) a point explosion where a fixed amount of energy is supplied impulsively at t=0 and where no energy losses occur afterwards (blast shock wave, SN explosions)

Ii) a constant energy supply at some luminosity L=dE/dt so that E(t)=L x t, which can serve as a crude model of the energy of a bubble blown into the ISM by a strong stellar wind

Blast shock waves

Blast wave generated by a 4.8 kiloton explosion

Explosions (eg nuclear detonations) generate blast shock wave, very much like a supernova: the piston is the exploding materal

Blast shock waves

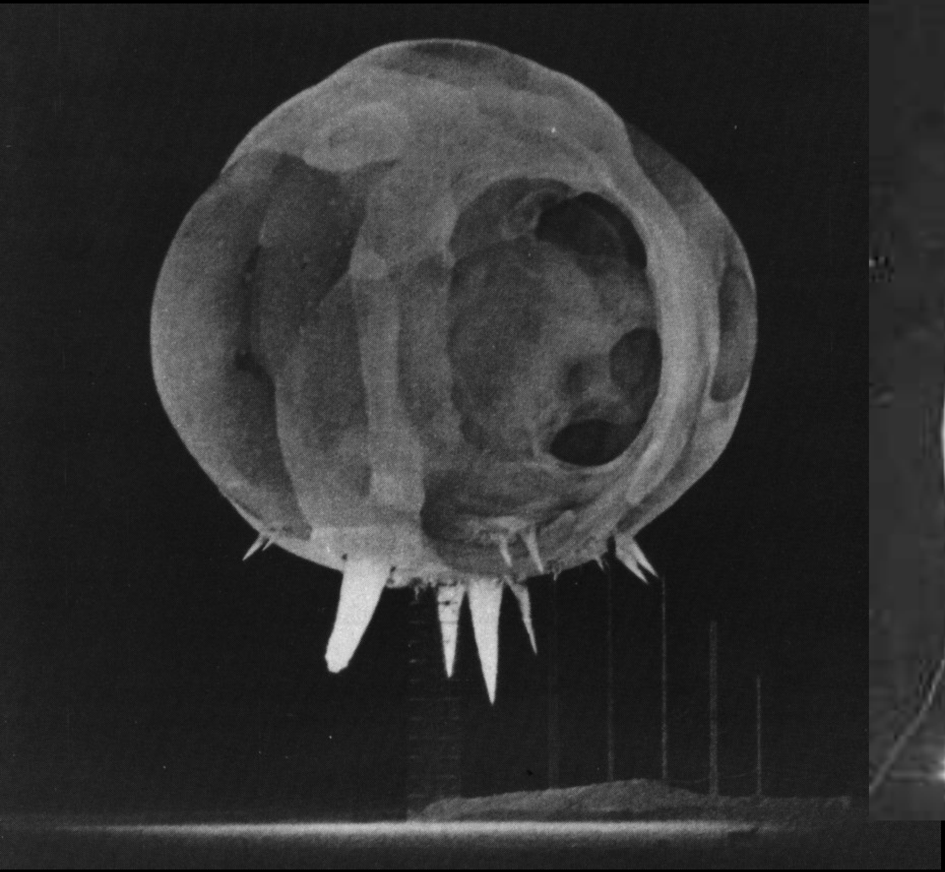
Most of the "experimental" knowledge comes from military nuclear tests during the cold war

For long time, and still now, most of the data on nuclear tests are classified top secret

Whatever the mechanism by which initial energy is liberated, the subsequent blast wave evolution does not depend on it!

Blast shock waves

10 ms from detonation



1 ms from detonation

Trinity test

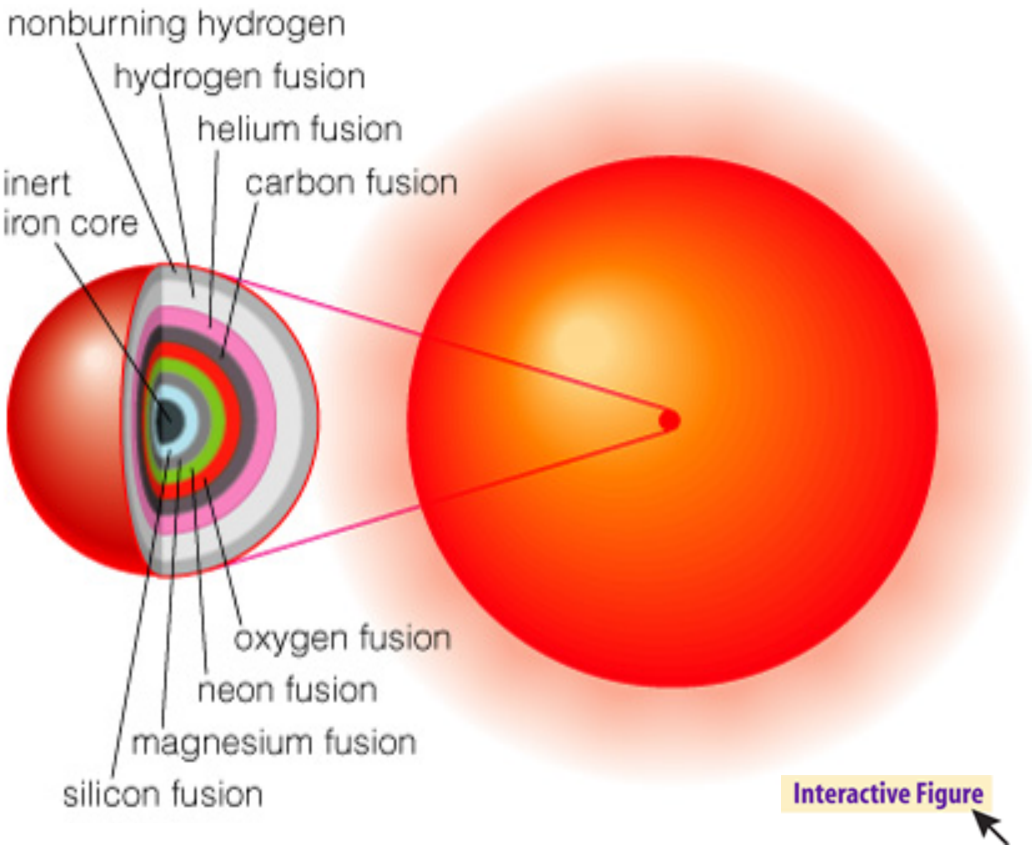
Supernova Blast Waves

Supernovae:	
Type Ia	Mechanism: explosive carbon burning in a mass-accreting white dwarf
Type Ib-Ic & Type II	Core collapse of massive star

We may neglect the details of the explosive event: for us is a sudden, point-like release (a $\delta(x,t)$) of energy

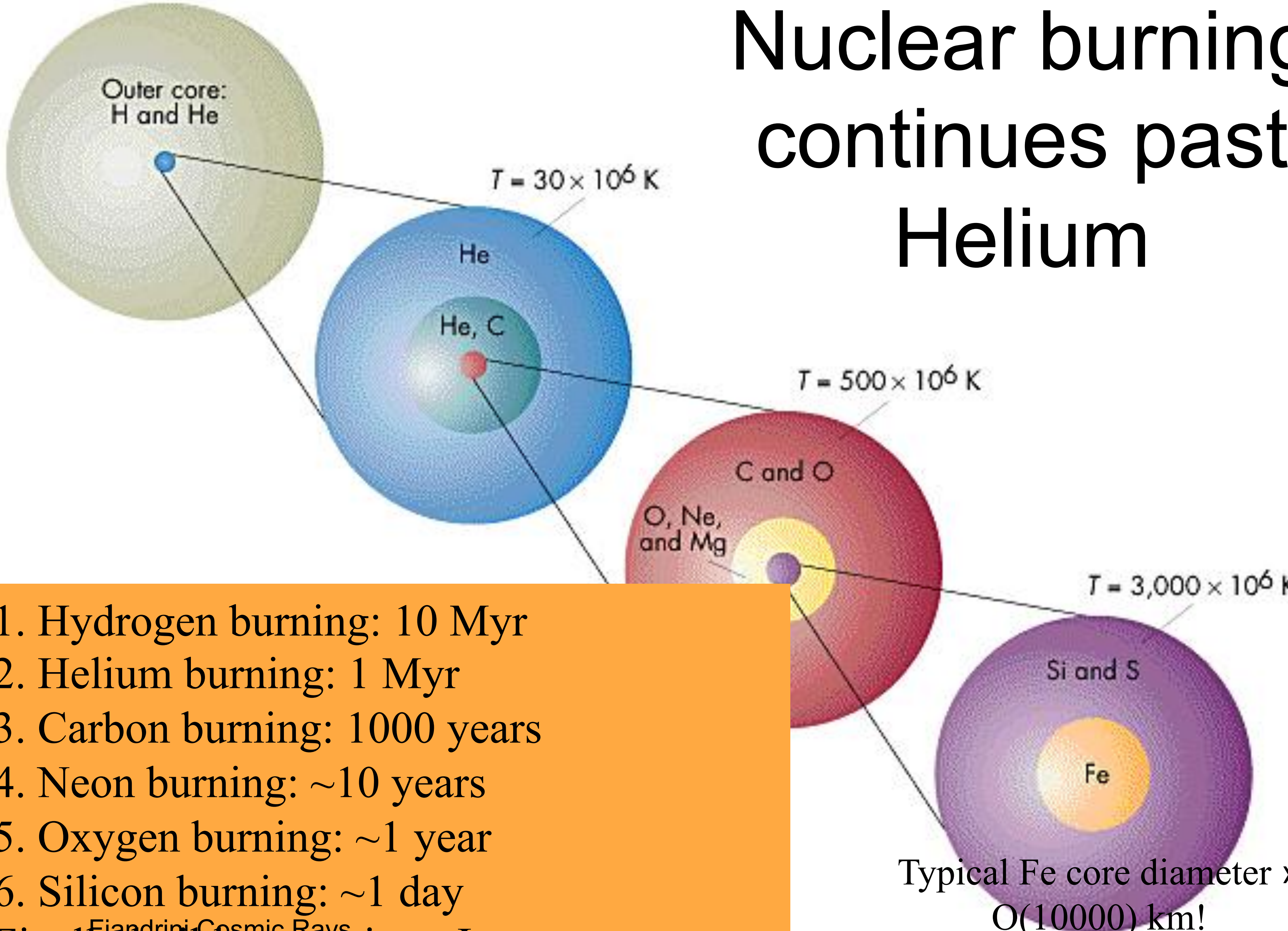
While the composition of the material of the SNR does depend on the mechanism (ie star composition) and on the surrounding ISM composition

Multiple Shell Burning



- Advanced nuclear burning proceeds in a series of nested shells, like an onion

Nuclear burning continues past Helium



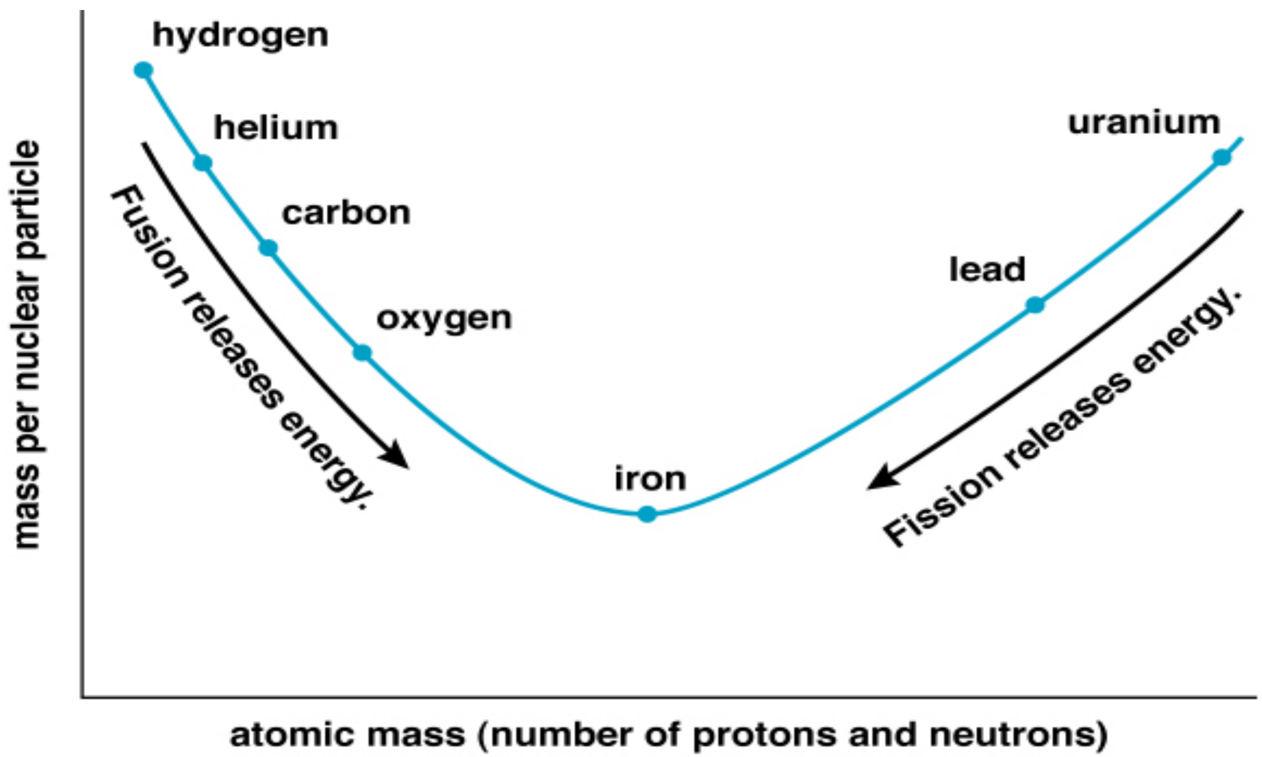
1. Hydrogen burning: 10 Myr
2. Helium burning: 1 Myr
3. Carbon burning: 1000 years
4. Neon burning: ~10 years
5. Oxygen burning: ~1 year
6. Silicon burning: ~1 day

Finally builds up an inert Iron core

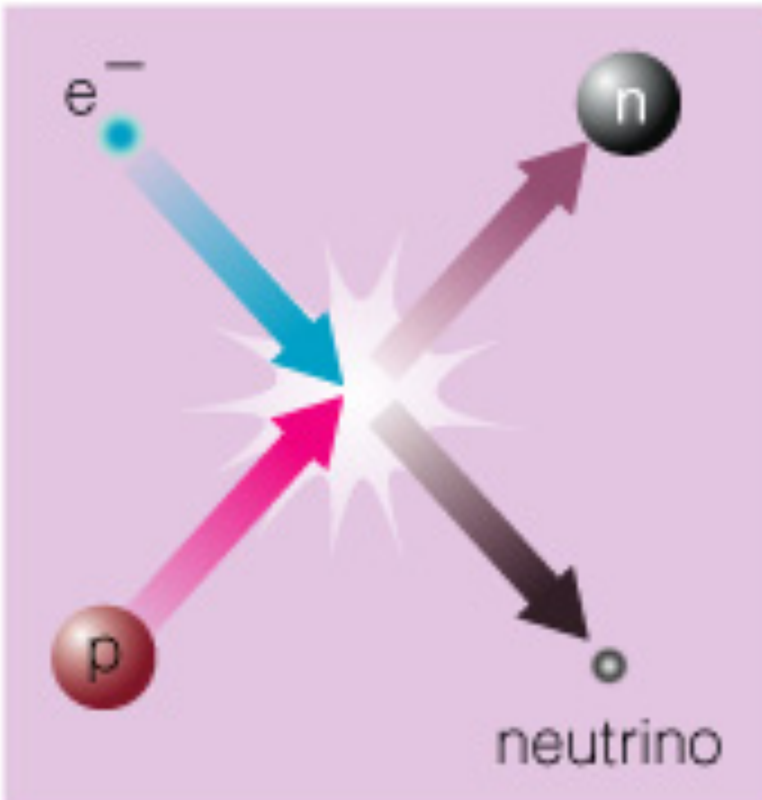
Typical Fe core diameter » O(10000) km!

- The supergiant has an inert Fe core which collapses & heats
  - Fe can not fuse
  - It has the lowest mass per nuclear particle of any element
  - It can not fuse into another element without *creating* mass

So the Fe core continues to collapse until it is stopped by electron degeneracy.
(like a White Dwarf)



Supernova Explosion



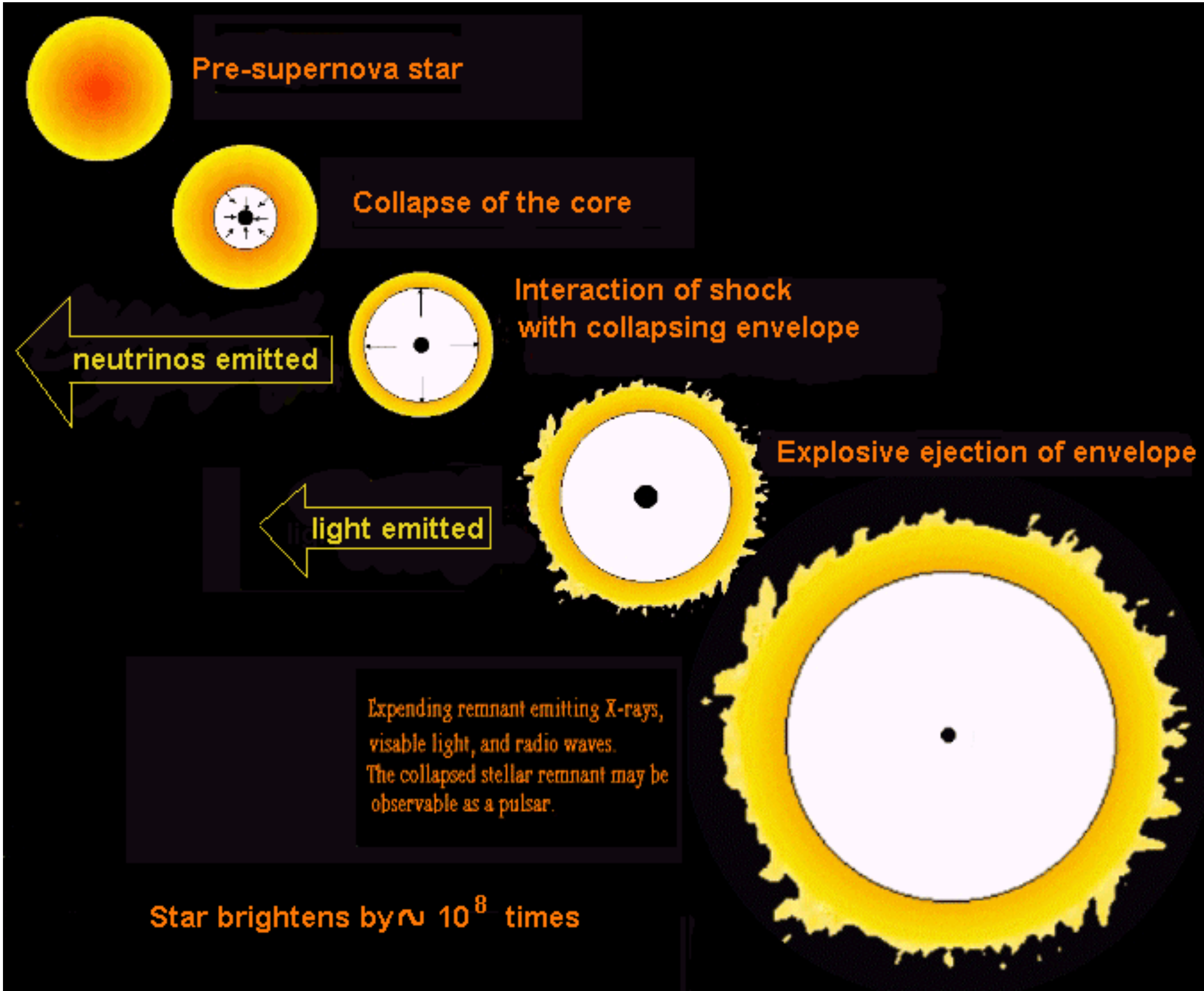
- Core degeneracy pressure goes away because electrons combine with protons, making neutrons and neutrinos

$P + e^- \rightarrow n + \nu_e$

- Neutrons collapse to the center, forming a **neutron star**

Core collapse

- Iron core is degenerate
- Core grows until it is too heavy to support itself ($M_{nuc} > M_{chandrasekhar}$)
- Core collapses, density increases, normal iron nuclei are converted into neutrons with the emission of neutrinos
- Core collapse stops, neutron star is formed
- Rest of the star collapses on the core, but bounces off the new neutron star (also pushed outwards by the neutrinos)


Pre-supernova star
Collapse of the core
Interaction of shock
with collapsing envelope
neutrinos emitted
Explosive ejection of envelope
light emitted
Expending remnant emitting X-rays,
visable light, and radio waves.
The collapsed stellar remnant may be
observable as a pulsar.
Star brightens by ~ 10^8 times

Shock Waves in Supernovae

- Discontinuity in velocity, density and pressure in a flow of matter.
- Unlike a sound wave, it causes a permanent change in the medium
- Shock speed >> sound speed - between 30,000 and 50,000 km/s.
- Shock wave may be *stalled* if energy goes into breaking-up nuclei into nucleons.
- This consumes a lot of energy, even though the pressure (nkT) increases because n is larger.

Supernova Energetics

Same source for supernovae (Ib/Ic and II),

- Explosion powered by the collapse (death) of a massive core
- Energy source: Potential Energy from the collapse of the iron core down to a neutron star or black hole:

$$\text{Energy} = G\,M^2_{core}/r_{NS,BH} - G\,M^2_{core}/r_{before\ collapse}$$

$$> 10^{53}\ \text{ergs}$$

$M_{core} \sim$ 1.4-3 solar masses

$R_{NS,BH} \sim$ 10 km

$R_{core} \sim$ 10,000 km

Supernova energetics

Core Collapse Supernova Energetics

Liberated gravitational binding energy of neutron star:

$E_b \approx 3 \times 10^{53}$ erg $\approx$ 17% $M_{SUN}c^2$

This shows up as

- 99% Neutrinos
- 1% Kinetic energy of explosion (1% of this into cosmic rays)
- 0.01% Photons (outshine host galaxy)

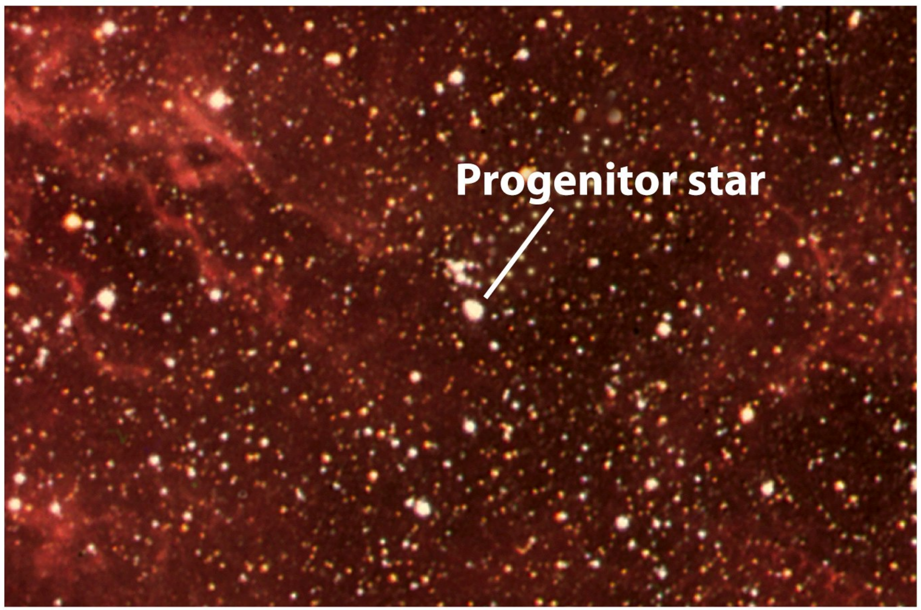
Neutrino luminosity

$L_\nu \approx 3\times10^{53}$ erg / 3 sec $\approx 3\times10^{19}\ L_{SUN}$

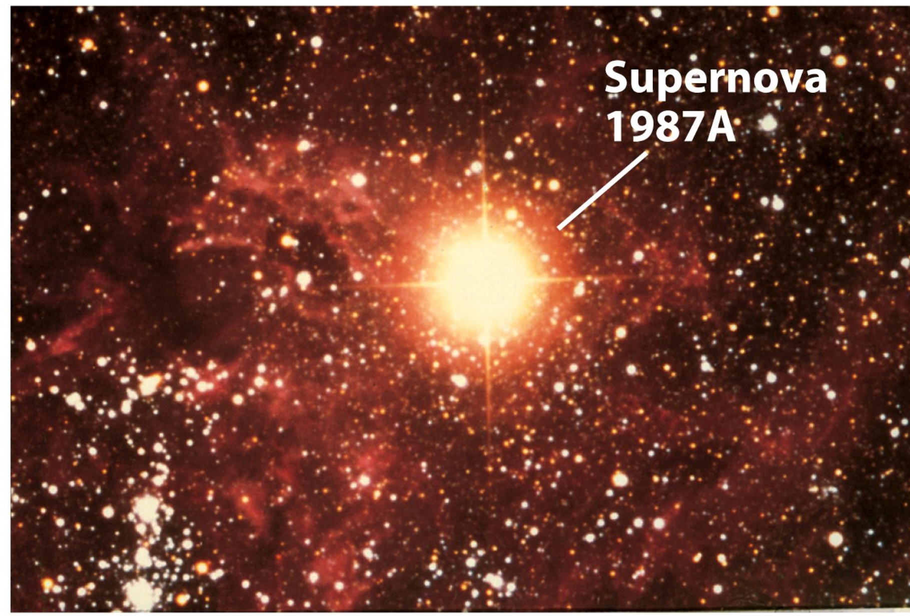
While it lasts, outshines the photon luminosity of the entire visible universe!

Georg Raffelt, Max-Planck-Institut für Physik (München)

In 1987 a nearby supernova gave us a close-up look at the death of a massive star



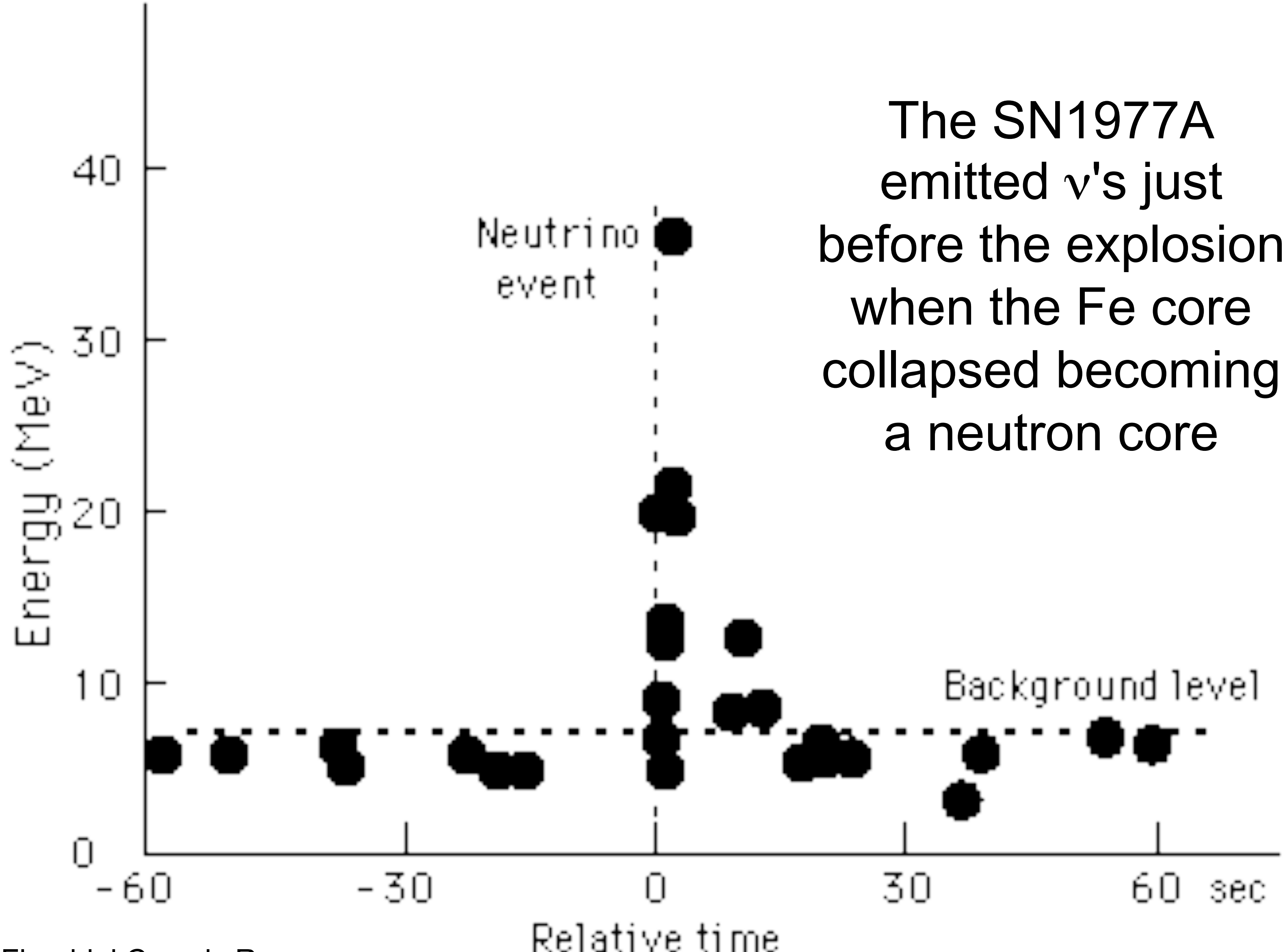
Before the star exploded



After the star exploded

At peak its luminosity was greater than host galaxy luminosity (Large magellanic Cloud)

Neutrinos from SN1987A

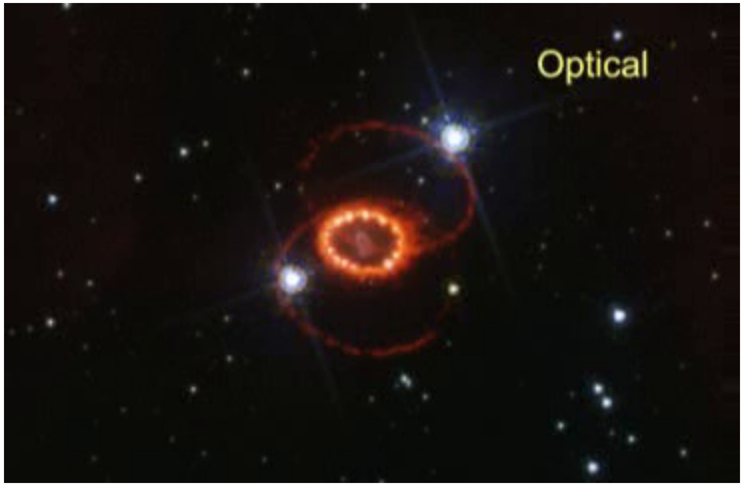


The SN1977A emitted ν's just before the explosion when the Fe core collapsed becoming a neutron core

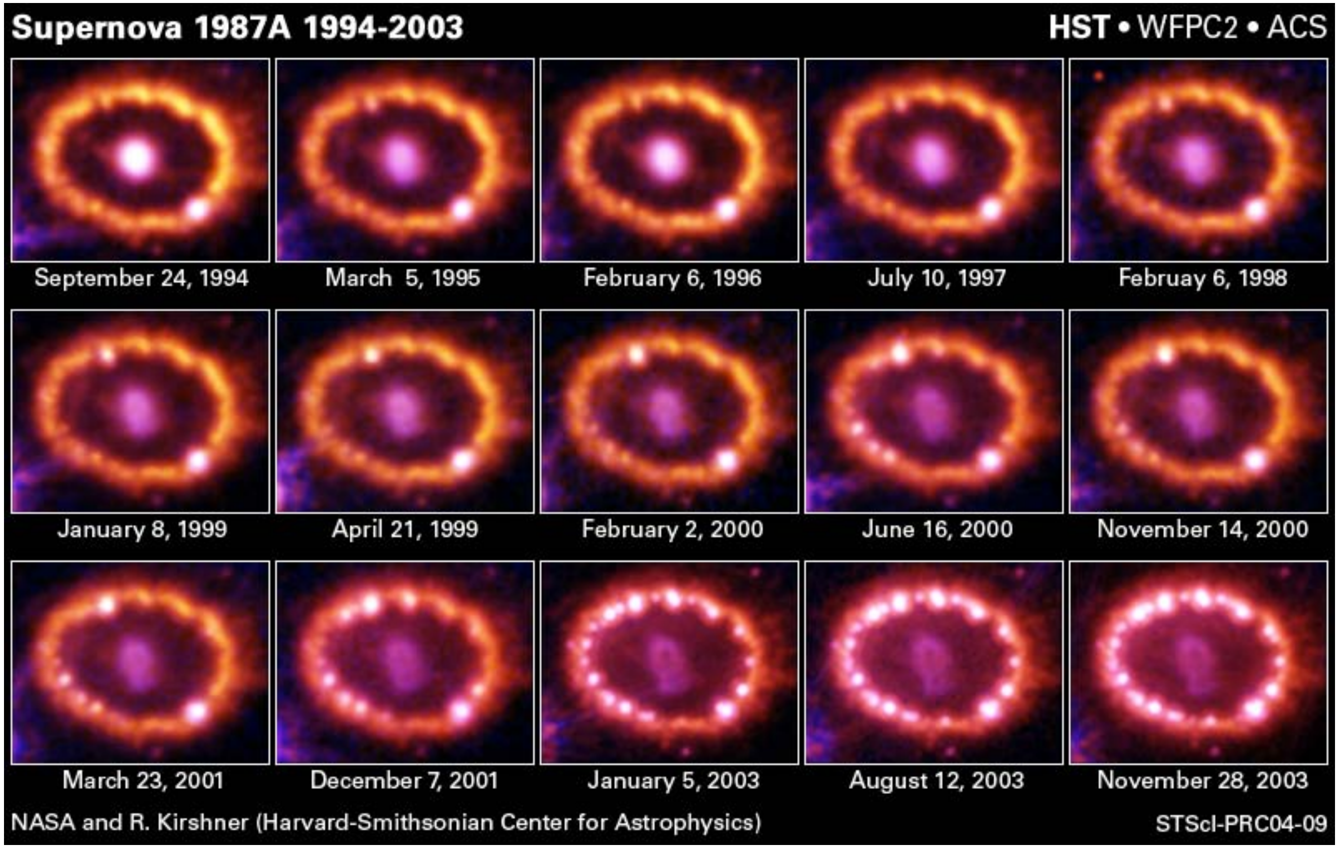
SN1987A – Blue Supergiant Supernova?

The progenitor of SN1987A was a blue giant with a mass of about 18 M_{sun}.

- Probably, the high-mass progenitor of SN1987A lost most of its outer layer by a slow stellar wind long before the supernova explosion.
- Right before the supernova explosion, a fast wind pushes the envelop to make a cavity around the star. Making the outer layer of the star unusually thin and warm
- The outer gas cloud forms a ring.
- The shockwave from the supernova explosion was expected to hit the edge of the ring around 1999.
- Chandra X-ray images from 1999 to 2005 shows brightening of the ring.



Shock hits inner ring

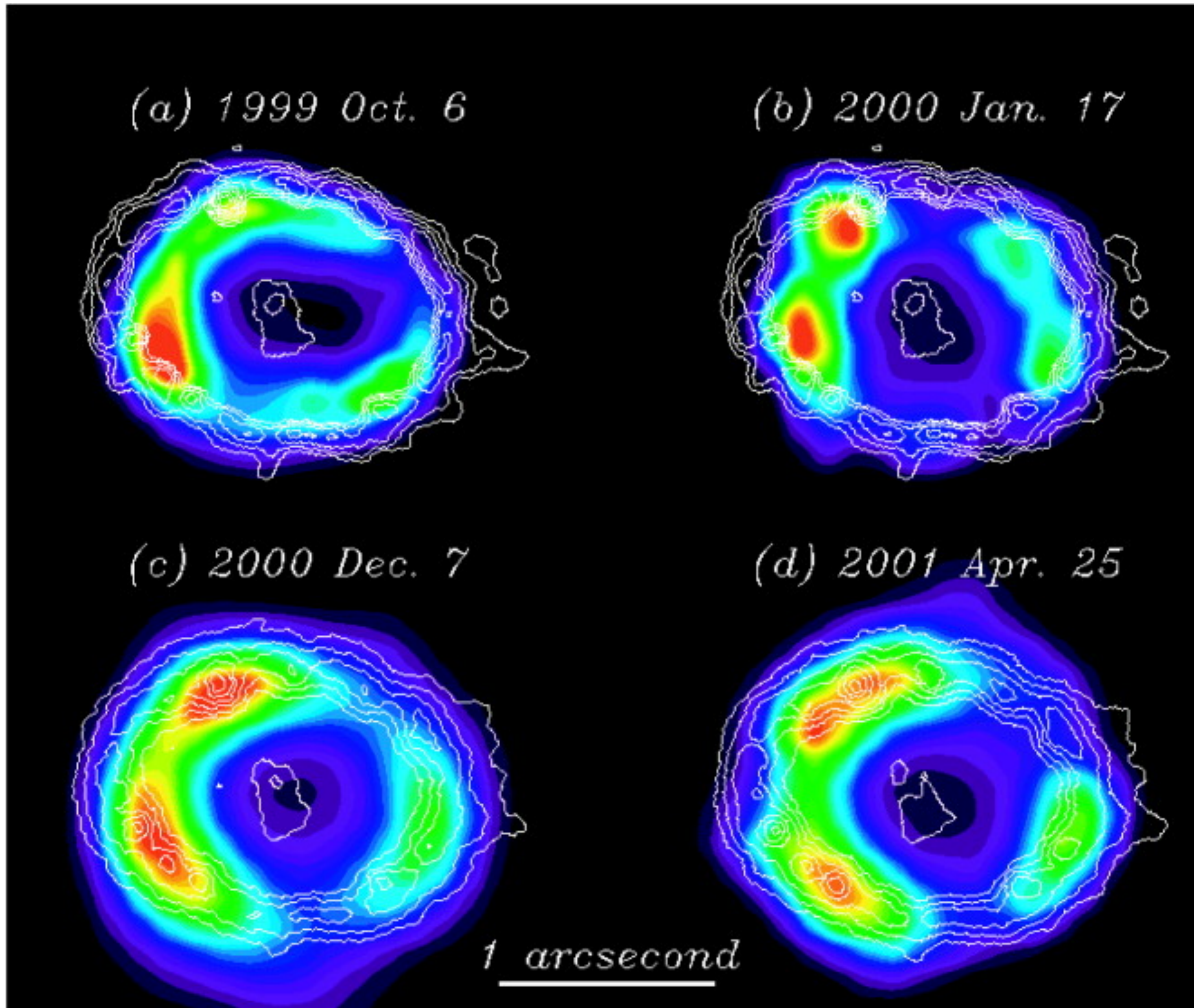


The shock has hit the inner ring at 20,000 km/s, lighting up knots of shocked, compressed and heated material in the ring which is 160 billion km wide.

Chandra X-ray Images of SN 1987A

- X-ray intensities (0.5 – 8.0 keV) in colour with HST Hα images as contours

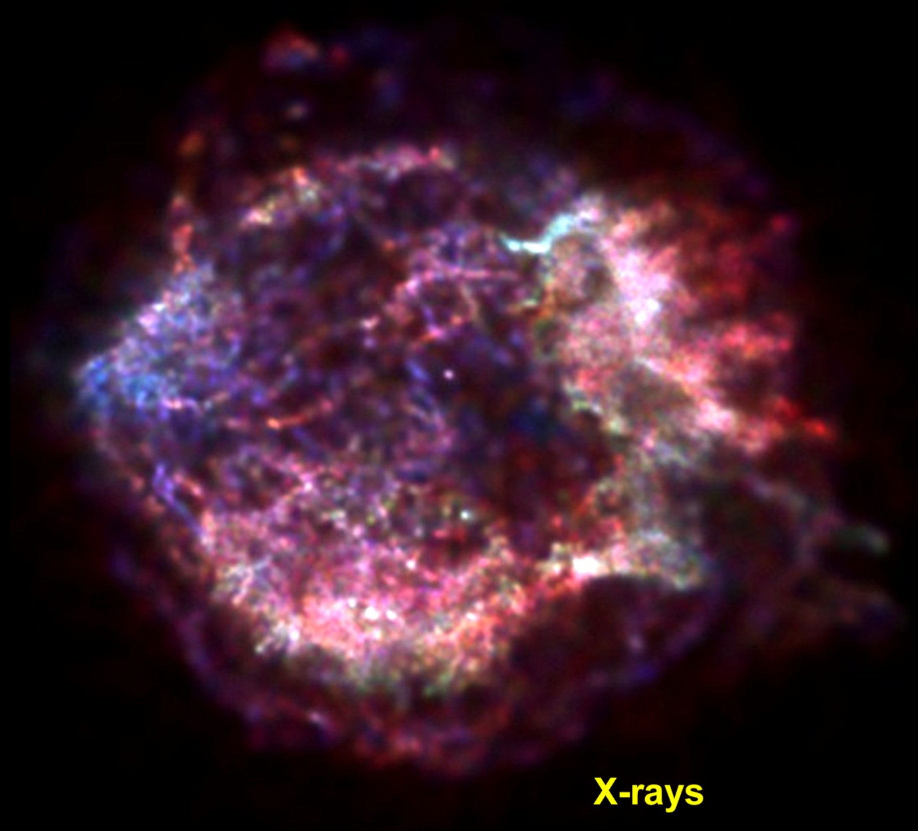
- Low energy X-rays are well correlated with optical knots in ring – dense gas ejected by progenitor?
- Higher energy X-rays well correlated with radio emission – fast shock hitting circumstellar H II region?
- No evidence yet for emission from central pulsar

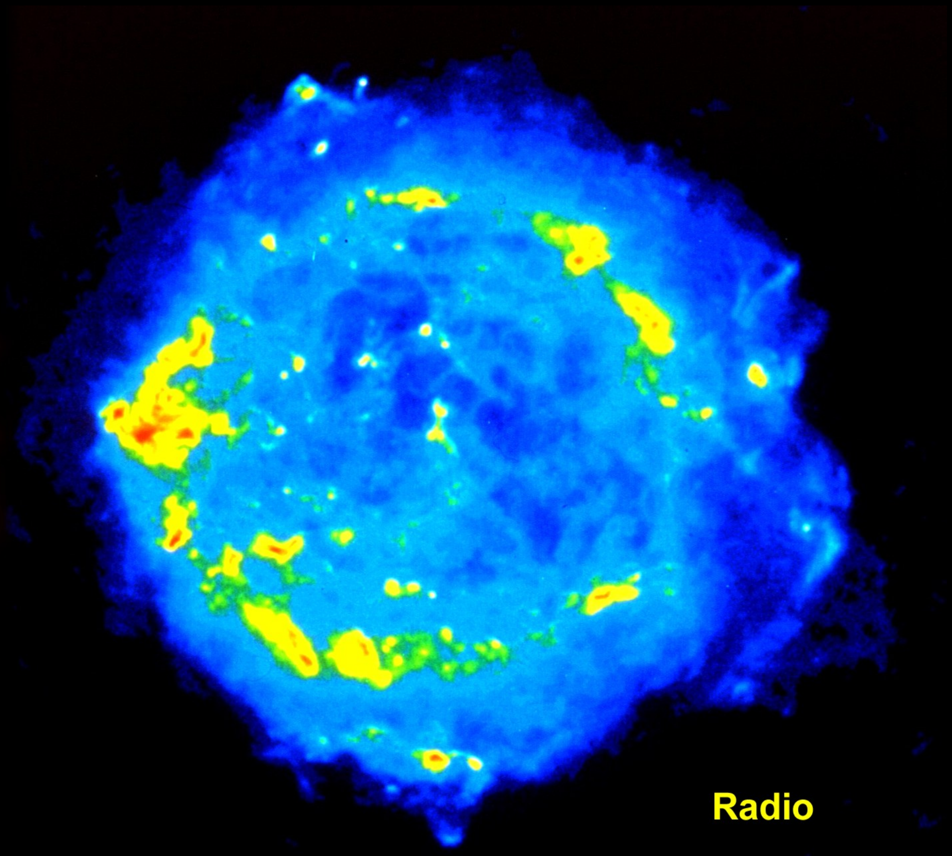


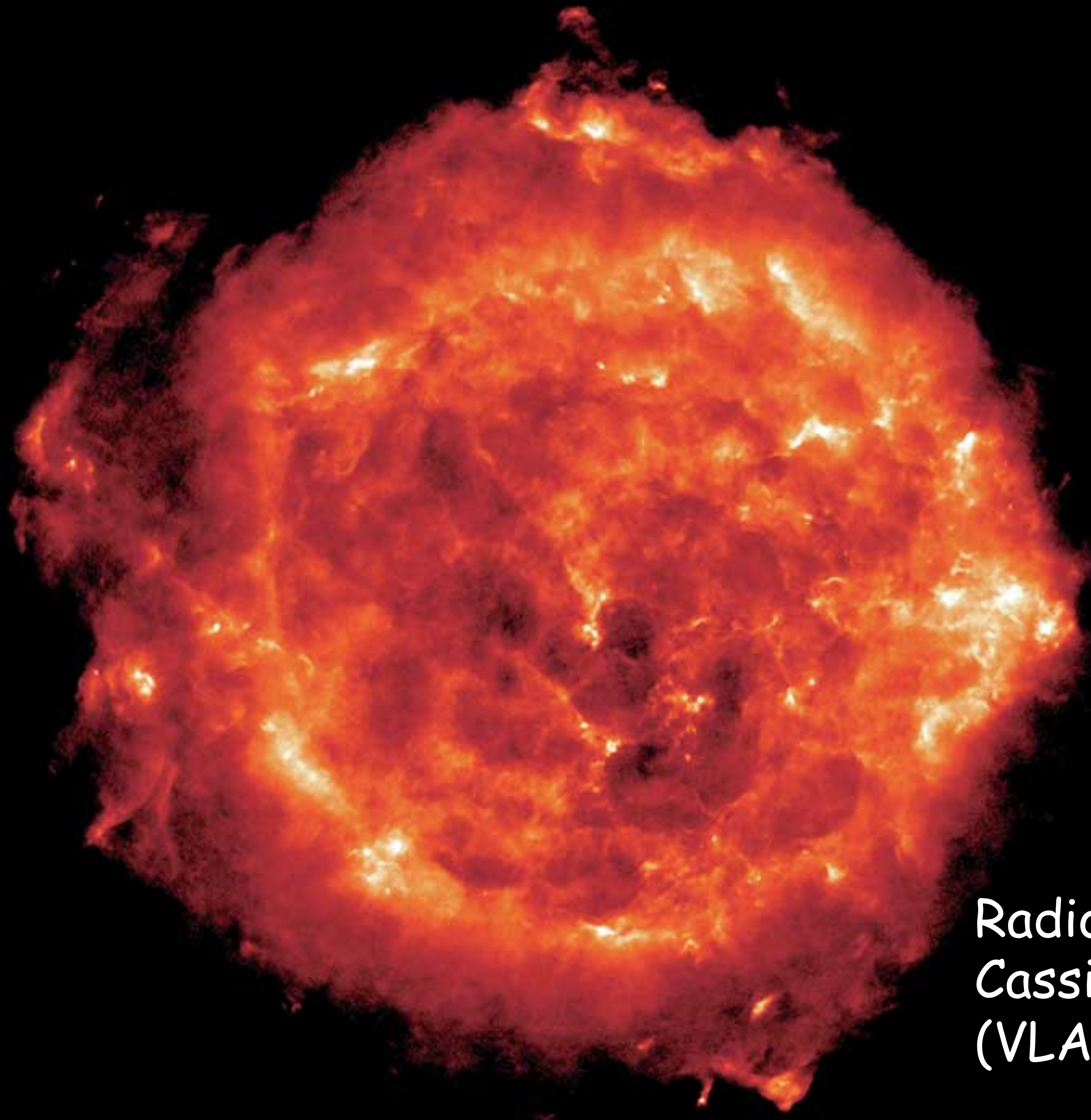
SuperNova Remnants (SNRs)

A supernova remnant (SNR) consists essentially of the stellar ejecta of the SN explosion embedded in a hot expanding bubble, preceded by swept-up interstellar material and an outer blast wave (strong shock) propagating into the interstellar medium

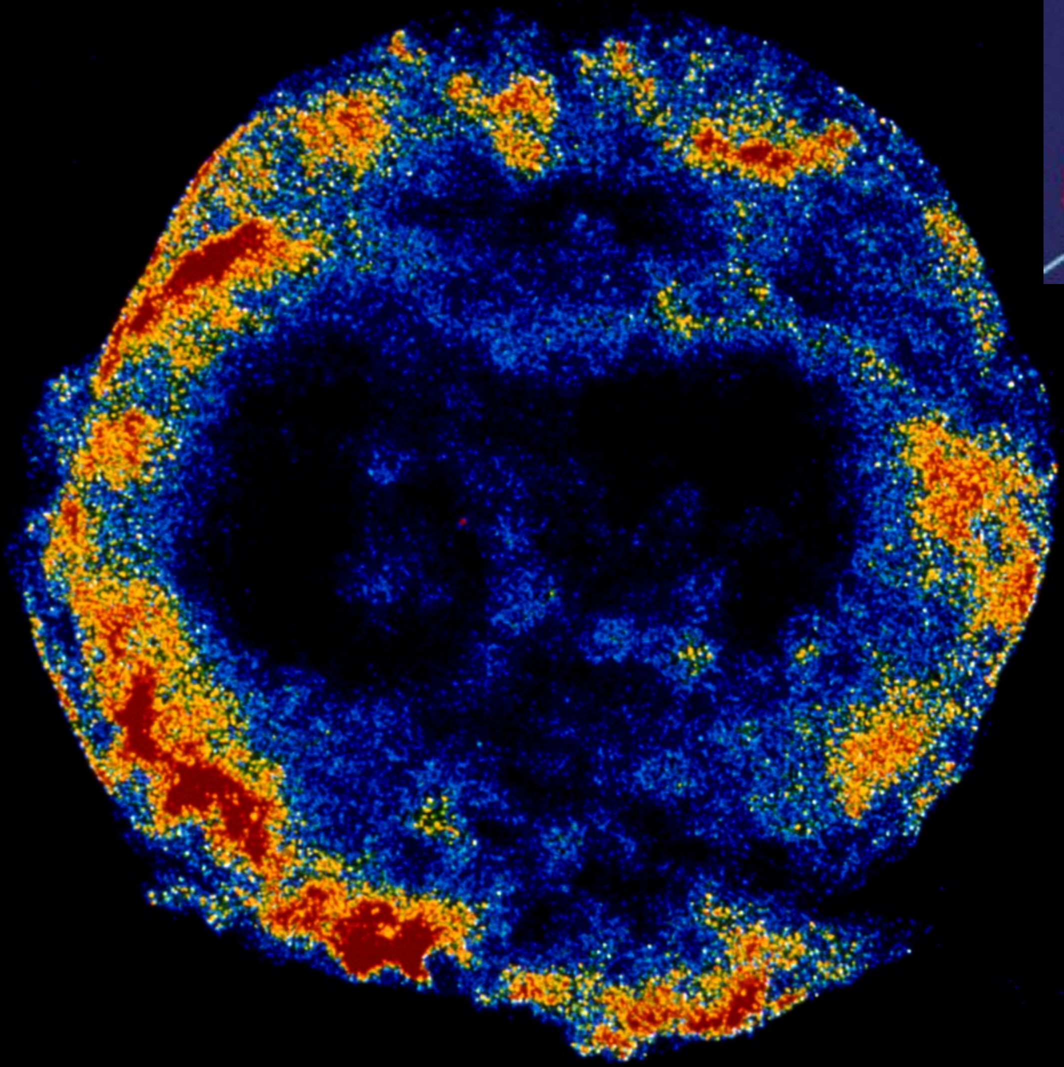
Supernova Remnant Cassiopeia A

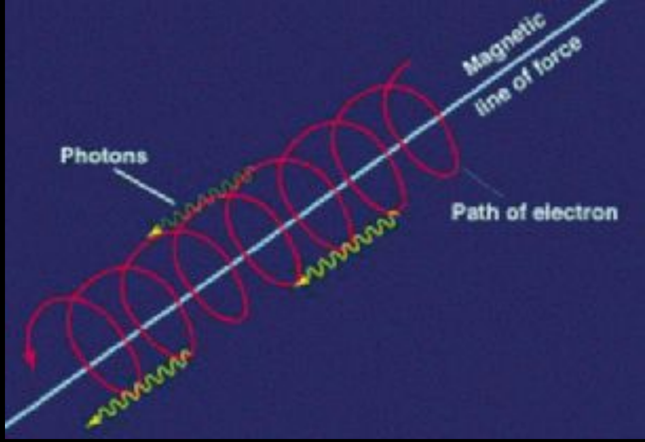






Radio map
Cassiopeia A
(VLA)





Remnant of Tycho's supernova of 1572 AD

SNR evolution: energy budget

The mechanical energy of the ejecta is of the order $E_{snr} \sim 0.01x\ E_{sn} \sim 10^{51}$ erg

This is the energy that fuels the explosive event and ultimately creates a SNR

The typical speed V_s of this material can be estimated by energy conservation

Let us assume that all the mechanical energy is converted into kinetic energy of the remnant and that energy loss is negligible

$$E_{snr} = \frac{1}{2} M_{snr} V_s^2 \quad \Rightarrow \quad V_s = \left(\frac{2E_{snr}}{M_{snr}}\right)^{1/2}$$

The mass is $\mathbf{M_{snr} = M_{ej} + M_{sw}}$, the sum of explosively ejected mass from the star at the time of explosion and the mass swept up added later as the remnant sweeps up more and more ISM material

SNR evolution: free expansion phase

If the density of the ISM is constant, then $M_{snr} = M_{ej} + \frac{4\pi}{3}\rho_o R_s^3$

Since we know the typical energy involved and that the mass of the remnant must be several solar masses, we can estimate the typical expansion speed

$$V_s \simeq 3000(\frac{E_{snr}}{10^{51}\ erg})^{1/2}(\frac{M_{snr}}{10M_{sun}})^{-1/2}\ km/s$$

Initially, the mass consists almost entirely of the ejecta mass $M_{snr} \sim M_{ej} \sim 2 - 10\ M_{\odot}$

➔ the expansion speed is almost constant $V_s \approx (\frac{2E_{snr}}{M_{ej}})^{1/2} = 10000\ km/s$

This is >> of sound speed (~10-100 km/s)→ shock must form

The bubble expands as if there is not surrounding medium, therefore this phase is called free expansion phase and lasts few hundreds of years until the swept up mass increases enough to start the deceleration

Sedov-Taylor phase

As more and more ISM gas is swept up, the mass of remnant increases

After few hundreds of years, the mass is dominated by the swept up material so that

$$M_{snr} = M_{ej} + \frac{4\pi}{3}\rho_o R_s^3 \approx \frac{4\pi}{3}\rho_o R_s^3 \qquad V_s \approx (\frac{2E_{snr}}{M_{ej}})^{1/2} \approx (\frac{6E_{snr}}{4\pi\rho_o})^{1/2} R_s^{-3/2}$$

Speed decreases as $R^{-3/2}$, result of the increasing remnant mass ~ R^3

This is the so called Sedov-taylor phase or energy-conserving phase

The typical expansion speed remains supersonic for a considerable time, typically >100000 years, so that the shock at the outer boundary persists in this evolutionary phase

Sedov-Taylor phase

The transition between the free expansion and the Sedov-taylor phases occurs gradually when the radius of the remnant reaches the deceleration radius, defined as the radius at which $M_{ej} = M_{sw}$

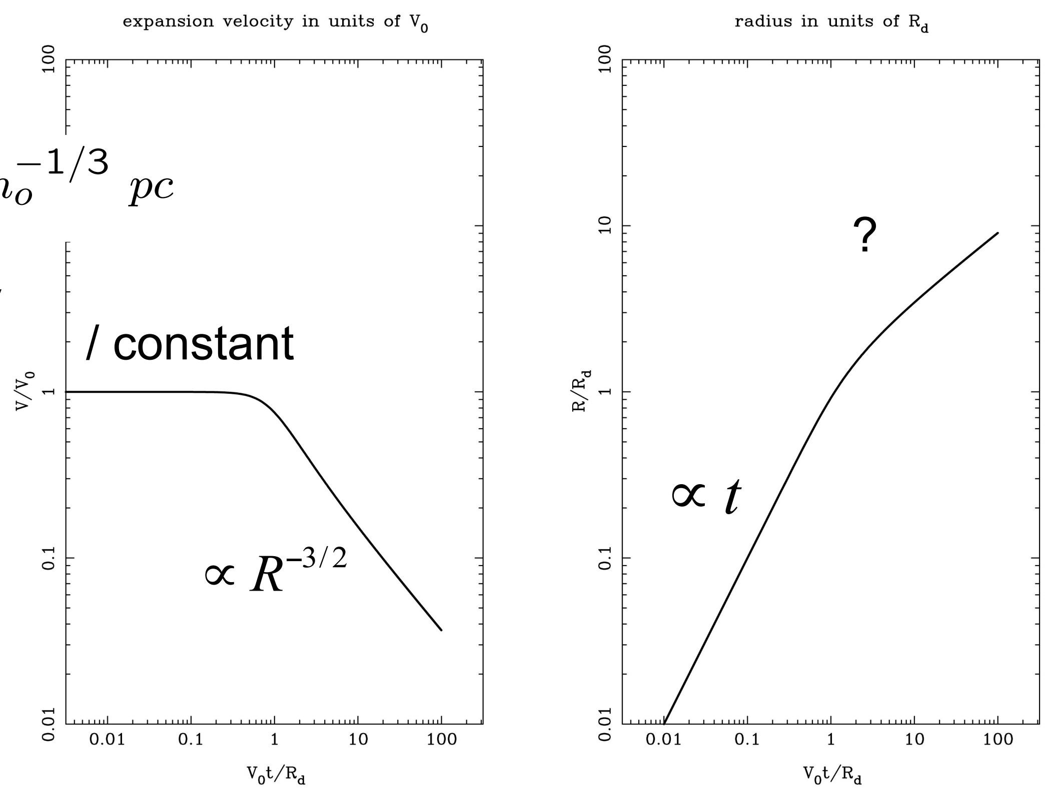
$$\frac{4\pi}{3}\rho_o R_d^3 = M_{ej}$$

$$R_d = \left(\frac{3M_{ej}}{4\pi\rho_o}\right)^{1/3} \approx 2.2\left(\frac{M_{ej}}{M_{sun}}\right)^{1/3} \times n_o^{-1/3}\ pc$$

Here $n_o = \rho_o/m_p$ is the number density of ISM, which is typically ~1 cm^{-3}

Assuming $M_{ej} = 5M_{sun}$ we have R_d ~ 4 pc

A crude extimation of time spent during the free expansion phase is ~ R_d/V_{free} ~ 400 years



Sedov-Taylor phase

In a SN explosion, the mechanical energy E_0~ E_{snr} the drives the expansion is supplied impulsively in a point explosion at t=0

If no energy is lost, for istance through radiation losses, E remains constant for t>0

$$E(t) = C_\gamma M(t)(\frac{dR}{dt})^2 = constant \qquad C_\gamma = \frac{\gamma^2+3}{2(\gamma^2-1)}$$

Once the remnant has expanded to a radius larger than R_d, the mass is

$$M(t) \approx M_{sw} = \frac{4\pi}{3}\rho_o R^3(t)$$

→ the energy equation can written as $R^{3/2}\frac{dR}{dt} = (\frac{3E_{snr}}{4\pi C_\gamma \rho_o})^{1/2} = constant$

Sedov-Taylor phase

$$R^{3/2}\frac{dR}{dt} = (\frac{3E_{snr}}{4\pi C_\gamma \rho_o})^{1/2} = constant$$

This relationship between the speed and the radius of the bubble is the same one as derived above using a simple conservation law for the kinetic energy but in this derivation we also take into account of the thermal energy of the hot bubble material

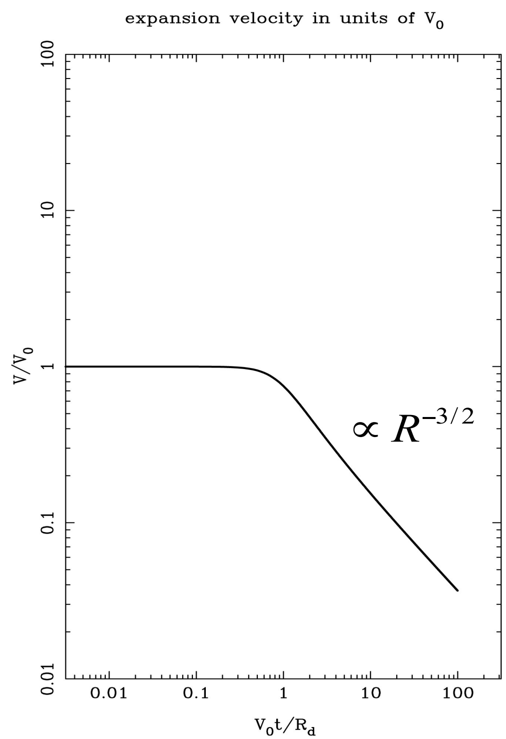
The integration is straightforward

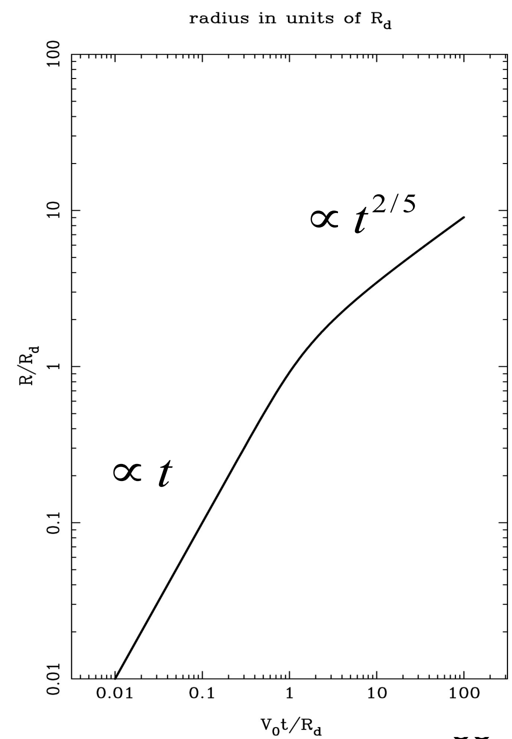
$$\frac{2}{5}R^{5/2} \approx (\frac{3E_{snr}}{4\pi C_\gamma \rho_o})^{1/2} t$$

Assuming R(0) = 0

$$R = (\frac{75}{16}\frac{E_{snr}}{\pi C_\gamma \rho_o})^{1/5} t^{2/5}$$

$$V_s = (2/5)(\frac{75}{16}\frac{E_{snr}}{\pi C_\gamma \rho_o})^{1/5} t^{-3/5}$$





S-T phase: exact solution

The solution interior to the shock obeys the equations

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r},$$

$$\frac{\partial}{\partial t}\left[\rho\left(\mathcal{E} + \frac{U^2}{2}\right)\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \rho u\left(\mathcal{E} + \frac{P}{\rho} + \frac{U^2}{2}\right)\right] = 0$$

with $\mathcal{E} = P/[\rho(\gamma - 1)]$, for an ideal gas.

Sedov recognised that the solution must be self-similar, *i.e.*, that at any time the pressure, density, velocity, *etc.* at all points interior to the shock can be expressed in terms of *single* similarity variable

$$\xi = r\left(\frac{\rho_0}{t^2 E}\right)^{1/5} = \xi_0 \frac{r}{r_s}.$$

Thus

$$\rho(r,t) = \rho_2 \alpha(\xi),$$

$$u(r,t) = u_2(t)\frac{r}{r_s} v(\xi),$$

$$P(r,t) = P_2(t)\left(\frac{r}{r_s}\right)^2 p(\xi),$$

where α, v & P are dimensionless *time independent* functions of the similarity variable ξ. All three functions are deliberately scaled so that they reach the value unity at $\xi = \xi_0$.

The value of the constant ξ_0 is determined by the condition that total energy is conserved, which can be written

$$\int_0^{r_s(t)} \rho\left(\mathcal{E} + \frac{U^2}{2}\right) 4\pi r^2 dr = E$$

or

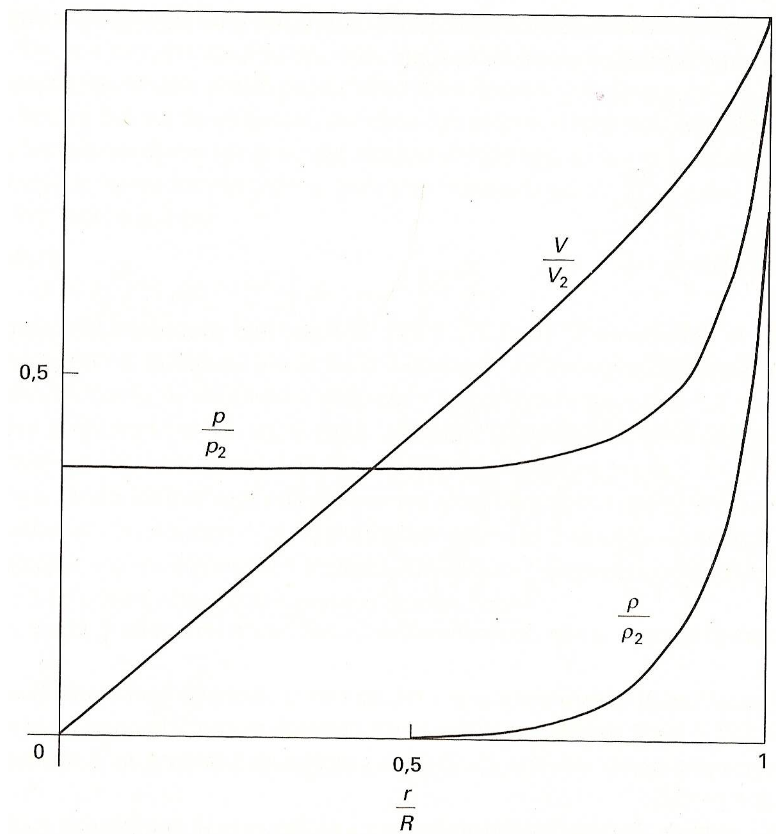
$$\frac{32\pi}{25(\gamma^2 - 1)}\int_0^{\xi_0} \left(p(\xi) + \alpha(\xi) v^2(\xi)\right) \xi^4 d\xi = 1.$$

Sedov-Taylor: ρ, p, V distribution in the bubble

Integration has to be made numerically

The solution is self-similar: at any stage of the expansion the functional form is the same

NB: the quantities P_2, V_2 and ρ_2 are the values immediately post-shock as given by the RH conditions



Our assumption that all the swept up mass is in a thin layer of thickness $\delta R/R \sim 0.1$ behind the shock is well justified
Also the pressure is concentrated in the layer between the contact discontinuity and the blast wave

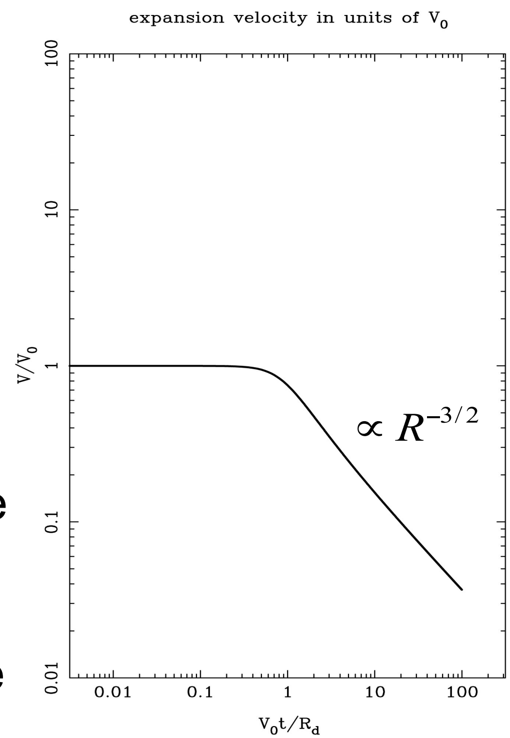
The pressure in the bubble decays as the bubble expands

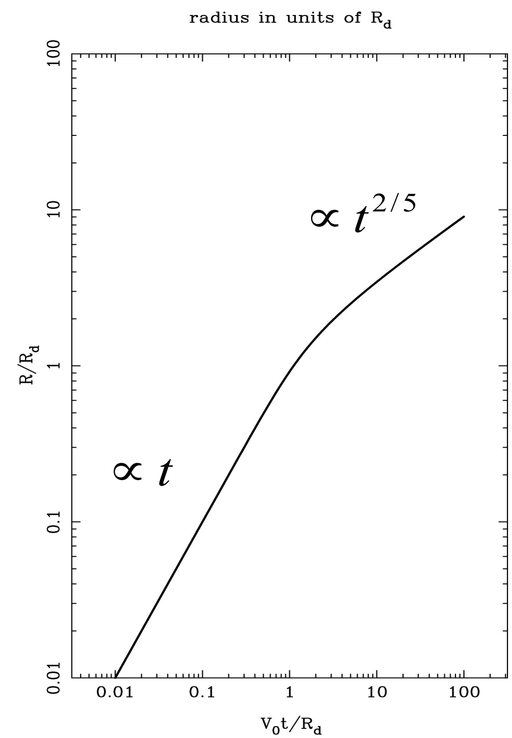
From $p_i \approx p_2 = \frac{2\rho_o V_s^2}{(\gamma+1)}$ One finds $p_i \sim t^{-6/5}$ or $p_i \sim R_s^{-3}$

This decay is simply an expansion loss as the internal pressure is converted into kinetic energy of the expanding shell

This energy conserving Sedov-taylor solution applies for R> R_d and until the radiation losses become important

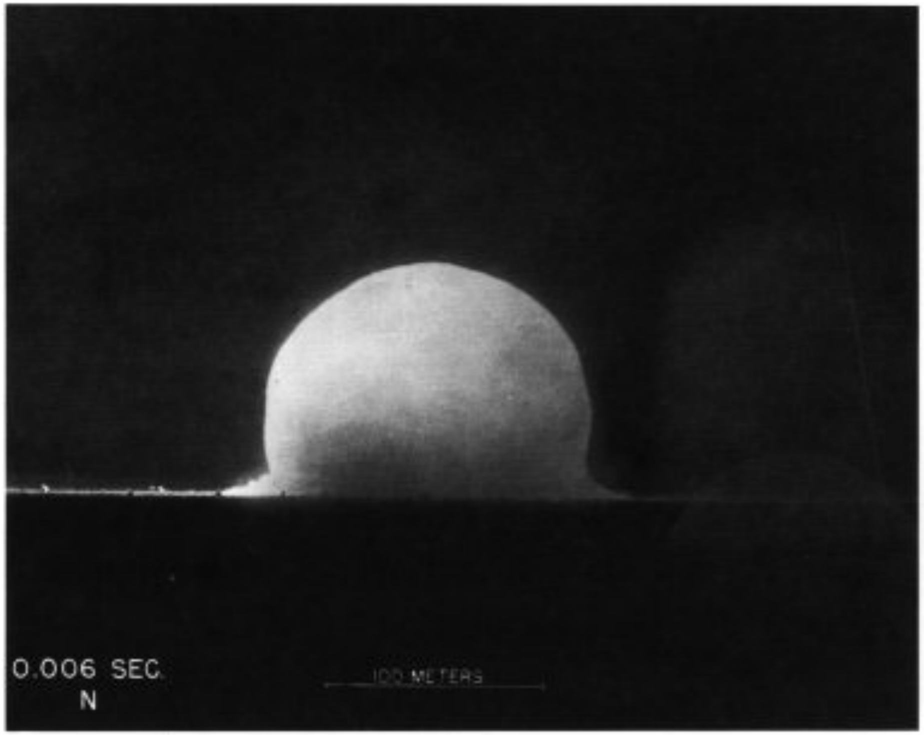
Radiative cooling makes the pressure inside the hot bubble decay faster and consequently the remnant looses energy and the expansion slows down more rapidly than in Sedov-taylor phase

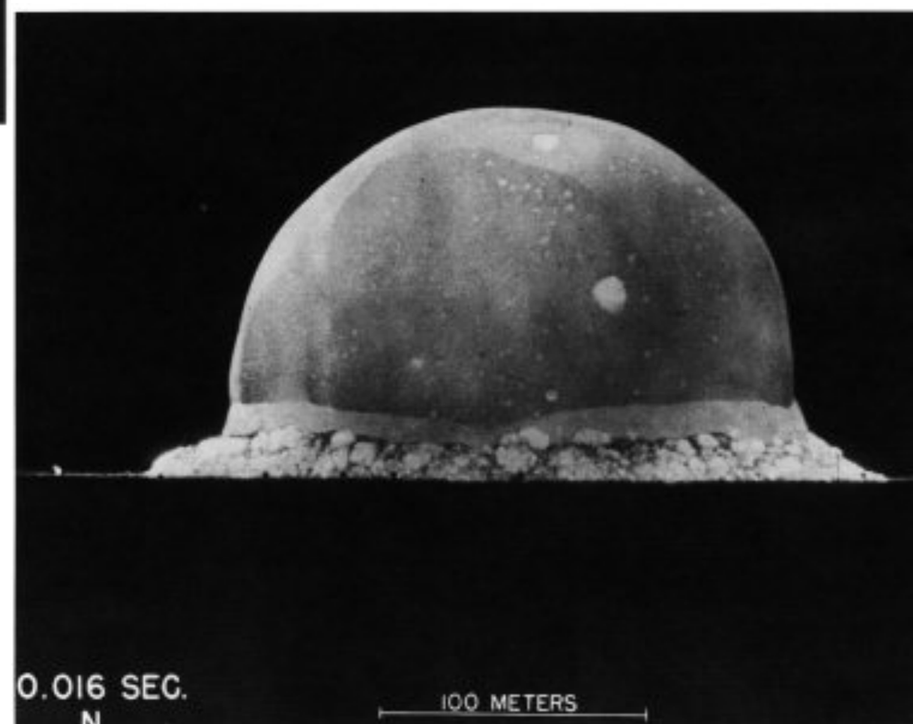




The cooling dominated stages of the evolution set in after about 10000 years

Sedov & Taylor





Sedov & Taylor

Crab Nebula: SNR exploded in 1054 ad
Distance 2000+-500 pc
Diameter ~1.7 pc

Temperature

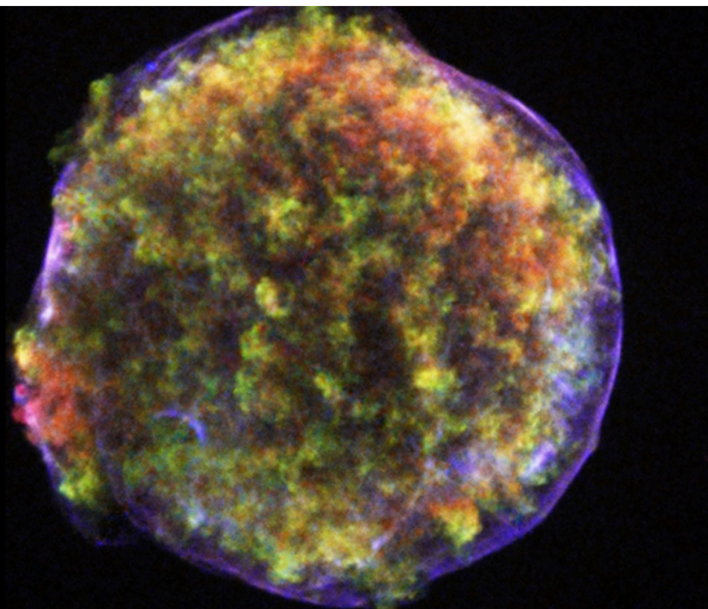
$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_s^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \frac{V_s^2 \rho_o}{\gamma p_o} \qquad T_2 = \frac{2(\gamma-1)}{(\gamma+1)^2} \frac{V_s^2 \rho_o T_1}{p_o} = \frac{2(\gamma-1)}{(\gamma+1)^2} V_s^2 \frac{\mu}{R_{gas}}$$

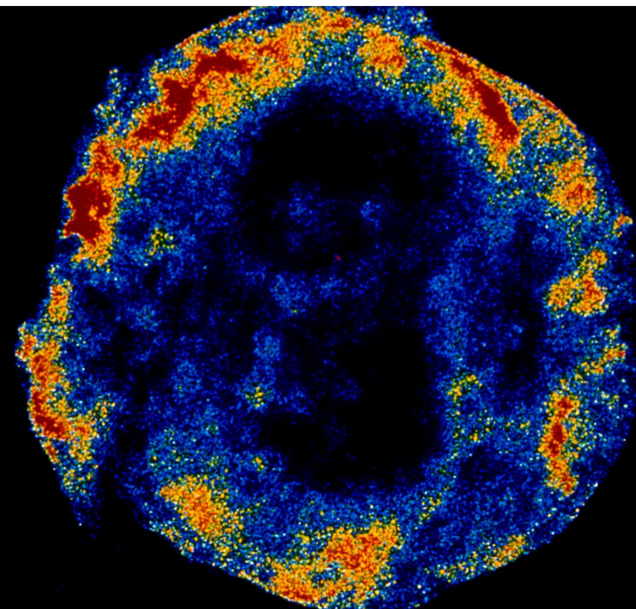
For typical values, T_2 is several million K (>10^7 K) after 10^4 years and the SNR is a bright X ray source

After 10^5 years the post-shock T decreases to ~ 2×10^5 K, when radiative cooling becomes important and the energy conservation assumption breaks down

This occurs because at sufficiently low T, nuclei and electrons recombine so that radiation is no longer trapped in the shell and radiates away, cooling the shell

Two pictures of the remnant of Tycho's supernova (AD 1572), a picture in X-Rays (left), made with the CHANDRA satellite, and a radio picture made with the Very Large Array *radio synthesis telescope (right). The X-ray picture shows the hot ($T \sim 10^8$ K) gas in the remnants interior in yellow. This is mostly* line emission *from exited nuclei. The blue radiation at the outer rim of the remnant is* synchrotron continuum emission, *caused by relativistic electrons moving in a weak magnetic field. The radio emission is also synchrotron radiation. It is believed that these relativistic electrons are accelerated at the outer shock.*
This is a 'classical' remnant with a nearly perfect spherical shape. It is believed to be entering the Sedov-Taylor phase. Note the sharp outer edge of the remnant, which is believed to coincide with the position of the outer blast wave.





Tycho's Remnant (SN 1572AD)

X-Rays (CHANDRA Observatory) Radio (21cm)

Snowplow phase

When the SNR becomes sufficiently old, radiative cooling becomes important and the total energy is no longer conserved

In the energy conserving Sedov-Taylor phase, pressure forces accelerate the swept up ISM converting thermal energy (which came from original explosion) into kinetic energy of the shell of swept up mass

Since radiative cooling depends on the particle density (~ n^2), ie the higher is particle density, the higher is the radiation, and since the density in the shocked shell is much higher than in the bubble interior, most of the cooling occurs in the shocked ISM layer

Snowplow phase

In the snowplow approximation, it is assumed that all the energy in the shocked shell is radiated away, but that the hot interior does not cool because there radiative processes are much less effective due to the very low density (ie the time scale of radiative cooling in the hot interior is much longer than in the shocked shell)

In such a case the shell must collapse until it becomes very thin because the pressure in the shell is decreased together with temperature but the pressure equilibrium at the contact discontinuity must still hold and the pressure in bubble interior is not changed much due to lack of radiative cooling, so the shell is compressed until a new pressure equilibrium is reached

The hot interior can be therefore considered as adiabatic → $p_i \sim \rho_i^\gamma$

Since the mass in the interior is conserved, one has $$\rho_i = \frac{M_{ej}}{(4\pi/3)R_s^3}$$

Combining the two relations one gets $$p_i = (\frac{M_{ej}}{(4\pi/3)})^\gamma R_s^{-3\gamma}$$ $\sim r_s^{-5}$ for $\gamma=5/3$

To be compared with the scaling law in the Sedov-Taylor phase $\sim R_s^{-3}$

Snowplow phase

$$p_i = (\frac{M_{ej}}{(4\pi/3)})^{\gamma} R_s^{-3\gamma}$$

The motion of the collapsed shell, containing most of the mass, is driven by the pressure p_i of the hot interior (neglecting the ISM pressure, which very low compared to p_i)

The motion equation of the expanding blast shock is then

$$\frac{d}{dt}(M(R_s)\frac{dR_s}{dt}) = 4\pi R_s^2 p_i(R_s) \quad = 4\pi(\frac{3M_{ej}}{4\pi})^{\gamma} R_s^{2-3\gamma} \equiv AR_s^{2-3\gamma}$$

Taking into account that $M_s(R_s) = \frac{4\pi}{3}\rho_o R_s^3$

$$\frac{4\pi}{3}\rho_o \frac{d}{dt}(R_s^3 \frac{dR_s}{dt}) = 4\pi(\frac{3M_{ej}}{4\pi})^{\gamma} R_s^{2-3\gamma} \quad \Rightarrow \quad \frac{d}{dt}(R_s^3 \frac{dR_s}{dt}) = \frac{3}{\rho_o}(\frac{3M_{ej}}{4\pi})^{\gamma} R_s^{2-3\gamma}$$

Snowplow phase

$$\frac{d}{dt}(R_s^3\frac{dR_s}{dt}) = \frac{3}{\rho_o}(\frac{3M_{ej}}{4\pi})^\gamma R_s^{2-3\gamma}$$

We look for power law solutions **$R_s(t) = Bt^\alpha$**, with B some constant and α to be determined

The index α is determined by substituting the trial solution into the equation

$$\alpha(4\alpha - 1)B^3 t^{(4\alpha-2)} = AB^\beta t^{(2-3\gamma)\alpha}$$

$$A = \frac{3}{\rho_o}(\frac{3M_{ej}}{4\pi})^\gamma$$

For the trial function to be solution, the two exponents must be equal

$$4\alpha - 2 = (2 - 3\gamma)\alpha$$

$$\alpha = \frac{2}{3\gamma + 2}$$

Which for γ=5/3 yields α = 2/7=0.286

The actual value of the index of this pressure driven phase obtained by numerical, more accurate, simulations yields a value closer to 3/10=0.3

We have assumed uniform pressure in the bubble, no mixing at the contact discontinuity between bubble and shocked ISM, and neglected radiative losses

Momentum driven phase

$$p_i = (\frac{M_{ej}}{(4\pi/3)})^{\gamma} R_s^{-3\gamma}$$

$$\frac{d}{dt}(M(R_s)\frac{dR_s}{dt}) = 4\pi R_s^2 p_i(R_s)$$

As the SNR evolves, the pressure inside the bubble decreases until becomes negligible since the remnant radiates away all its internal energy

In this limit, the total momentum is conserved since the total force acting the bubble is zero (hence the name momentum-conserving phase)

$$\frac{d}{dt}(M(R_s)\frac{dR_s}{dt}) = 0$$

$$M_s(R_s) = \frac{4\pi}{3}\rho_o R_s^3$$

This implies $M(R_s)\frac{dR_s}{dt} = constant$ ⇒ $\frac{dR_s}{dt} = (\frac{3}{4\pi\rho_o})R_s^{-3}$

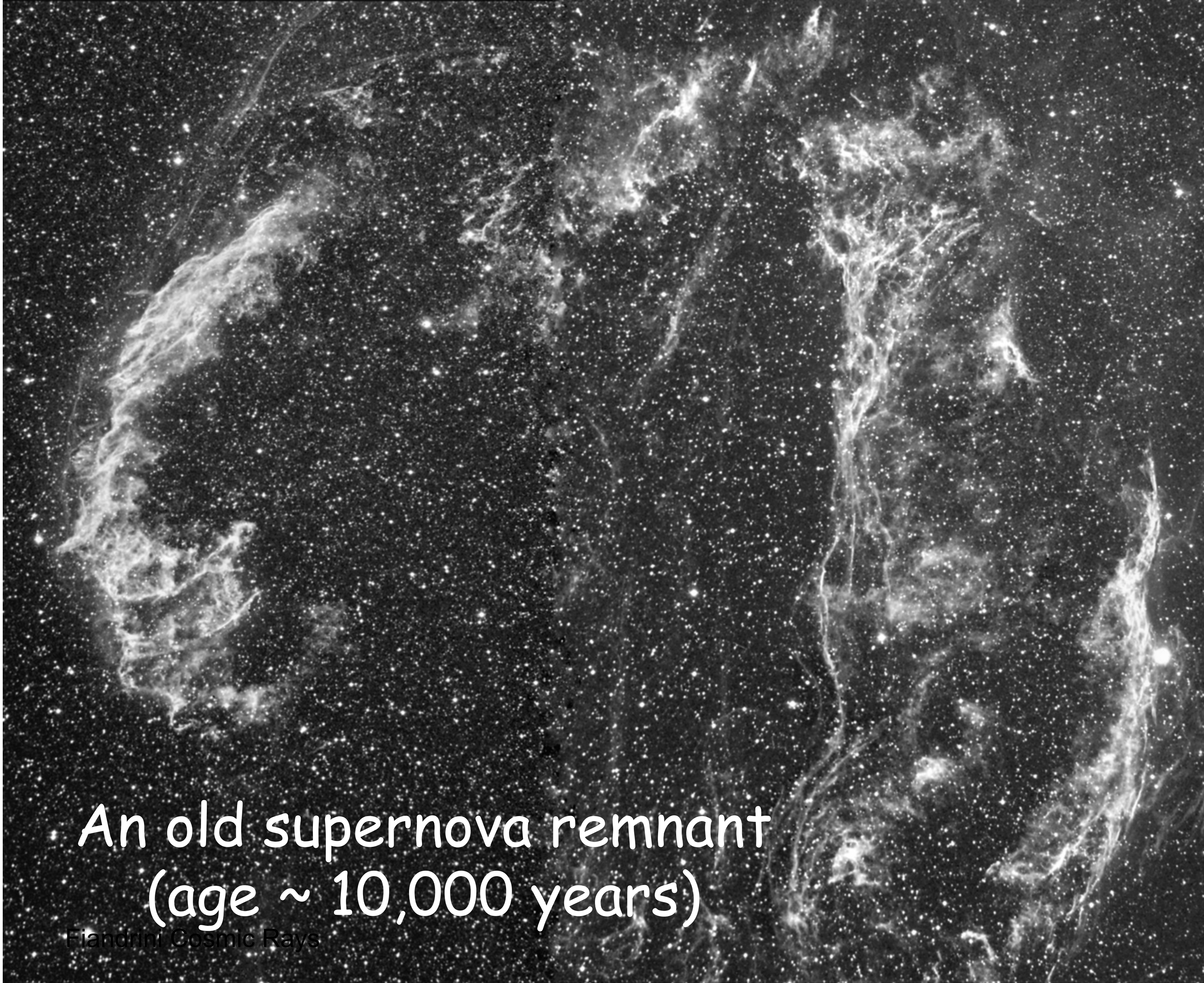
$R_s^3 dR_s = (\frac{3}{4\pi\rho_o})dt$ ⇒ $(1/4)R_s^4 dR_s = (\frac{3}{4\pi\rho_o})t$ ⇒ $R_s = [4(\frac{3}{4\pi\rho_o})]^{1/4} t^{1/4}$

Coalescence phase

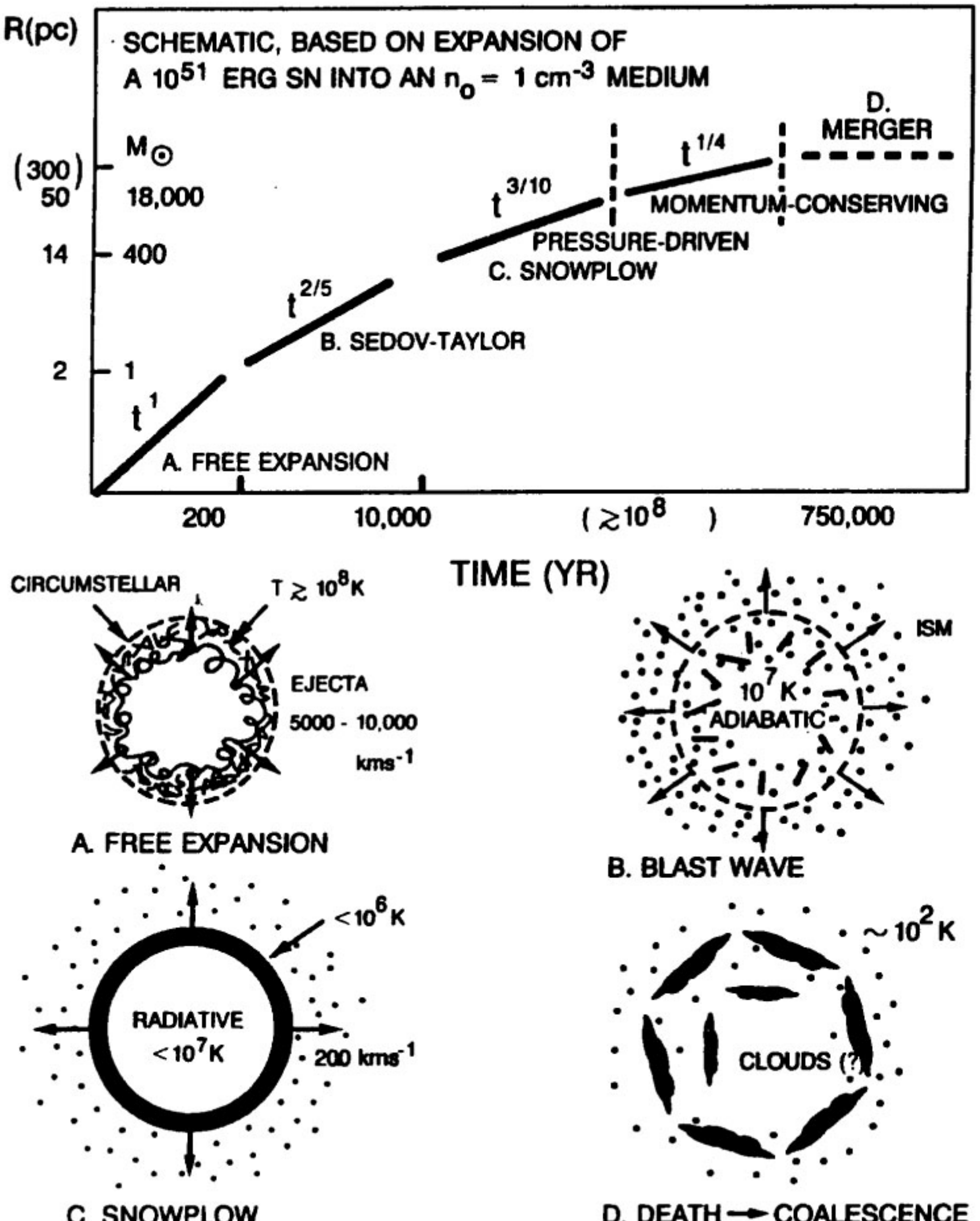
The momentum conserving phase lasts until the shock speed remains beyond sound speed in the ISM

The shock speed decreases as $R_s^{-3} \sim t^{-3/4}$, so at some point it approaches the sound speed and the shock itself disappear.
The material starts to straggle into the ISM (initially at sound speed) becoming part of it

Figure 7.15: *The old supernova remnant S147, which is in the process of dissolving into the general interstellar medium*. Photo credit: Robert Gendler


An old supernova remnant
(age ~ 10,000 years)

STANDARD SNR EVOLUTION



Main properties:

Different expansion stages:

- Free expansion stage (t<1000 yr) $R \propto t$

- Sedov-Taylor stage (1000 yr< t< 10,000 yr) $R \propto t^{2/5}$

- Pressure-driven snowplow (10,000 yr< t<250,000 yr) $R \propto t^{3/10}$

- Momentum-conserving (250,000<t<750,000 yr) $R \propto t^{1/4}$

MagnetoHydroDynamics (MHD)

We have seen the equations of hydrodynamics and some astrophysical applications of them, but we have neglected any electromagnetic phenomenon relative to the fluids

However, in astrophysics environments fluid temperatures are usually very high and the most part of the atoms are completely ionized

This is particularly true for H and He, which make the most part cosmic matter, since their ionization potentials are quite low

Plasmas

Plasmas are gases in which the constituent particles are electrically charged

The first consequence is that the electric fields are not important, because the abundances of free electric charges ensure that any E field would be short-circuited by their motion ie the charges move in such a way to cancel the (external) field

But if the fluid is in a B field, its motion wrt B builds up E fields too by induction and this generates currents

In turn, currents are influenced by the B fields and generate new B fields, that influence again the fluid motion

Plasmas

The plasmas are made of particles of opposite sign

In a volume containing many charged particles, we expect that volume to be close to charge-neutral, since any charge imbalance would produce strong electrostatic forces to restore charge neutrality

Charge imbalances may exist only over a short distance (called Debye length) or for a short period of time (the inverse of the so called plasma frequency)

What makes the macroscopic behavior of plasmas so different from that of neutral gases is the fact that an electromotive force applied to a plasma can drive large currents→ volumes of plasmas can substain large currents in spite of being nearly charge neutral

Since many charged particles in a plasma can interact simultaneously through long range electromag interactions, there can be many phenomena in plasma which are caused by collective interactions

Plasmas

In neutral gases, the interactions between partilces are mediate by collisions

In a completely ionized plasma, instead, the interactions are mediated by long range elm forces

In a partially ionized gas, both processes play a role

if we put a test charge q in the plasma, the presence of many charges of opposite sign around q screens the elm fields felt by "far" charges

The characteristic length over which the screening occurs is a measure of the assumption of charge neutrality of the overall gas: the Debye length

Plasmas: Debye length

To understand how good is the assumption of charge neutrality is, let us consider the charge separation produced by introducing a charge q inside a plasma

If n_e and n_i are the number densities of e- and ions, the charge density at a point x is

$$\nabla^2\Phi = -4\pi(n_i - n_e)e$$

If the plasma in thermodynamical equilibrium and n is the density of e- and ions far away from the charge q, then we expect, according to Maxwell-Boltzman distribution

$$n_i = n \times exp(-\frac{e\Phi}{kT}) \qquad n_e = n \times exp(\frac{e\Phi}{kT})$$

Substituting in the potential eqn $\nabla^2\Phi = -4\pi ne(exp(-\frac{e\Phi}{kT}) - exp(\frac{e\Phi}{kT}))e$

Usually $e\Phi \ll kT$ → we may expand in Taylor series the exp, neglecting higher terms

$$\nabla^2\Phi = -\frac{\Phi}{\lambda_D^2}$$ with $$\lambda_D = (\frac{kT}{8\pi ne^2})^{1/2}$$ Called Debye length

Plasmas: Debye length

$$\nabla^2\Phi = -\frac{\Phi}{\lambda_D^2} \qquad \lambda_D = (\frac{kT}{8\pi n e^2})^{1/2}$$

The solution is easy to find assuming a spherical symmetry $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$

We get $\Phi(r) = \frac{q}{r}e^{-r/\lambda_D}$

It thus appears that the effect of the charge is screened beyond a distance λ_D

So a plasma can be considered charge neutral when distances larger than the Debye lehgth are considered

Altough the E field of a charge in principle extends to ∞, the influence on a charged particle in a plasma is effectively felt to a distance λ_D, ie within a volume λ_D^3, called the Debye volume

Hence the nbr of particles interacting with q is $n\lambda_D^3$: this is a measure of the number of particles which can interact simultaneously

Plasma parameter

The plasma parameter is defined as $g = 1/n\lambda_D^3 = \frac{(8\pi)^{3/2} e^3 n^{1/2}}{(kT)^{3/2}}$

When g is smaller, there is more collective interaction in the plasma (note g is smaller for smaller n)

Therefore the nbr of particles interacting collectively is more for a low density plasma, the Debye length being less effective so that Debye volume is much larger

The average distance between the particles of a plasma is of the order of $n^{-1/3}$ → the average potential energy between a pair of nearby particles is of the order of $e^2 n^{1/3}$

Hence the ratio of potential and average kinetic energy (~kT) is

$$\frac{<PE>}{<KE>} \sim \frac{e^2 n^{1/3}}{kT} \propto n^{1/3}$$

Another interpretation of g is, therefore, that it is a measure of the potential energy of interactions compared to kinetic energy: when g is small (as for low n), the interaction amongst particles is weak, but a large nbr of particles interact simultaneously. On the other hand, a larger g implies few particles interacting collectively, but interacting strongly

The limit of small g is referred to as the ***plasma limit***

$$g = \frac{(8\pi)^{3/2} e^3 n^{1/2}}{(kT)^{3/2}}$$

$$\lambda_D = (\frac{kT}{8\pi n e^2})^{1/2}$$

Types of plasmas

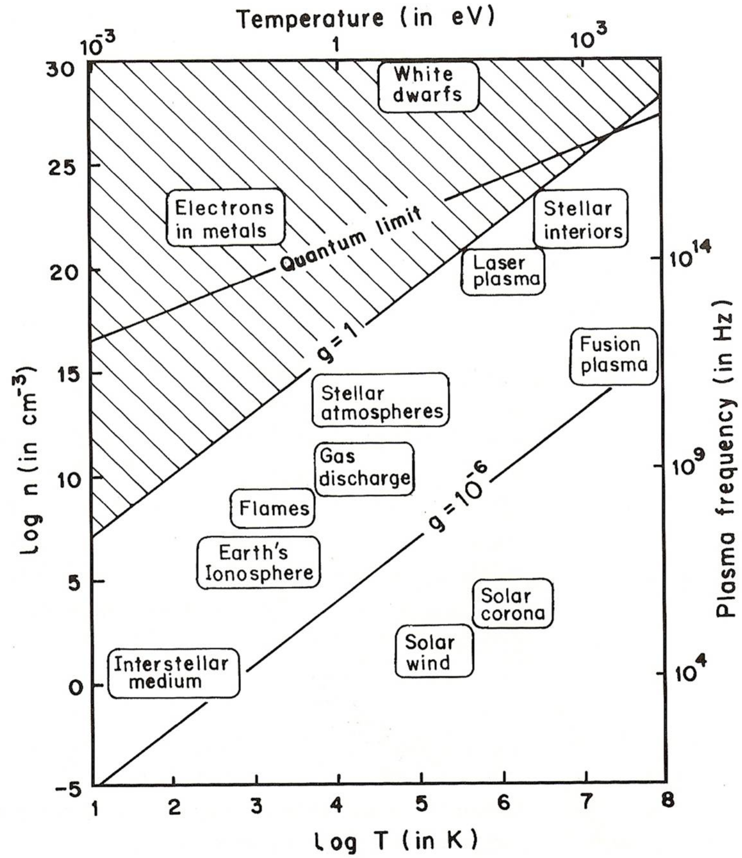
The characteristics of a plasma are determined by n and T

The condition g<1 is a requirement for a gas to be defined a plasma

In the region above g=1, the gas do not behave like a plasma

Above quantum limit, we have to use proper quantum mechanics, as for electrons in a metal and white dwarfs interiors (as for neutron stars)

In between QL and g=1 we find solids, liquids and crystals which can not be considered plasmas



The ratio

$$\frac{<PE>}{<KE>} \sim \frac{e^2 n^{1/3}}{kT} \propto n^{1/3}$$

Implies that for small g plasmas kinetic energy exceeds potential energy, so that it can be treated as a perfect gas

This holds even for material at the centres of stars!

Plasmas

Plasma

Density low to neglect particles collisions…

…but high enough to allow elm interactions.

The overall charge is nearly zero ➔ quasi-neutrality

Free charged particles→ currents→self magnetic fields→ auto-interactions

➔ Magnetohydrodynamics

Plasma is the more common state - the 4th state - of matter in universe:

99% of "normal" matter is plasma

MHD

The MHD studies the interactions of ensembles of charged particles and electromagnetic fields

Limiting cases:

1) external fields assigned→ determine the particle motion from Lorentz force, $ma=(q/c)(E+v \times B)$ or $\rho dv/dt = \rho_c E + j \times B$, $\rho_c = qn$, $\rho = mn$

2) charges and current distributions assigned→ determine the fields from maxwell equations

The MHD is different for two reasons:

(a) it deals with many particles systems (plasmas) which exhibit collective behavior (bulk motion, oscillations, instabilities,...)

(b) fields E and B are not prescribed but determined by the positions and motions of the particles

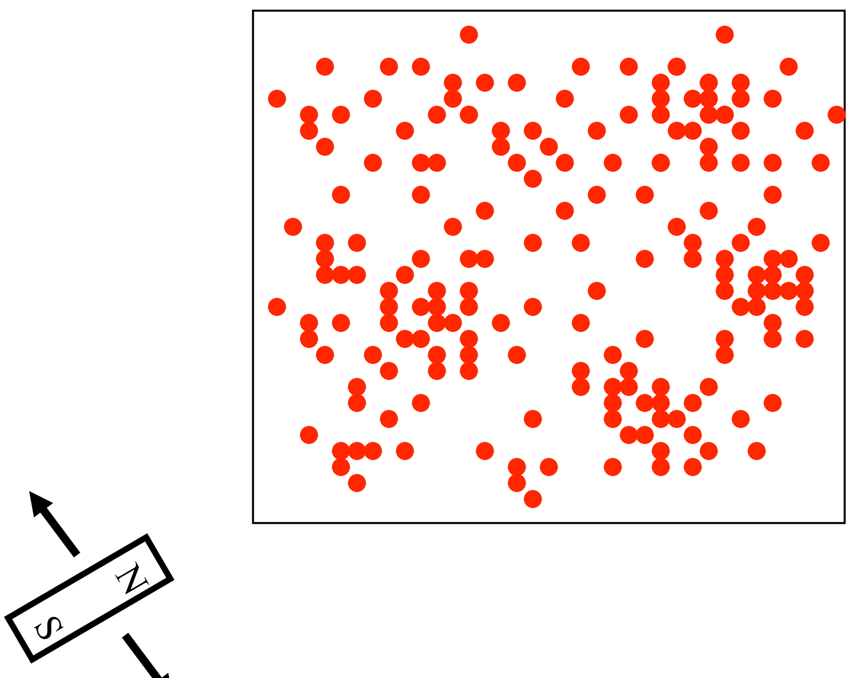
➔ field and motion equations must be solved simultaneously and self-consistently: we are looking for a set of particle trajectories and fields patterns such that the particles generate the field patterns as they move along their orbits and the fields patterns force the particles to move in exactly these orbits

In a plasma the generated self-fields act as coupling device between the individual particles

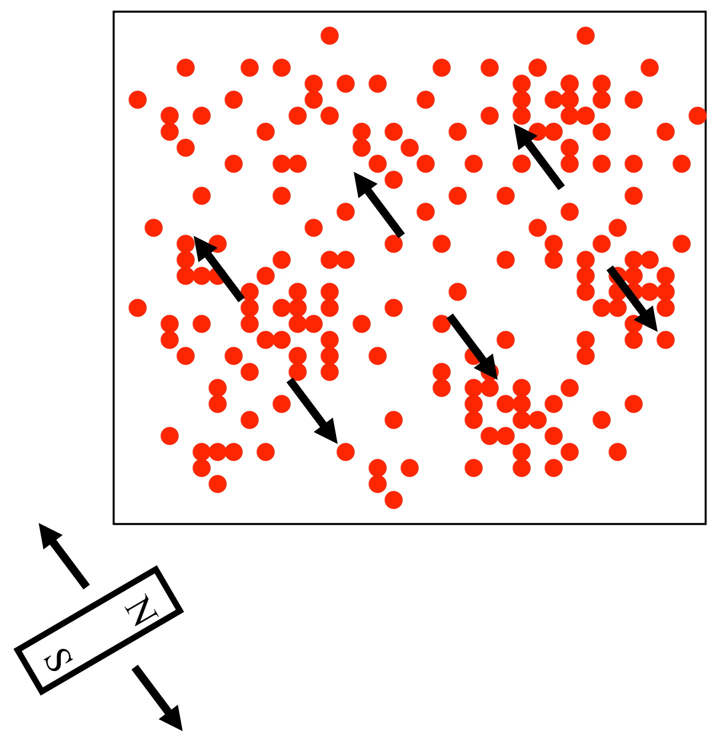
Plasmas Respond to B-Fields

- ***Magneto-HydroDynamics (MHD) = hydrodynamics flow+electro-magnetic phenomena***

Regular Gas



Plasma



MHD

The problem is mechanical, thermodynamical and electromagnetic

We need to find the matter distribution ρ, charge distr. ρ_c, current density j, speed v and the fields E and B, with given boundary conditions

Assumptions:
(a) the medium cannot be magnetized nor polarized $\varepsilon = \mu = 0$
(b) small velocities compared to c
(c) highly conductive medium, ie $E/B >> 1$
(d) displacement current small compared to induction current: $\partial E/\partial t << \mu_o j$

Motion equations

To get the motion equations for the fluid we have to modify the hydrodynamic equations

The mass conservation law remains unchanged $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

In the momentum equation a new term appears due to the Lotentz force acting on the moving charges

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}$$

The energy equation must be modified too, since the presence of currents implies electrical resistance and this, in turn, implies dissipation and therefore heating

From elementary physics we know that the rate of heating (ie power dissipated) is given by Joule's law P = dE/dt= j^2/σ, being σ the fluid conductivity, therefore

While in absence of dissipation, we got for a fixed mass element $T\frac{Ds}{Dt} = 0$

In presence of resistive dissipation it becomes $\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$

$$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$$

Motion equations

Why should we have currents in the fluid?

Because elm fields could be present: from elementary physics, we get Ohm's law

In a stationary medium, an electric field generates a current density $j=\sigma E'$

In our case, the fluid is in motion wrt the laboratory frame (or, for astronomers, of the observer!) and the electric and magnetic fields are measured in this frame

From this it follows that the electric and magnetic fields felt by the fluid in its reference (rest) frame are different from those measured in laboratory

Remember that the Lorentz transformations for the fields are

$$\vec{E}' = \gamma[\vec{E} + (\vec{v} \times \vec{B})/c]$$

being E and B the fields in the lab

We assumed that v<<c, therefore we can approximate as $\vec{E}' \approx [\vec{E} + (\vec{v} \times \vec{B})/c]$

This is the field felt by the fluid in its rest frame

Motion equations

$$\vec{E}' \approx [\vec{E} + (\vec{v} \times \vec{B})/c]$$

Therefore the Ohm's law becomes $\vec{j}' \approx \sigma[\vec{E} + (\vec{v} \times \vec{B})/c]$

Generally speaking, the conductivity is a tensor and it's easy to argue that, in presence of strong B fields, σ (as diffusion coefficients, like heat transport or viscosity) is different if we consider the directions parallel or normal to the field, but it can be shown that the conduction coefficients along the two directions differ by $3\pi/32=0.295$, so that we will omit the distinction and treat σ as a scalar

Of course, j is the spatial component of a 4-vector (cρ,**j**) and is expected to transform when we go from rest frame to the lab frame

But the Lorentz transf mixes charge density and current, and we have seen that we can safely assume $\rho=0$ because of charge-neutrality, so in the limit v<< c→

$$\vec{j}' \approx \gamma\vec{j} \approx \vec{j}$$

In these hypothesis, then, current density does not change when passing from rest frame to the lab frame and the Ohm's law is

$$\vec{j}' \approx \sigma[\vec{E} + (\vec{v} \times \vec{B})/c]$$

Motion equations

Together with the fluid equations, we have to take into account also the Maxwell equations to describe the elm fields

$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \approx 0$ Due to charge neutrality $\rho \sim 0$

$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ It follows that the only way to have E fields in MHD is by induction!

And that the induced field strength is, to order of magnitude, E~vB/c, being V a typical speed

$\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} + \frac{1}{c}\frac{\partial E}{\partial t} \approx (4\pi/c)\vec{j}$ Since the displacement current is negligible

Since the E field in MHD is due exclusively to induction, from rotor equation we find that E~ LB/cT, where L and T are typical lengths and time scale over which the elm varies and L/T is clearly a typical speed

From this it follows that the displacement current is, to order of magnitude, ~ LB/c^2T^2

This must be compared with rotor of B, which, to order of magnitude is B/L

The ratio of the two terms is $L^2/(cT)^2 \sim (v/c)^2$, so that for v<< c can be neglected

$\vec{\nabla} \cdot \vec{B} = 0$ Remains unchanged

MHD eqns summary

In the simplest case of a single charge specie the equations are

Maxwell field equations

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

Ohm's law

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}/c)$$

Equations of motion (Euler eqn for ideal fluids)

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \vec{j} \times \vec{B}/c$$

Mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0$$

Energy equation

$$\rho T\frac{Ds}{Dt} = \frac{j^2}{\sigma}$$

Equation of state

$$\frac{d(\frac{p}{\rho^{\gamma a}})}{dt} = 0$$

MHD induction equation

Let assume that σ = const. in time and space.

Combining Faraday and Ohm law we eliminate E and J from equations

$$\vec{E} = \vec{j}/\sigma - \vec{v} \times \vec{B}/c$$

$$\vec{\nabla} \times (\vec{j}/\sigma - \vec{v} \times \vec{B}/c) = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{j} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c}\frac{\partial B}{\partial t})$$

From Ampere law $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j}$ $\Rightarrow$ $\frac{c}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{j}$

$$\frac{c}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c}\frac{\partial B}{\partial t})$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\nabla^2 \vec{B} + \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) = -\nabla^2 \vec{B} \quad \Rightarrow \quad -\frac{c^2}{4\pi}\nabla^2 \vec{B} = \sigma(\vec{v} \times \vec{B} - \frac{\partial B}{\partial t})$$

$$\boxed{\frac{c^2}{4\pi\sigma}\nabla^2 \vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial B}{\partial t}}$$

The coefficient of laplacian $\lambda = \frac{4\pi\sigma}{c^2}$ Is called magnetic diffusivity

Or D = $c^2/4\pi\sigma$ is the magnetic diffusion coefficient

This equation, called induction equation, is very useful because only the B field appears in it