Lecture 12 211118

- Il pdf delle lezioni puo' essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/

The slides are taken from http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/bibliography/hydrodynamics_achterberg.pdf

Idrodinamica (non relativistica)

La maggior parte dei fenomeni astrofisici concerne il rilascio di energia all'interno di stelle, o al loro esterno, nel mezzo interstellare.

In entrambi i casi, il mezzo in cui avviene il rilascio di energia comincia a muoversi, a espandersi o contrarsi, a riscaldarsi o raffreddarsi

Le proprieta' della radiazione emessa (fotoni, particelle cariche), e rivelata a Terra, dipendono in dettaglio dalle condizioni termodinamiche e di moto del fluido in questione

Ne segue che un requisito necessario per l'astrofisica delle alte energie e' lo studio dell'idrodinamica e della magnetoidrodinamica

Idrodinamica (non relativistica)

In idrodinamica il fluido e' considerato un sistema termodinamico macroscopico.
Viene idealizzato come un mezzo continuo

La descrizione del fluido in quiete richiede la conoscenza delle sue proprieta' termodinamiche locali (p, ρ, T) → occorre quindi avere un'equazione di stato che lo caratterizza

Lo stato di moto di un fluido generico, non in quiete, sara' descritto dalla velocita' istantanea dell'elemento di fluido $\mathbf{v}(\mathbf{x},t)$

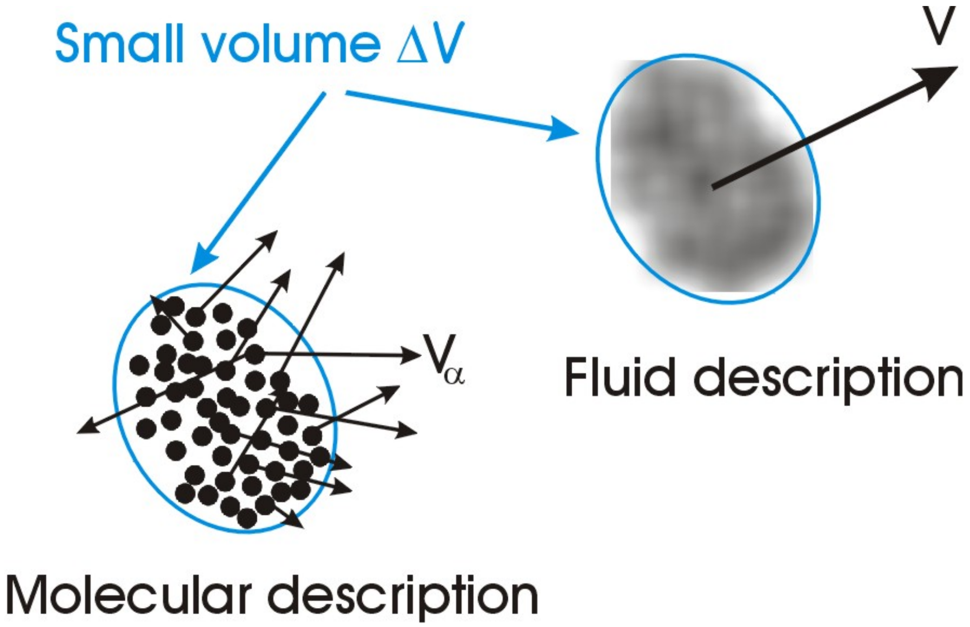
→ lo stato del fluido e' quindi determinato dalla conoscenza di:

$p(\mathbf{x},t)$, $\rho(\mathbf{x},t)$, $T(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$

Classical Mechanics vs. Fluid Mechanics

Single-particle (classical) Mechanics	Fluid Mechanics
Deals with <u>single</u> particles with a <u>fixed mass</u>	Deals with a <u>continuum</u> with a <u>variable mass-density</u>
Calculates a <u>single particle trajectory</u>	Calculates a <u>collection of flow lines</u> (flow field) in space
Uses a position *vector* and velocity *vector*	Uses a *fields* : Mass density, velocity field..
Deals only with <u>externally applied</u> forces (e.g. gravity, friction etc)	Deals with <u>internal</u> AND <u>external</u> forces
Is formally linear (superposition principle for solutions)	Is intrinsically <u>non-linear</u> <u>No</u> superposition principle

Mass, mass-density and velocity



$$\rho(\boldsymbol{x}\,,\,t) = \lim_{\Delta\mathcal{V}\to 0} \frac{\Delta m}{\Delta\mathcal{V}}$$

$$\Delta m = \sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha$$

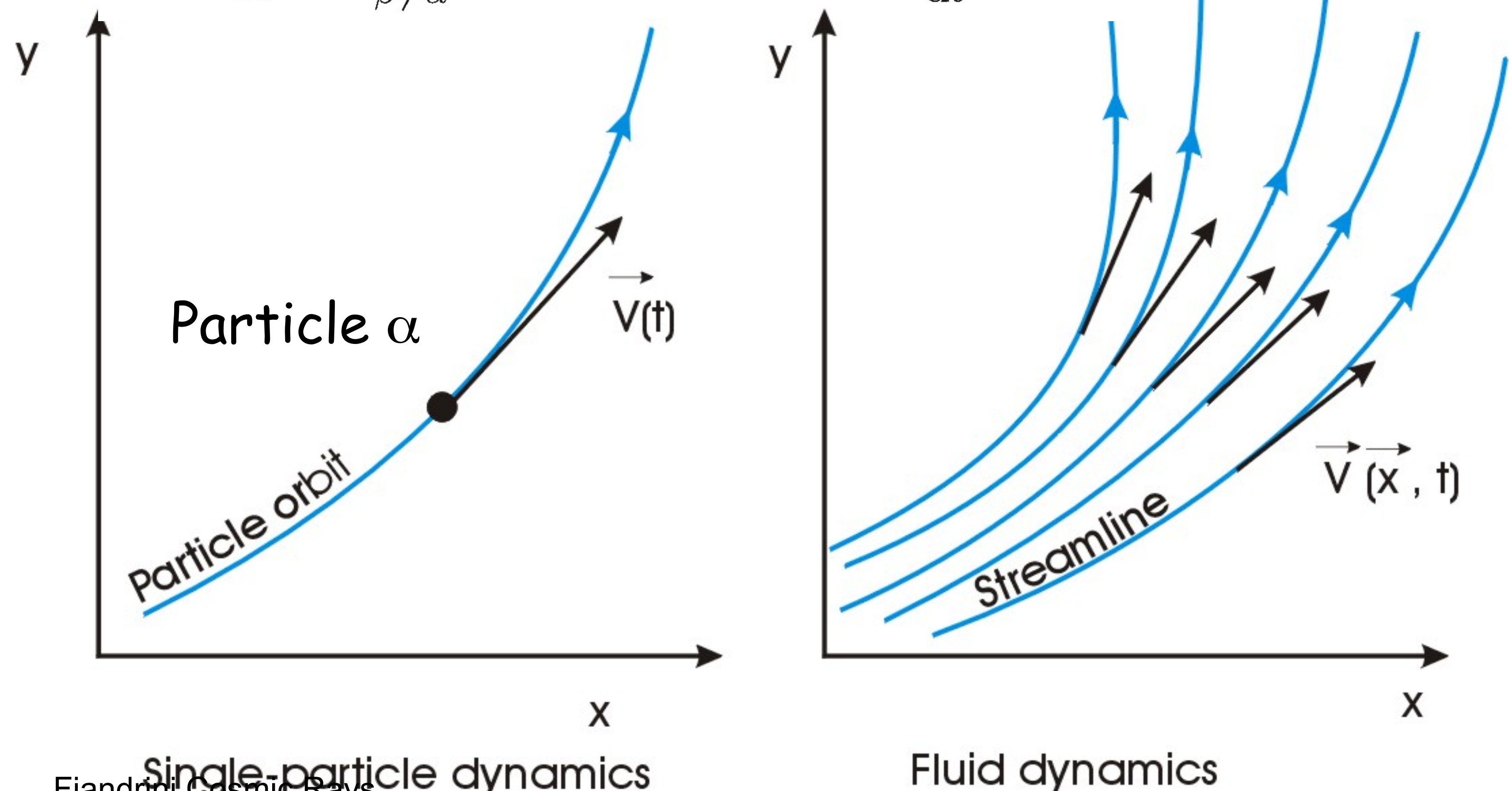
$$\boldsymbol{V} = \frac{\sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha \boldsymbol{V}_\alpha}{\Delta m}$$

Dal punto di vista dei costituenti la velocita' ***V*** di un elemento di fluido e' la velocita' media delle particelle contenute nel volume (che sono tante in modo che la media statistica abbia senso)

Equation of Motion: from Newton to Navier-Stokes/Euler

$$m_\alpha \frac{\mathrm{d}\boldsymbol{V}_\alpha}{\mathrm{d}t} = \sum_{\beta \neq \alpha} \boldsymbol{F}_{\alpha\beta}$$

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$

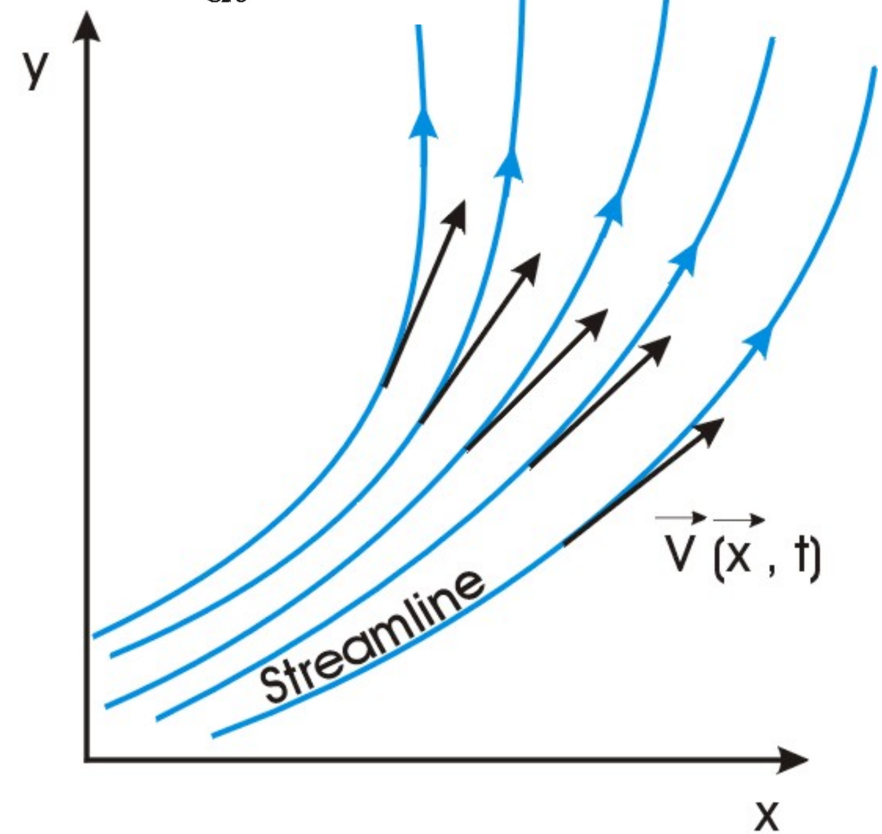


Equation of Motion: from Newton to Navier-Stokes/Euler

You have to work with a velocity <u>field</u> that depends on position *and* time!

$$\boldsymbol{V} = (V_x\,,\ V_y\,,\ V_z) = \boldsymbol{V}(\boldsymbol{x}\,,\ t)$$

$$\rho\,\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$



Fluid dynamics

Derivatives, derivatives...

Eulerian change:
fixed position

$$\delta Q = Q(\boldsymbol{x}\,,\ t+\Delta t) - Q(\boldsymbol{x}\,,\ t) \approx \frac{\partial Q}{\partial t}\,\Delta t$$

Lagrangian change:
shifting position

$$\Delta Q = Q(\boldsymbol{x}+\Delta\boldsymbol{x}\,,\ t+\Delta t) - Q(\boldsymbol{x}\,,\ t) \approx \frac{\mathrm{d}Q}{\mathrm{d}t}\,\Delta t$$

Shift along streamline:

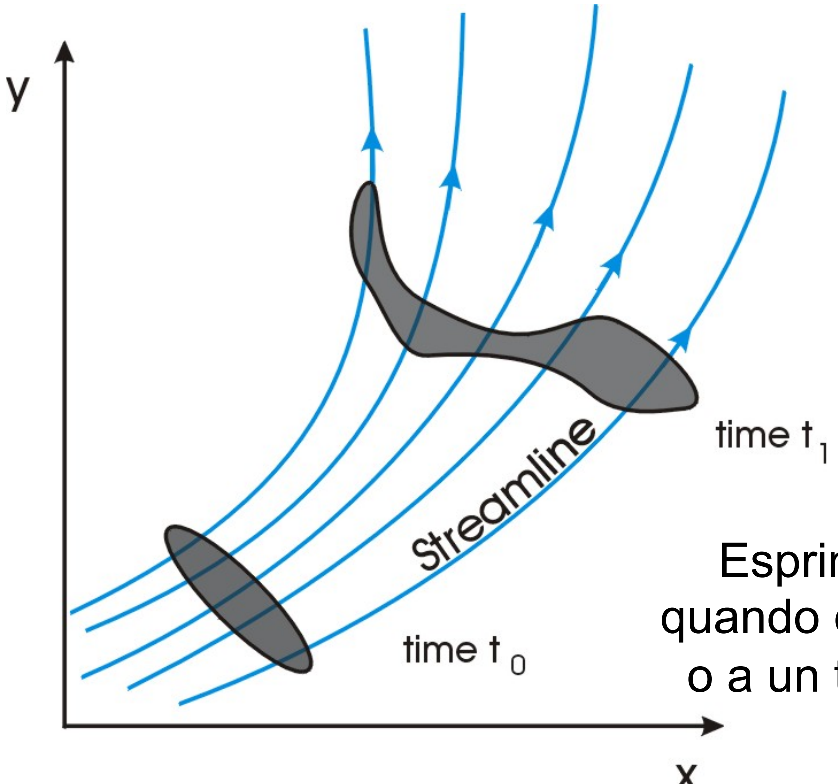
$$\Delta\boldsymbol{x} = \boldsymbol{V}\,\Delta t$$

Comoving derivative d/dt

$$\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t$$

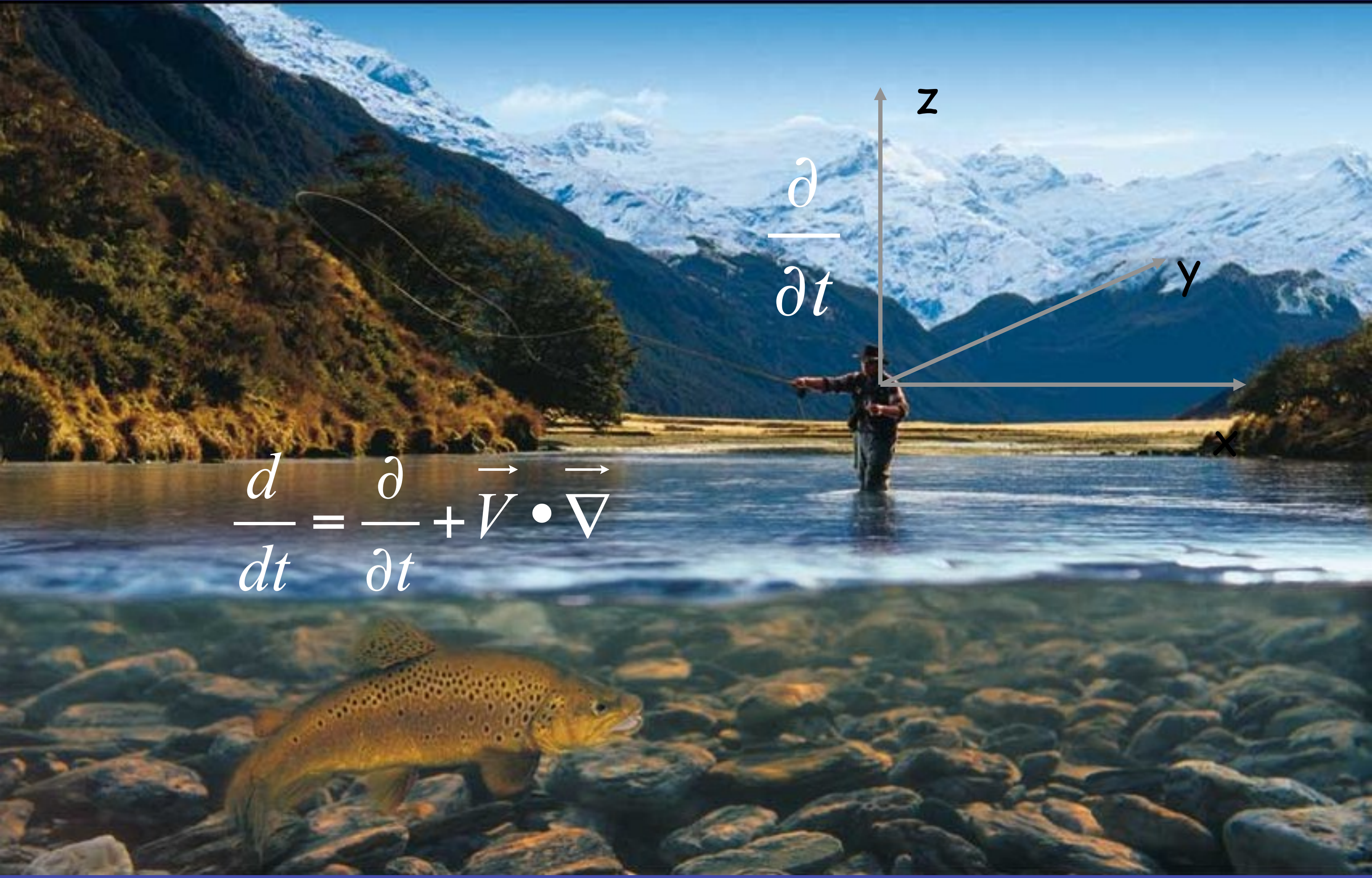
$$\begin{aligned}\Delta Q &= Q(t+\Delta t\,,\ \boldsymbol{x}+\Delta\boldsymbol{x}) - Q(t\,,\ \boldsymbol{x}) \\ &\approx \frac{\partial Q}{\partial t}\Delta t + (\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla})Q \\ &= \left[\frac{\partial Q}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})Q\right]\Delta t \\ &\equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)\Delta t\ .\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})$$



Esprime come varia una quantita' fisica nel tempo quando consideriamo la sua variazione non in un posto o a un tempo fissato ma a elemento di massa fissato

In altre parole, se ci concentriamo su un certo elemento di massa e lo seguiamo nel suo moto, d/dt esprime la variazione col tempo come la vedremmo se ci trovassimo a cavalcioni dell'elemento di massa in questione


z
$$\frac{\partial}{\partial t}$$
y
x
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \vec{\nabla}$$

Equazione del moto per un fluido

Variazione di p di un elemento = F totale applicata sull'elemento

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}$$

<u>termine non lineare!</u>

Rende molto piu' difficile trovare soluzioni "semplici"

E' il prezzo da pagare per lavorare con un <u>campo</u> di velocita'

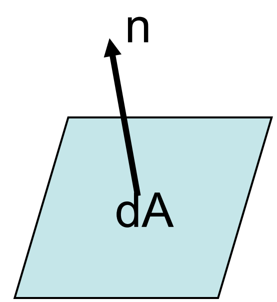
$$\boldsymbol{V} = (V_{\mathrm{x}}\,,\, V_{\mathrm{y}}\,,\, V_{\mathrm{z}}) = \boldsymbol{V}(\boldsymbol{x}\,,\, t)$$

<u>Densita' di forza</u>

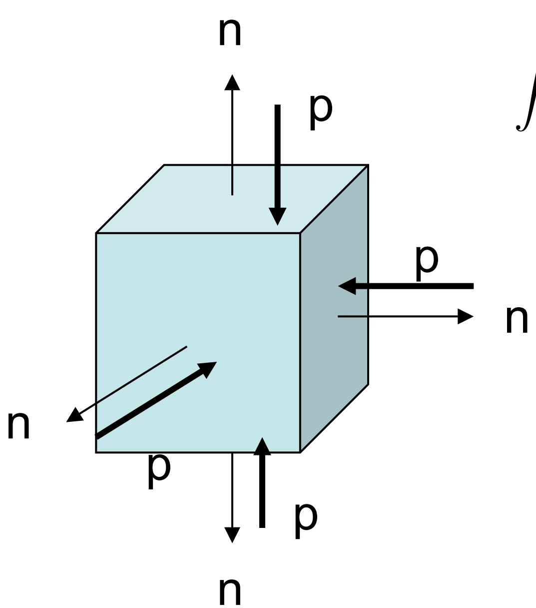
Puo' essere:
- interna:
  - pressione
  - viscosita' (frizione)
  - self-gravity
- esterna

Equazione del moto per un fluido ideale

In un fluido su un elemento di superficie arbitrario viene esercitata una forza d**F**=pd**A**=pdA**n**



→ la forza a cui e' soggetto un elemento di fluido in un volume V e' $\vec{F} = -\int_S p d\vec{A}$



$$\int_S p d\vec{A} = \int_V \nabla p dV$$

→ la densita' di forza e' $\vec{f} = \nabla p$

$$\rho \frac{d\vec{V}}{dt} = -\nabla p \qquad \rho(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla})\vec{v} = -\nabla p$$

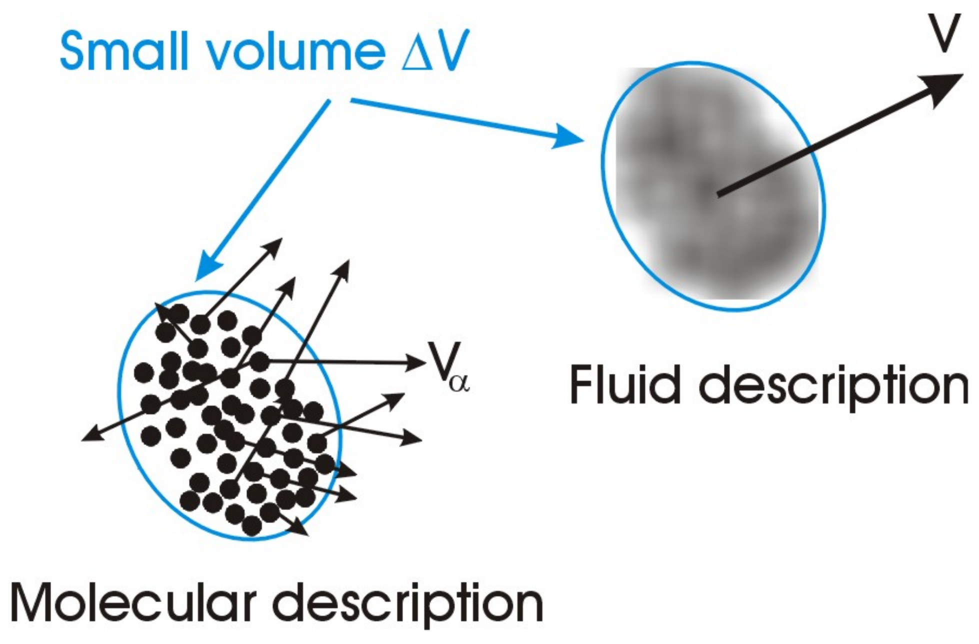
$$(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla})\vec{v} = -\frac{\nabla p}{\rho} + \frac{\vec{f}_{ext}}{\rho}$$

E' l'equazione del moto del fluido ideale (detta di Eulero)

P. Es **f** puo' essere forza di gravita' $\vec{f} = -\rho \nabla \Phi$

Connessione microscopica

La pressione deriva dal fatto che i costituenti (ie particelle) sono in agitazione termica per cui le singole particelle possiedono una distribuzione di velocita' (p. Es. Maxwelliana all'equilibrio) intorno al valore medio V della velocita' dell'elemento di fluido



Questo significa che l'impulso esatto istantaneo di una particella NON coincide con quello medio del fluido

Questa differenza "genera", in ultima analisi, una forza: la pressione

Pressure force and thermal motions

Split velocity into the
average velocity
V(**x**, t),

and an

Isotropically
distributed
deviation
from average, the
random velocity:
σ(**x**, t)

Individual particle:

$$\boldsymbol{v}_\alpha = \boldsymbol{V}(\boldsymbol{x}\,,\ t) + \boldsymbol{\sigma}_\alpha(\boldsymbol{x}\,,\ t)\ .$$

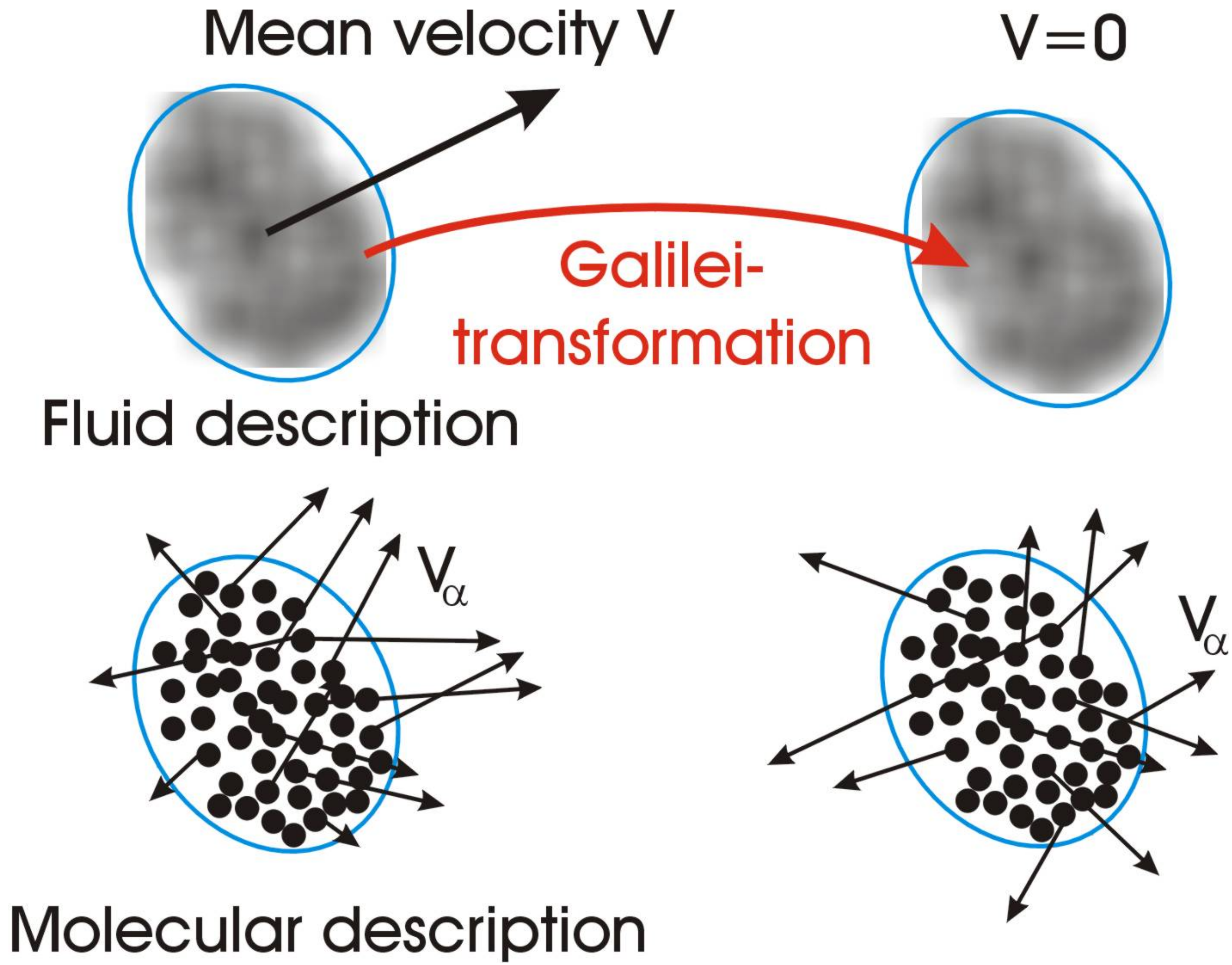
Average properties of random velocity $\boldsymbol{\sigma}$:

$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{v}} - \boldsymbol{V} = \boldsymbol{0}\ ;$$

$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = \tfrac{1}{3}\overline{\sigma^2}\ ,$$

and

$$\overline{\sigma_x\sigma_y} = \overline{\sigma_x\sigma_z} = \overline{\sigma_y\sigma_z} = \cdots = 0\ .$$



Nel sistema di quiete dell'elemento di fluido le molecole sono in moto a causa dell'agitazione termica

Acceleration of particle α

$$\frac{\mathrm{d}\boldsymbol{v}_\alpha}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}_\alpha}{\partial t} + (\boldsymbol{v}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{v}_\alpha$$

$$= \frac{\partial(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)}{\partial t} + ((\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \cdot \boldsymbol{\nabla})(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)$$

$$= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \underbrace{\frac{\partial \boldsymbol{\sigma}_\alpha}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}_\alpha}_{\text{linear in } \boldsymbol{\sigma}} + \underbrace{(\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}_\alpha}_{\text{quadratic in } \boldsymbol{\sigma}}$$

Effect of average over many particles in small volume:

$$\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}}$$

$$= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \underbrace{\left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\right)\overline{\boldsymbol{\sigma}}}_{\text{vanishes: } \overline{\boldsymbol{\sigma}}=0!} + \underbrace{\overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}}}_{\text{remains: quadratic in } \boldsymbol{\sigma}}$$

Average equation of motion

$$\rho \, \overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\boldsymbol{f}}$$

$$\rho \left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right) = \underbrace{\overline{\boldsymbol{f}}}_{\text{mean ext. force}} \quad - \rho \, \overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \, \boldsymbol{\sigma}}$$

For isotropic fluid: $\rho \, \overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla} \left(\frac{\rho \overline{\sigma^2}}{3} \right) \equiv \boldsymbol{\nabla} P$

Some tensor algebra:

Vector

$$\boldsymbol{A} \equiv A_i\, \boldsymbol{e}_i = A_x\, \boldsymbol{e}_1 + A_y \boldsymbol{e}_2 + A_z\, \boldsymbol{e}_3 = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Three notations for the same animal!

the divergence of a vector in cartesian coordinates

Scalar

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Rank 2 Tensor

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Rank 2 Tensor and Tensor Divergence

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Vector

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = \left(\frac{\partial T_{ij}}{\partial x_i}\right)\,\boldsymbol{e}_j = \begin{pmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{pmatrix}$$

Special case:
Dyadic Tensor = Direct Product of two Vectors

$$\boldsymbol{A} \otimes \boldsymbol{B} \equiv A_i B_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \; \boldsymbol{B} + (\boldsymbol{A} \cdot \boldsymbol{\nabla}) \, \boldsymbol{B}$$

Application: Pressure Force

Tensor divergence

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot\,(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\;\boldsymbol{\sigma}$$

Isotropy of Random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

Second term = scalar x vector!

This <u>must</u> vanish upon averaging!!

Application: Pressure Force

Tensor divergence

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot\,(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\,\boldsymbol{\sigma}$$

Isotropy of Random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

$$\overline{\sigma_i\sigma_j} = \tfrac{1}{3}\overline{\sigma^2}\,\delta_{ij} = \begin{cases} \tfrac{1}{3}\overline{\sigma^2} & \text{when } i=j \\ 0 & \text{when } i\neq j \end{cases}$$

$$\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} = \rho\begin{pmatrix} \tfrac{1}{3}\overline{\sigma^2} & 0 & 0 \\ 0 & \tfrac{1}{3}\overline{\sigma^2} & 0 \\ 0 & 0 & \tfrac{1}{3}\overline{\sigma^2} \end{pmatrix} = \frac{\rho\overline{\sigma^2}}{3}\,\boldsymbol{I}$$

Diagonal Pressure Tensor

Pressure force, continued

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\left(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}\right) = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P$$

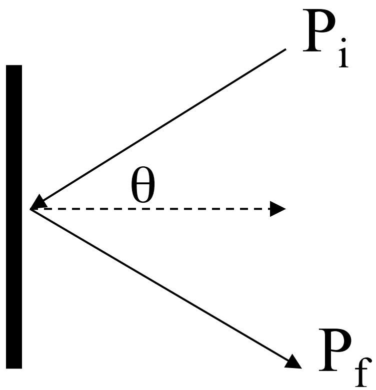
Equation of motion for frictionless ('ideal') fluid:

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = -\boldsymbol{\nabla}P + \text{other (external) forces}$$

$$P(\boldsymbol{x}\,,\,t) \equiv \tfrac{1}{3}\rho\,\overline{\sigma^2}$$

Pressure, statistical derivation

We have to find the momentum transfer to a wall from a gas of particles with number density n in the hypothesis of elastic collisions $|p_i|=|p_f|$



A single particle with momentum p arriving from direction θ with respect to the normal n to the wall gets a $\Delta p = 2p\cos\theta$

In the time dt, the # of particles coming from θ direction impinging the wall are $dN = nv\cos\theta\, Sdt$, ie all the particles in the volume $(v\cos\theta)dtS$

Then the net momentum transfer to the wall is dP = (single part Δp)x(# of part imping. the wall) = $2p\cos\theta \times nv\cos\theta\, Sdt = 2nvp\cos^2\theta\, Sdt$

Pressure

The force is $F = dP/dt = 2nvp\cos^2\theta\, S$ The pressure is $p = dF/dS$

So $p(\theta) = 2nvp\cos^2\theta$ is the pressure due to to particles arriving from direction θ

If the distribution of directions is isotropic, then the probability for a particle to arrive in a solid angle $d\Omega$ along θ is $d\Omega/4\pi$, so the total pressure is

$$P_{tot} = \int p(\theta)\frac{d\Omega}{4\pi} = \int_0^1 p(\theta)\frac{2\pi dcos\theta}{4\pi} = nvp\int_0^1 \cos^2\theta d\cos\theta = nvp/3$$

non relativistic particle $p = mv$ $P_{tot} = nmv^2/3 = \frac{2}{3}u \quad u = \frac{1}{2}nmv^2$

From equipartition theorem $\langle E \rangle = 3kT/2$ Energy density

$P_{tot} = nkT$ Ideal gas law

Relativistic particle (as photons) E_{tot}' cp and v' c

$P_{tot} = ncp/3 = u/3$ $u = ncp$ Energy density

Summary:

- We know how to treat the time-derivative
- We know what the equation of motion looks like
- We know where the pressure term comes from
- We <u>still</u> need:
  - A way to link the pressure to density and temperature
  - A way to calculate how the density of the fluid changes

Connection with thermodynamics: Ideal Gas Law

Isotropic gas of point particles in Thermodynamic Equilibrium:

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\rm b}T$$

Temperature is defined in terms of kinetic energy of the thermal motions!

Connection with thermodynamics: Ideal Gas Law

Isotropic gas of point particles in Thermodynamic Equilibrium:

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\, m\overline{\sigma^2} = \frac{1}{2}\, k_{\rm b}T$$

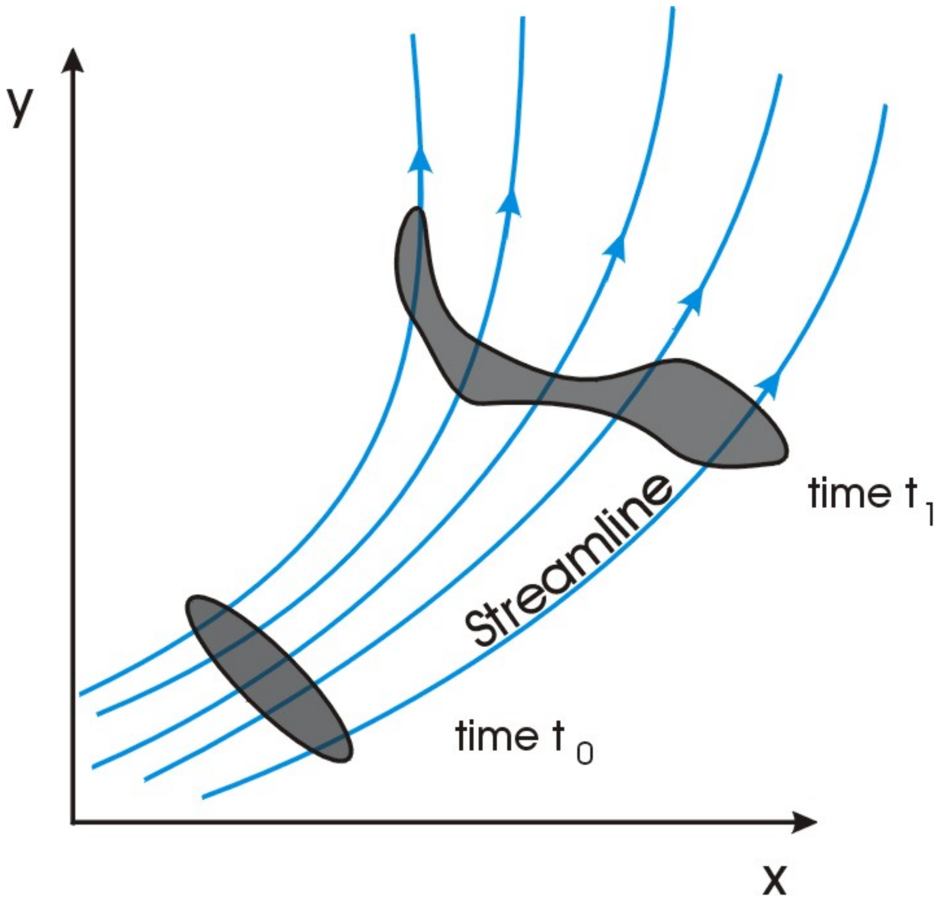
Ideal Gas Law: in terms of temperature T and number-density n: (ρ = nm = μnm_H , $R = k_b / m_H$)

$$p = \frac{1}{3}\rho\sigma^2$$

↓

$$P(\rho\,,\; T) = nk_{\rm b}T = \frac{\rho \mathcal{R} T}{\mu}$$

Density Changes and Mass Conservation

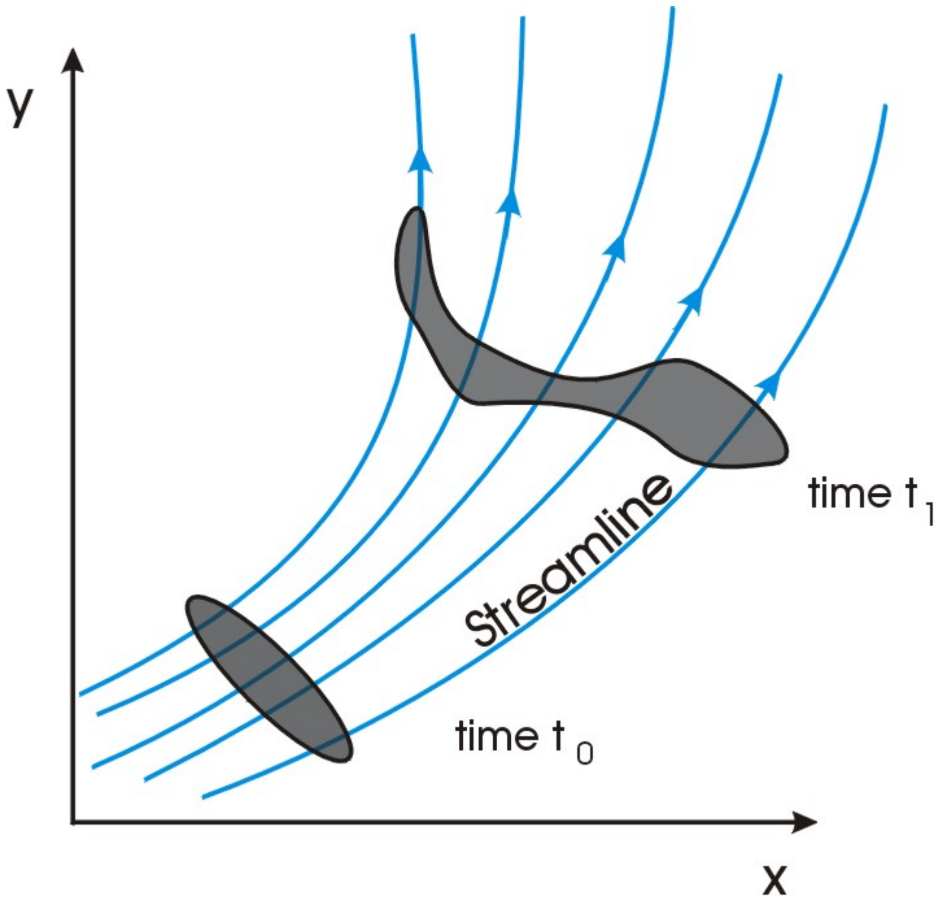


Two-dimensional example:

A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

Density Changes and Mass Conservation



Fiandrini Cosmic Rays

Two-dimensional example:

A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

So: the mass density must change in response to the flow!

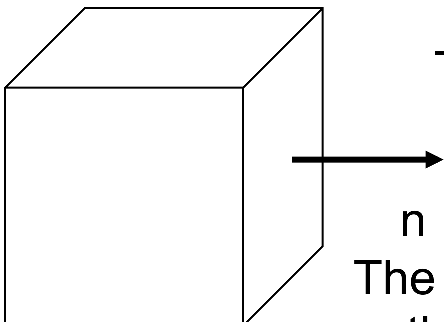
Mass conservation law

The mass cannot be created nor destroyed (in non-relativistic classical dynamics)

Therefore in a volume V the mass can change only because some of it leaves or enters the volume

The amount of mass through d**A** per time unit is $dF = \rho\vec{v}\cdot d\vec{A}$

Convention: dF>0 if outgoing



The flux $(\frac{dM}{dt})_{out} = \int_S \rho\vec{v}\cdot d\vec{A}$

The outgoing flux must be balanced by the change of mass in the volume $\frac{\partial M_V}{\partial t} = -(\frac{dM}{dt})_{out}$

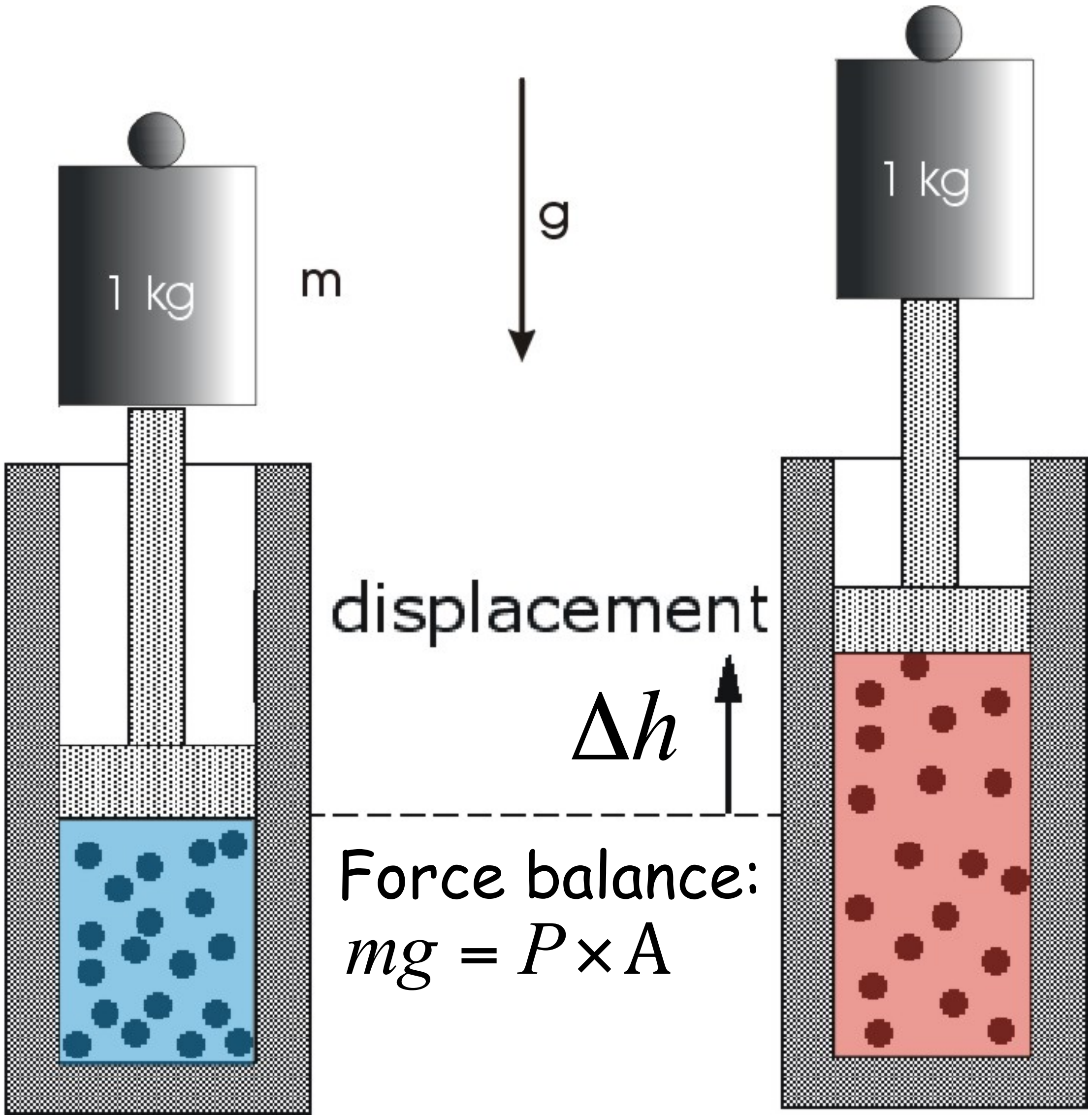
$$\frac{\partial M_V}{\partial t} = \frac{\partial}{\partial t}(\int_V \rho dV)$$

$$\frac{\partial}{\partial t}(\int_V \rho dV) = -\int_S \rho\vec{v}\cdot d\vec{A} \quad = -\int_V \nabla\cdot(\rho\vec{v})dV$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{v})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

Thermodynamics



$$\Delta W = mg\Delta h$$

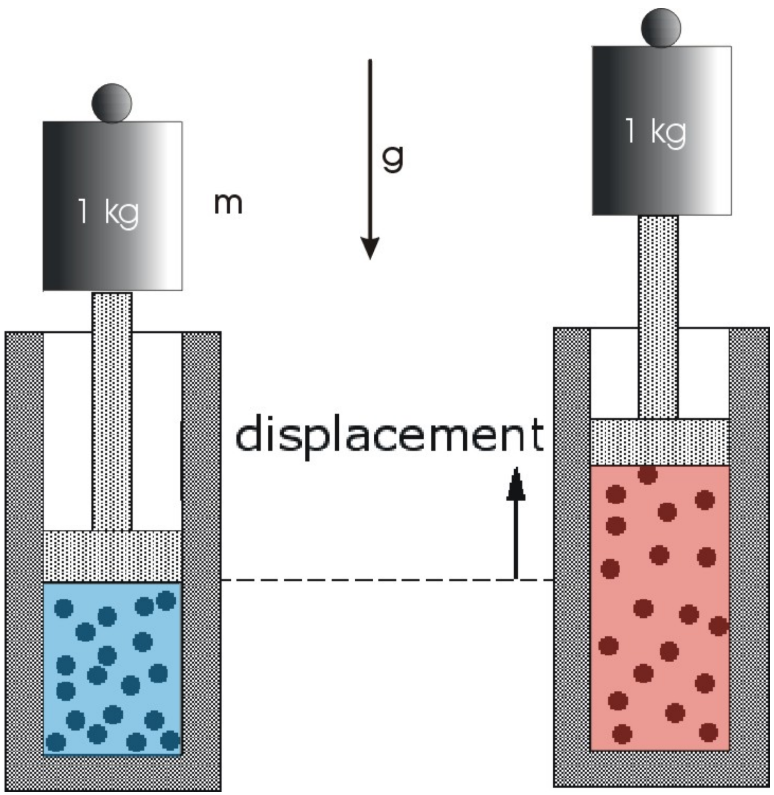
$$\Delta U_{gas} = \Delta Q - \Delta W$$

$$= \Delta Q - mg\Delta h$$

$$= \Delta Q - PA\ \Delta h$$

$$= \Delta Q - P\ \Delta V$$

First law of thermodynamics:



$$dU_{gas} = dQ - pdV$$

Change in internal energy
=
heat added by external sources
-
work done by gas

Entropy:

A measure of
'disorder'

Second Thermodynamic law:

$$dQ = TdS$$

$$dS \geq 0$$

$$\frac{dS}{dt} \geq 0$$

For an isolated system

The Adiabatic Gas Law: the behaviour of pressure

Thermodynamics:

$$dQ \equiv T\,dS = dU + P\,d\mathcal{V}$$

Special case: adiabatic change

$$dQ = T\,dS = 0$$

U = internal energy, *T* = temperature, *S* = entropy and V = volume

The Adiabatic Gas Law: the behaviour of pressure

Thermodynamics:

$$dQ \equiv T\,dS = dU + P\,d\mathcal{V}$$

Special case: adiabatic change

$$dQ = T\,dS = 0$$

Gas of point particles of mass *m* :

Internal energy: $U = n\,\mathcal{V} \times \left[\frac{1}{2}m\left(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}\right)\right] = \frac{1}{2}\,\rho\,\mathcal{V}\overline{\sigma^2}$

Pressure: $P = \frac{1}{3}nm\,\left(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}\right) = \frac{1}{3}\,\rho\,\overline{\sigma^2}$

Thermal equilibrium: $\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\rm b}T$

$$\rightarrow \quad P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \frac{3}{2}\,\frac{\rho \mathcal{R} T\,\mathcal{V}}{\mu}$$

Thermal equilibrium:

$$\tfrac{1}{2}m\overline{\sigma_x^2} = \tfrac{1}{2}m\overline{\sigma_y^2} = \tfrac{1}{2}m\overline{\sigma_z^2} = \tfrac{1}{6}\, m\overline{\sigma^2} = \tfrac{1}{2}\, k_\mathrm{b}T$$

$$P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \tfrac{3}{2}\, \frac{\rho \mathcal{R} T\, \mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P\, \mathrm{d}\mathcal{V} = 0 \longrightarrow \mathrm{d}\left(\frac{3\rho \mathcal{R} T \mathcal{V}}{2\mu}\right) + \left(\frac{\rho \mathcal{R} T}{\mu}\right)\, \mathrm{d}\mathcal{V} = 0$$

Adiabatic Gas Law: a polytropic relation

Thermal equilibrium:

$$\tfrac{1}{2}m\overline{\sigma_x^2} = \tfrac{1}{2}m\overline{\sigma_y^2} = \tfrac{1}{2}m\overline{\sigma_z^2} = \tfrac{1}{6}\,m\overline{\sigma^2} = \tfrac{1}{2}\,k_{\rm b}T$$

$$P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \frac{3}{2}\,\frac{\rho \mathcal{R} T\,\mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P\,\mathrm{d}\mathcal{V} = 0$$

$$\mathrm{d}\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\mathcal{V} = 0$$

Chain rule for 'd'-operator:

$$\mathrm{d}(f\,g) = (\mathrm{d}f)\,g + f\,(\mathrm{d}g)$$

$$\tfrac{5}{3}\,P\,\mathrm{d}\mathcal{V} + \mathcal{V}\,\mathrm{d}P = 0\,.$$

(just like differentiation!)

$$\frac{\mathrm{d}P}{P} + \frac{5}{3}\,\frac{\mathrm{d}\mathcal{V}}{\mathcal{V}} = \mathrm{d}\log\left(P\,\mathcal{V}^{5/3}\right) = 0$$

Adiabatic Gas Law: a polytropic relation

Adiabatic pressure change:

$$P \times \mathcal{V}^{5/3} = \text{constant}$$

For <u>small</u> volume:
mass conservation!

$$M = \rho\, \mathcal{V} = \text{constant}$$

$$\Rightarrow P\, \rho^{-5/3} = \text{constant}$$

Specific Heat and Entropy

Specific Volume:
contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics
of unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific Heat and Entropy

Specific Volume
contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics
of a unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific energy *e*
and pressure *P* :

$$e \equiv \tfrac{3}{2}\,\frac{\mathcal{R}T}{\mu} = \tfrac{3}{2}\,\frac{k_{\mathrm{b}}T}{m}\ , \quad P = \frac{\rho\,\mathcal{R}\,T}{\mu}$$

Specific heat coeff.
at constant volume

$$c_{\mathrm{v}} = \frac{\partial e}{\partial T} = \tfrac{3}{2}\,\frac{k_{\mathrm{b}}}{m}$$

ρ is kept constant!
d(1/ρ) =0

Specific Heat and Entropy

Specific Volume contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics of a unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific energy e and pressure P :

$$e \equiv \frac{3}{2}\frac{\mathcal{R}T}{\mu} = \frac{3}{2}\frac{k_{\mathrm{b}}T}{m} \;, \quad P = \frac{\rho\,\mathcal{R}\,T}{\mu}$$

Specific heat coeff. at constant volume

$$c_{\mathrm{v}} = \frac{\partial e}{\partial T} = \frac{3}{2}\frac{k_{\mathrm{b}}}{m}$$

$$c_{\mathrm{p}} - c_{\mathrm{v}} = \frac{k_{\mathrm{b}}}{m} = \frac{\mathcal{R}}{\mu}$$

$$\mathrm{d}q = \mathrm{d}\left(e + \frac{P}{\rho}\right) - \frac{\mathrm{d}P}{\rho}$$

Specific heat coeff. at constant pressure: dP =0

$$c_{\mathrm{p}} = \frac{\partial(e + P/\rho)}{\partial T} = \frac{5}{2}\frac{k_{\mathrm{b}}}{m}$$

$$\begin{aligned} \mathrm{d}q &= c_{\mathrm{v}}\,\mathrm{d}T + \left(\frac{\rho \mathcal{R} T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right) \\ &= c_{\mathrm{v}}\,\mathrm{d}T - \left(\frac{\mathcal{R} T}{\rho\mu}\right)\,\mathrm{d}\rho \\ &= c_{\mathrm{v}} T\,\left[\frac{\mathrm{d}T}{T} - \boxed{\left(\frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} - 1\right)}\frac{\mathrm{d}\rho}{\rho}\,\right] \end{aligned}$$

Thermodynamic law for a unit mass,
rewritten in terms of specific heat coefficients

Definition specific entropy s

$$\mathrm{d}q = c_\mathrm{v}\,\mathrm{d}T + \left(\frac{\rho \mathcal{R} T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

$$= c_\mathrm{v}\,\mathrm{d}T - \left(\frac{\mathcal{R} T}{\rho \mu}\right)\,\mathrm{d}\rho$$

$$= c_\mathrm{v} T\left[\frac{\mathrm{d}T}{T} - \left(\frac{c_\mathrm{p}}{c_\mathrm{v}} - 1\right)\frac{\mathrm{d}\rho}{\rho}\right]$$

$$T\,\mathrm{d}s = c_\mathrm{v} T\left[\frac{\mathrm{d}T}{T} - (\gamma - 1)\frac{\mathrm{d}\rho}{\rho}\right]$$

$$s = c_\mathrm{v}\,\log\left(\frac{T}{\rho^{\gamma-1}}\right) + \mathrm{constant}$$

$$s = c_\mathrm{v}\,\log\left(P\,\rho^{-\gamma}\right) + \mathrm{constant}$$

γ is the specific heat ratio
= 5/3 for ideal gas of point particles

Definition specific entropy s

$$dq = c_v\, dT + \left(\frac{\rho \mathcal{R} T}{\mu}\right) d\left(\frac{1}{\rho}\right)$$

$$= c_v\, dT - \left(\frac{\mathcal{R} T}{\rho \mu}\right) d\rho$$

$$= c_v T \left[\frac{dT}{T} - \boxed{\left(\frac{c_p}{c_v} - 1\right)} \frac{d\rho}{\rho}\right]$$

$$T\, ds = c_v T \left[\frac{dT}{T} - \boxed{(\gamma - 1)} \frac{d\rho}{\rho}\right]$$

$$s = c_v \log\left(\frac{T}{\rho^{\gamma - 1}}\right) + \text{constant}$$

$$s = c_v \log\left(P\, \rho^{-\gamma}\right) + \text{constant}$$

Case of constant entropy (adiabatic gas) : $ds = 0$

$$\log T - (\gamma - 1) \log \rho = \text{constant}$$

with

$$\gamma \equiv \frac{c_p}{c_v} = \frac{5}{3}\,.$$

$$T \rho^{-(\gamma - 1)} = \text{constant} \quad , \quad P \rho^{-\gamma} = \text{constant}$$

(Self-)gravity

$$\boldsymbol{f}_{\rm gr} = \rho\, \boldsymbol{g} = -\rho\, \boldsymbol{\nabla}\Phi$$

$$\boldsymbol{g}(\boldsymbol{x},t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x},t) = -\begin{pmatrix} \dfrac{\partial\Phi}{\partial x} \\ \\ \dfrac{\partial\Phi}{\partial y} \\ \\ \dfrac{\partial\Phi}{\partial z} \end{pmatrix}$$

$$\rho\, \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right] = -\boldsymbol{\nabla}P - \rho\,\boldsymbol{\nabla}\Phi$$

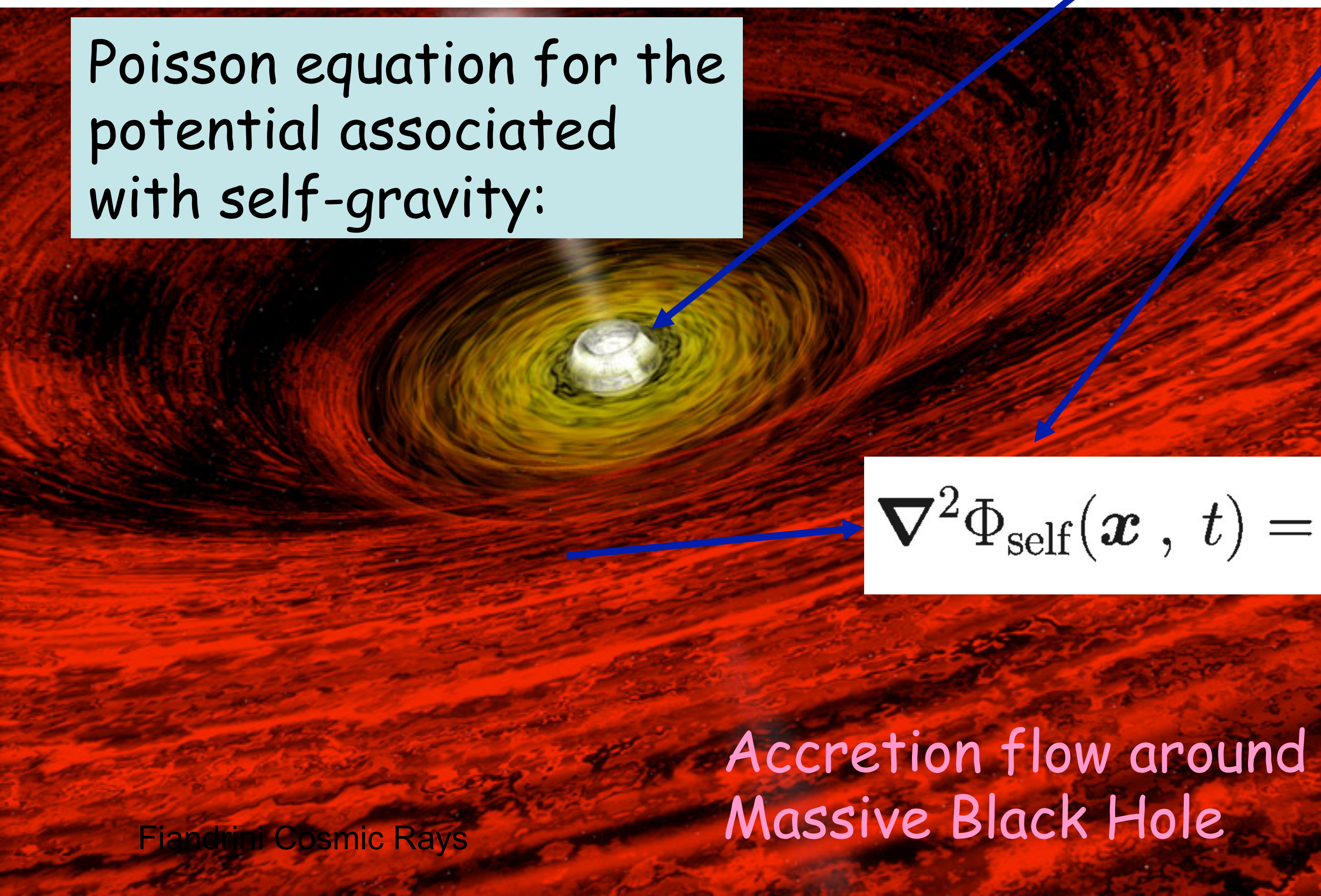
Self-gravity and Poisson's equation

Potential: two contributions!

$$\Phi(\boldsymbol{x}\,,t) = \Phi_{\text{ext}}(\boldsymbol{x}\,,t) + \Phi_{\text{self}}(\boldsymbol{x}\,,t)$$

Poisson equation for the potential associated with self-gravity:

$$\boldsymbol{\nabla}^2 \Phi_{\text{self}}(\boldsymbol{x}\,,\; t) = 4\pi G\,\rho(\boldsymbol{x}\,,\; t)$$

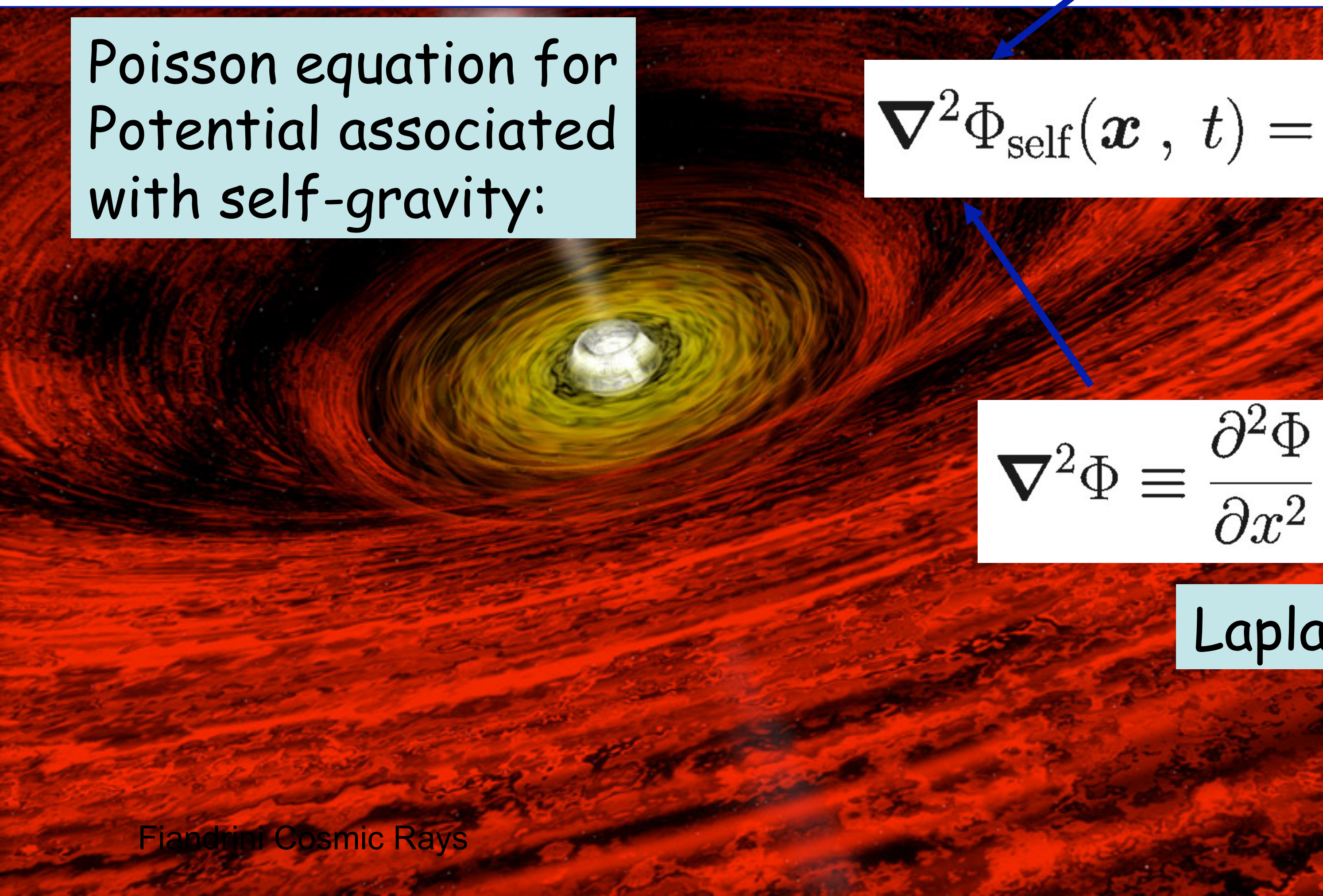


Accretion flow around Massive Black Hole

Self-gravity and Poisson's equation

Potential: two contributions!

$$\Phi(\boldsymbol{x}\,,t) = \Phi_{\rm ext}(\boldsymbol{x}\,,t) + \Phi_{\rm self}(\boldsymbol{x}\,,t)$$



Poisson equation for Potential associated with self-gravity:

$$\boldsymbol{\nabla}^2\Phi_{\rm self}(\boldsymbol{x}\;,\;t) = 4\pi G\,\rho(\boldsymbol{x}\;,\;t)$$

$$\boldsymbol{\nabla}^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Laplace operator

Summary: Equations describing ideal (self-)gravitating fluid

Equation of Motion:

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla} \Phi$$

Continuity Equation:
behavior of mass-density

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V}) = 0$$

Ideal *gas* law
&
Adiabatic law:
Behavior of pressure
and temperature

$$P(\rho\,,\, T) = n k_{\rm b} T = \frac{\rho \mathcal{R} T}{\mu}$$

$$P\, \rho^{-5/3} = \text{constant}$$

Poisson's equation: self-gravity

$$\boldsymbol{\nabla}^2 \Phi_{\rm self}(\boldsymbol{x}\,,\, t) = 4\pi G\, \rho(\boldsymbol{x}\,,\, t)$$

Conservative Form of the Equations

Aim: To cast all equations in the same *generic form:*

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

<u>Reasons</u>:

1. Allows quick identification of conserved quantities
2. This form works best in constructing numerical codes for *Computational Fluid Dynamics*
3. Shock waves are best studuied form a conservative point of view

Generic Form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix} + \nabla \cdot \begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix} = \begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Transported quantity is a scalar S, so flux **F** must be a vector!

$$\frac{\partial S}{\partial t} + \nabla \cdot \boldsymbol{F} = q(\boldsymbol{x}\ ,\ t)$$

Component form:

$$\frac{\partial S}{\partial t} + \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = q$$

Generic Form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Transported quantity is a vector *M* , so the flux must be a tensor *T* .

$$\frac{\partial \boldsymbol{M}}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{T}=\boldsymbol{Q}(\boldsymbol{x}\,,\,t)$$

Component form:

$$\frac{\partial}{\partial t}\begin{pmatrix}M_x\\ \\ M_y\\ \\ M_z\end{pmatrix}+\begin{pmatrix}\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{yx}}{\partial y}+\frac{\partial T_{zx}}{\partial z}\\ \\ \frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{zy}}{\partial z}\\ \\ \frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{yz}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\end{pmatrix}=\begin{pmatrix}Q_x\\ \\ Q_y\\ \\ Q_z\end{pmatrix}$$

The fact the the flux of a vector field is a rank 2 tensor can be understood as follows: the transported quantity is a vector with 3 arbitrary components, each of them can be transported in 3 indipendent directions → so there are 3x3 indipendent quantitites...exactly the nbr of components of a rank 2 tensor

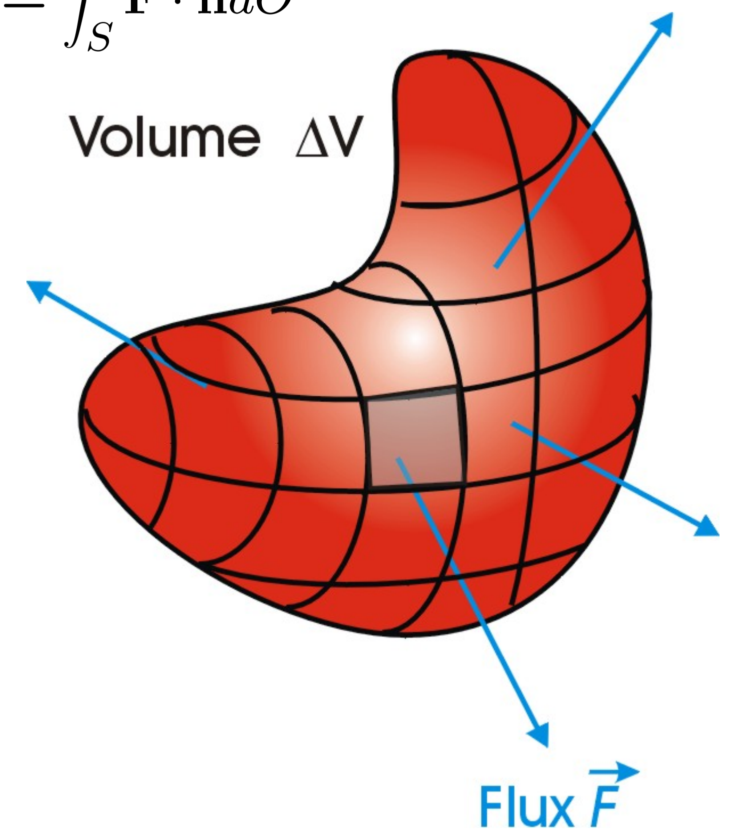
Integral properties: Stokes Theorem

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Let integrate the equation over a volume V and use the Stokes theorem

$$\int_V \nabla\cdot\mathbf{F}dV = \int_S \mathbf{F}\cdot\mathbf{n}dO$$

$$\frac{\partial}{\partial t}\left(\int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, S\right) = \int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, q(\boldsymbol{x}\,,\, t) - \oint_{\partial\mathcal{V}} \mathrm{d}\boldsymbol{O}\cdot\boldsymbol{F}$$



The integral relation states the amount of quantity S in a volume can change only due to sources in V or by a flux of S into or out from V

Examples: mass and momentum conservation

Mass conservation: already in conservation form!

Continuity Equation:
transport of the scalar ρ

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Excludes 'external mass sources' due to processes like two-photon pair production etc.

Examples: mass- and momentum conservation

Mass conservation: already in conservation form!

Continuity Equation:
transport of the scalar ρ

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V}) = 0$$

Momentum conservation: transport of a vector!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla} \Phi$$

↓ Algebraic Manipulation

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\, \boldsymbol{I}\right) = -\rho \boldsymbol{\nabla} \Phi$$

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \, \boldsymbol{V} \right] = -\boldsymbol{\nabla} P - \rho \, \boldsymbol{\nabla} \Phi$$

Starting point: Equation of Motion

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\mathbf{V}\right] = -\boldsymbol{\nabla} P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{V}) = 0$$

$$\rho\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial(\rho\mathbf{V})}{\partial t} - \mathbf{V}\frac{\partial \rho}{\partial t}$$

$$= \frac{\partial(\rho\mathbf{V})}{\partial t} + \mathbf{V}\,(\boldsymbol{\nabla}\cdot(\rho\mathbf{V}))$$

Use:
1. chain rule for differentiation
2. continuity equation for density

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V}) = 0$$

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} = \frac{\partial(\rho\boldsymbol{V})}{\partial t} - \boldsymbol{V}\frac{\partial \rho}{\partial t}$$

$$= \frac{\partial(\rho\boldsymbol{V})}{\partial t} + \boldsymbol{V}\,(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t} + (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi$$

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P-\rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t}+(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V}+\rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=-\boldsymbol{\nabla}P-\rho\boldsymbol{\nabla}\Phi$$

$$(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\,\boldsymbol{V}+\rho\,(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=\boldsymbol{\nabla}\cdot\,(\rho\boldsymbol{V}\otimes\boldsymbol{V})$$

Use divergence chain rule for dyadic tensors

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + (\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi$$

$$(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\, \boldsymbol{V} + \rho\, (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V})$$

$$\boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot (P\, \boldsymbol{I})$$

Rewrite pressure gradient as a divergence

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + (\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi$$

$$(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\, \boldsymbol{V} + \rho\, (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V})$$

$$\boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot (P\, \boldsymbol{I})$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\, \boldsymbol{I}) = -\rho \boldsymbol{\nabla}\Phi$$

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P-\rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t}+(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V}+\rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=-\boldsymbol{\nabla}P-\rho\boldsymbol{\nabla}\Phi$$

Momentum density

Stress tensor = momentum flux

Momentum source: gravity

$$\frac{\partial\boxed{(\rho\boldsymbol{V})}}{\partial t}+\boldsymbol{\nabla}\cdot\boxed{(\rho\boldsymbol{V}\otimes\boldsymbol{V}+P\,\boldsymbol{I})}=\boxed{-\rho\boldsymbol{\nabla}\Phi}$$

The tensor $\mathbf{R_{ik}= \rho V_i V_k + p\delta_{ik}}$ is the Reynolds stress tensor for an ideal fluid and represents the momentum flux

Energy Conservation

The total energy of a fluid element is given by the mechanical energy (kinetic + potential) and the internal energy (thermodynamical)

$$\epsilon_{tot} = \epsilon_{mech} + \epsilon_{int} = \epsilon_{kin} + \epsilon_{pot} + \epsilon_{int}$$

The mechanical part can be obtained by the motion equation

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right] = -\boldsymbol{\nabla}P - \rho\,\boldsymbol{\nabla}\Phi$$

Using the vectorial identity

$$(\vec{V}\cdot\nabla)\vec{V} = \nabla(\frac{V^2}{2}) - \vec{V}\times(\nabla\times\vec{V})$$

And multiplying scalarly the motion equation by **V**, we get

$$\rho\vec{V}\cdot(\frac{\partial\vec{V}}{\partial t}+\nabla(\frac{V^2}{2})-\vec{V}\times(\nabla\times\vec{V})) = -\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

The right hand side corresponds to the power of the force densities acting on the system

Energy Conservation

$$\rho\vec{V}\cdot\left(\frac{\partial\vec{V}}{\partial t}+\nabla\left(\frac{V^2}{2}\right)-\vec{V}\times(\nabla\times\vec{V})\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

The scalar prod with 2nd term in the lefthand side is = 0 because the vectors are perp →

$$\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}+\rho\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

$$\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}=\frac{\partial}{\partial t}(\rho V^2/2)-\frac{V^2}{2}\frac{\partial\rho}{\partial t}\quad=\frac{\partial}{\partial t}(\rho V^2/2)+\frac{V^2}{2}\nabla\cdot(\rho\vec{V})$$

Using the mass conservation law to eliminate ∂ρ/∂t

$$\frac{\partial}{\partial t}(\rho V^2/2)+\frac{V^2}{2}\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

$$=\nabla\cdot\left(\frac{\rho V^2\vec{V}}{2}\right)$$

$$\Rightarrow\quad\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot\left(\frac{\rho V^2\vec{V}}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

This equation shows how the kinetic energy of the fluid changes due to work done by pressure forces and by gravitational force: they act as sources of kinetic energy

The energy flux merely redistributes kinetic energy over space

Energy Conservation

$$\frac{\partial}{\partial t}(\rho V^2/2) + \nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p - \boxed{\rho\vec{V}\cdot\nabla\Phi}$$

Now it's the turn of potential energy

$$\rho\vec{V}\cdot\nabla\Phi = \nabla\cdot(\rho\Phi\vec{V}) - \Phi\nabla\cdot(\rho\vec{V}) \quad = \nabla\cdot(\rho\Phi\vec{V}) + \Phi\frac{\partial\rho}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p-\nabla\cdot(\rho\Phi\vec{V})-\boxed{\Phi\frac{\partial\rho}{\partial t}}$$

$$\Phi\frac{\partial\rho}{\partial t} = \frac{\partial(\rho\Phi)}{\partial t} - \rho\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p-\nabla\cdot(\rho\Phi\vec{V})-\frac{\partial(\rho\Phi)}{\partial t}+\rho\frac{\partial\Phi}{\partial t}$$

Rearranging terms: $$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2})+\nabla\cdot(\rho\Phi\vec{V})+\frac{\partial(\rho\Phi)}{\partial t} = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

This equation states how the mechanical energy changes due to the work done by the pressure forces and due to non conservative gravitational field (ie explicitly time dependent)

Energy Conservation

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

The thermodynamics is in the pressure term

Let rewrite the I principle of TD in terms of unit mass variables (specific variables)

$$dq = Tds = d\epsilon + pdv$$

Where dq is the specific heat exchanged, ds the specific entropy, dε is the specific internal energy and v the specific volume **v= V/m=1/ρ**

a) $Tds = d\epsilon + pd(1/\rho) \equiv d(\epsilon + p/\rho) - \frac{dp}{\rho}$ But ε +p/ρ = h = specific enthalpy ➔

b) $Tds = dh - \frac{dp}{\rho}$ ➔ $\rho dh - \rho T ds = dp$ ➔ The pressure gradient is $\rho\nabla h - \rho T\nabla s = \nabla p$

Energy Conservation

To complete the energy balance we have to add the internal energy to the equation

The explicit time variation of internal energy density is $\frac{\partial(\rho\epsilon)}{\partial t} = \rho\frac{\partial\epsilon}{\partial t} + \epsilon\frac{\partial\rho}{\partial t}$

Taking the time derivative of a), we get

$$T\frac{\partial s}{\partial t} = \frac{\partial\epsilon}{\partial t} + p\frac{\partial(1/\rho)}{\partial t} = \frac{\partial\epsilon}{\partial t} - \frac{p}{\rho^2}\frac{\partial\rho}{\partial t} \quad \Rightarrow \quad \rho T\frac{\partial s}{\partial t} = \rho\frac{\partial\epsilon}{\partial t} - \frac{p}{\rho}\frac{\partial\rho}{\partial t}$$

$$\rho\frac{\partial\epsilon}{\partial t} = \rho T\frac{\partial s}{\partial t} + \frac{p}{\rho}\frac{\partial\rho}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\nabla\cdot(\rho\vec{V}) \qquad \frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{V})$$

$$\frac{\partial(\rho\epsilon)}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\nabla\cdot(\rho\vec{V}) - \epsilon\nabla\cdot(\rho\vec{V}) = \rho T\frac{\partial s}{\partial t} - (\frac{p}{\rho} + \epsilon)\nabla\cdot(\rho\vec{V}) = \rho T\frac{\partial s}{\partial t} - h\nabla\cdot(\rho\vec{V})$$

We add ∂ε/∂t on both sides of mech E equation and substitute the grad(p)

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2} + \rho\Phi) + \nabla\cdot[(\frac{\rho V^2}{2} + \rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p + \rho\frac{\partial\Phi}{\partial t} \qquad \rho\nabla h - \rho T\nabla s = \nabla p$$

$$\frac{\partial(\rho\epsilon)}{\partial t} + \frac{\partial}{\partial t}(\frac{\rho V^2}{2} + \rho\Phi) + \nabla\cdot[(\frac{\rho V^2}{2} + \rho\Phi)\vec{V}] = \rho T\frac{\partial s}{\partial t} - h\nabla\cdot(\rho\vec{V}) - \vec{V}\cdot(\rho\nabla h - \rho T\nabla s) + \rho\frac{\partial\Phi}{\partial t}$$

Energy Conservation

To complete the energy balance we have to add the internal energy to the equation

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = \boxed{\rho T(\frac{\partial s}{\partial t}+\vec{V}\cdot\nabla s)} - \boxed{(h\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla h)} + \rho\frac{\partial\Phi}{\partial t}$$

$$= \frac{Ds}{Dt} \qquad\qquad = \nabla\cdot(\rho h\vec{V})$$

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = \rho T\frac{Ds}{Dt}-\nabla\cdot(\rho h\vec{V})+\rho\frac{\partial\Phi}{\partial t}$$

Move the grad on right hand side to left side and get

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi+\rho h)\vec{V}] = \rho T\frac{Ds}{Dt}+\rho\frac{\partial\Phi}{\partial t}$$

Energy density Energy flux "Net heating rate density"

The 1st term in RHS in the true heating (or cooling) due to "external" irreversible processes as radiation losses. The 2nd "gravitational heating" $\rho\partial\Phi/\partial t$ corresponds to the process known as violent relaxation in a time-varying gravitational potential, which plays an important role in the dynamics of galaxies, where it acts in a way analogous to a heating mechanism

Energy Conservation

Energy density is a scalar! → the energy flux is a vector

Kinetic energy density

Internal energy density

Gravitational potential energy density

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\rho \boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\rm eff}$$

$$\mathcal{H}_{\rm eff} \equiv \mathcal{H} + \rho\frac{\partial \Phi}{\partial t}$$

Irreversibly lost/gained energy per unit volume

Energy Conservation

$$e = \frac{P}{(\gamma - 1)\,\rho} \;\; \text{if} \;\; P\,\rho^{-\gamma} = \text{constant} \qquad h = e + \frac{P}{\rho} = \frac{\gamma P}{(\gamma - 1)\,\rho}$$

Internal energy per unit mass

Specific enthalpy

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho\,V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\text{eff}}$$

$$\mathcal{H}_{\text{eff}} \equiv \mathcal{H} + \rho\,\frac{\partial\Phi}{\partial t}$$

Irreversible gains/losses, e.g. radiation losses

“Dynamical Friction”

Conservazione dell'energia

Nei fluidi ideali, in cui sono assenti fenomeni di dissipazione dovuti ad attrito "interno" (cioe' viscosita') e nell'ipotesi di assenza di conduzione termica (che puo' trasferire calore da una regione all'altra), l'unico fenomeno di scambio di energia non meccanica (ie non esprimibile come pdV) puo' essere solo attraverso l'irraggiamento

Nel caso in cui anche i processi radiativi siano assenti o trascurabili, il processo e' adiabatico (se reversibile) → dQ = 0 → TdS = 0

In tal caso la conservazione dell'energia di un elemento di massa e' equivalente alla conservazione dell'entropia del sistema dello stesso elemento

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Conservazione dell'energia

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

La densita' dei fluidi astrofisici e' tipicamente molto bassa (~1 idrogeno/cm^3) per cui un fotone emesso dalle particelle del fluido non viene mai riassorbito (a meno che vi siano certe condizioni (cfr. Autoassorbimento e spessore ottico) e lascia il sistema, facendo perdere energia attraverso processi radiativi

In tal caso l'entropia non e' piu' conservata

Anche in questo caso pero' l'equazione di sopra ha un suo ambito di validita': infatti i processi radiativi hanno tempi scala caratteristici e se l'evoluzione del sistema avviene su scale di tempo << di quelli radiativi, il processo puo' essere considerato adiabatico

Conservazione dell'energia

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Vi sono situazioni in cui conduzione e viscosita' giocano un ruolo importante (p es all'interno delle stelle, in dischi di accrescimento e piu' in generale in oggetti compatti e/o densi in cui il libero cammino medio diventa "piccolo" e quindi lo spessore ottico diventa "grande" (cfr. Autoassorbimento)

Ma in genere il meccanismo per cui un sistema si discosta dalla isoentropia e' il fatto che il fluido viene riscaldato o raffreddato da una varieta' di processi radiativi

Se definiamo Γ e Λ i coefficienti di riscaldamento e raffreddamento per unita' di massa e di tempo

$$\frac{Dq}{Dt} = T\frac{Ds}{Dt} = T(\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s) = \Gamma - \Lambda$$

Steady Flows:
no explicit time-dependence:

$$\frac{\partial}{\partial t} = 0$$

$$\text{mass conservation:} \quad \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}) = 0 \; ;$$

$$\text{momentum conservation:} \quad \boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{V} \otimes \boldsymbol{V} + P \, \boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi \; ;$$

$$\text{energy conservation:} \quad \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{V} \left(\tfrac{1}{2} V^2 + h + \Phi \right) \right] = 0 \; .$$

Energy conservation

mass conservation: $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}) = 0 ;$

momentum conservation: $\boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V} \otimes \boldsymbol{V} + P\, \boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi ;$

energy conservation: $\boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{V} \left(\frac{1}{2}V^2 + h + \Phi\right)\right] = 0 .$

$$\boldsymbol{\nabla} \cdot (f \boldsymbol{A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{A} \cdot \boldsymbol{\nabla}) f$$

$$\boldsymbol{A} = \rho \boldsymbol{V} \quad , \quad f = \frac{1}{2}V^2 + h + \Phi$$

$$\nabla \cdot \mathbf{A} = 0$$

For mass conservation law

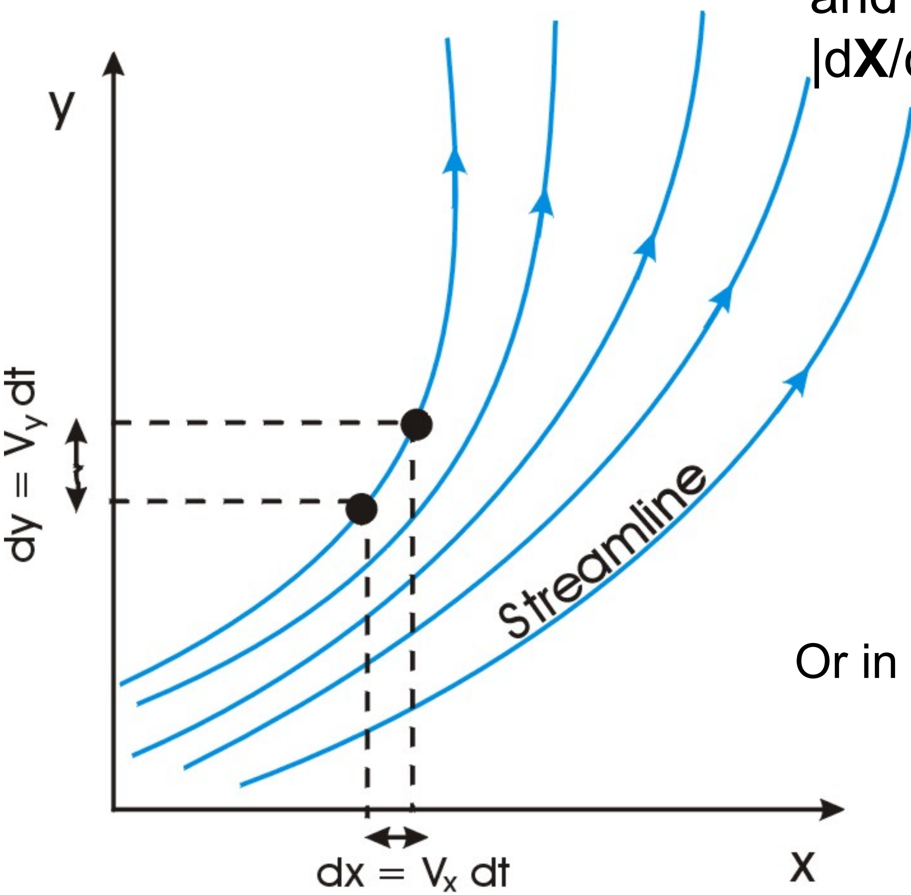
$$(\boldsymbol{V} \cdot \boldsymbol{\nabla}) \left(\frac{1}{2}V^2 + h + \Phi\right) = 0$$

The quantity f is (obviously) the specific energy (energy/mass)

Variation along flow lines in steady flows

$$(\boldsymbol{V} \cdot \boldsymbol{\nabla})\left(\tfrac{1}{2}V^2 + h + \Phi\right) = 0$$

Consider the flowlines defined as a trajectory **x**=**X**(l) such that the tangent vector (versor), d**X**/dl, is || the local **V,** d**X**/dl || **V**(**X**) and it can be choosen such that |d**X**/dl| = 1 → dl/dt=V



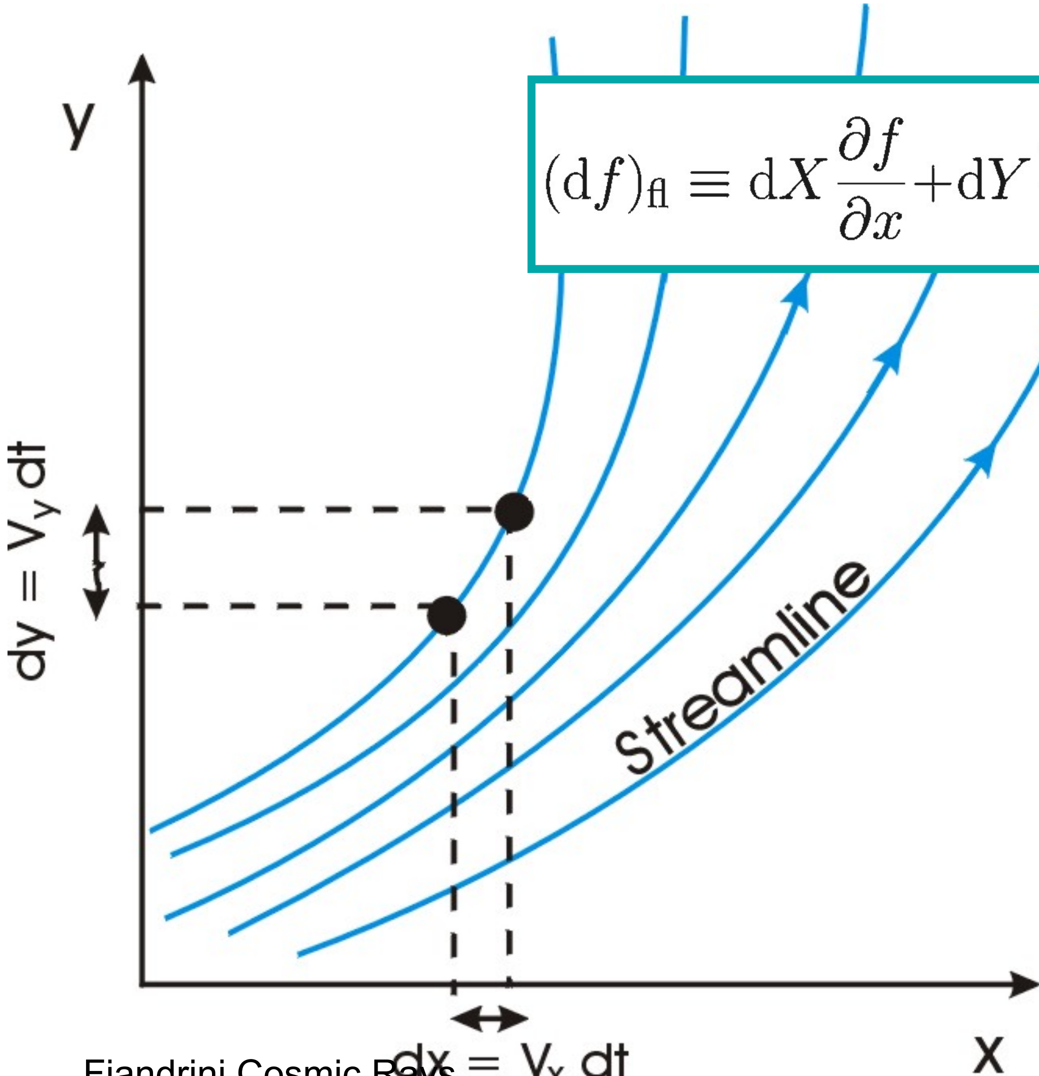
The coordinates of points on a given flow line satisfy the relation

$$\mathrm{d}\boldsymbol{X} = \boldsymbol{V}(\boldsymbol{x} = \boldsymbol{X})\,\mathrm{d}t$$

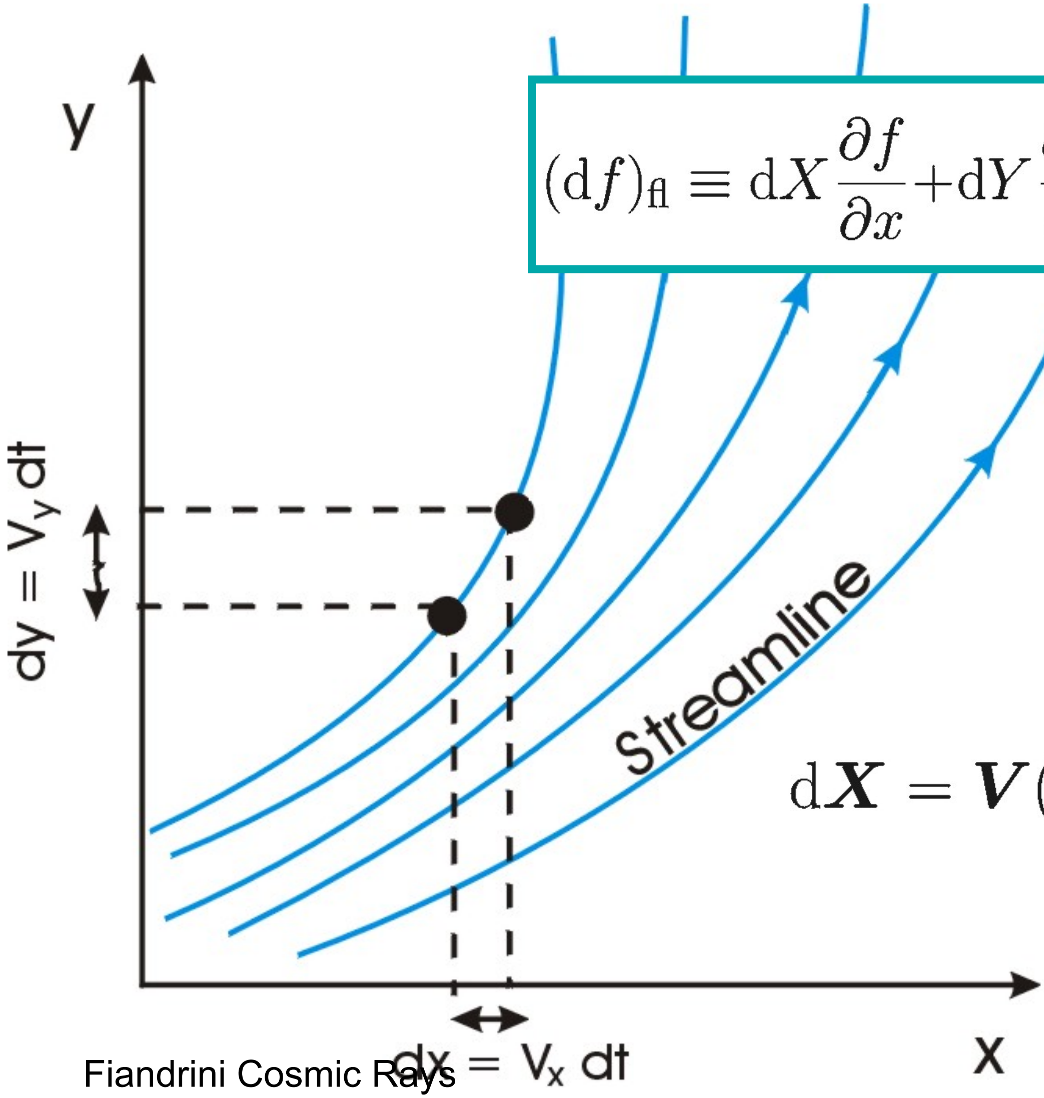
Or in components $\dfrac{\mathrm{d}X}{V_x(\boldsymbol{X})} = \dfrac{\mathrm{d}Y}{V_y(\boldsymbol{X})} = \dfrac{\mathrm{d}Z}{V_z(\boldsymbol{X})} = \mathrm{d}t$

When is a function *f(x,y,z)* constant along flow lines ?

$$(\mathrm{d}f)_{\mathrm{fl}} \equiv \mathrm{d}X\frac{\partial f}{\partial x}+\mathrm{d}Y\frac{\partial f}{\partial y}+\mathrm{d}Z\frac{\partial f}{\partial z} = (\mathrm{d}\boldsymbol{X}\cdot\boldsymbol{\nabla}f) = 0$$



When is a function *f(x,y,z)* constant along flow lines ?



$$(\mathrm{d}f)_{\mathrm{fl}} \equiv \mathrm{d}X\frac{\partial f}{\partial x} + \mathrm{d}Y\frac{\partial f}{\partial y} + \mathrm{d}Z\frac{\partial f}{\partial z} = (\mathrm{d}\boldsymbol{X}\cdot\boldsymbol{\nabla}f) = 0$$

$$\mathrm{d}\boldsymbol{X} = \boldsymbol{V}(\boldsymbol{x} = \boldsymbol{X})\,\mathrm{d}t$$

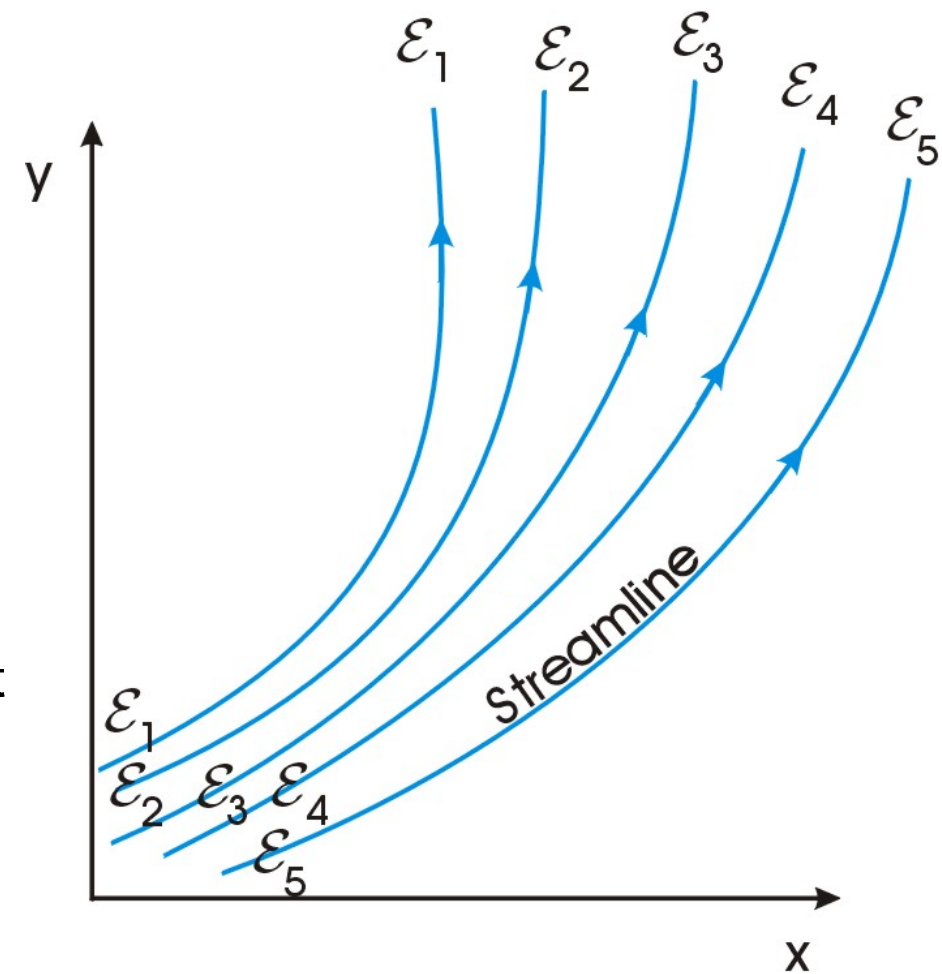
$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})f(\boldsymbol{x}) = 0$$

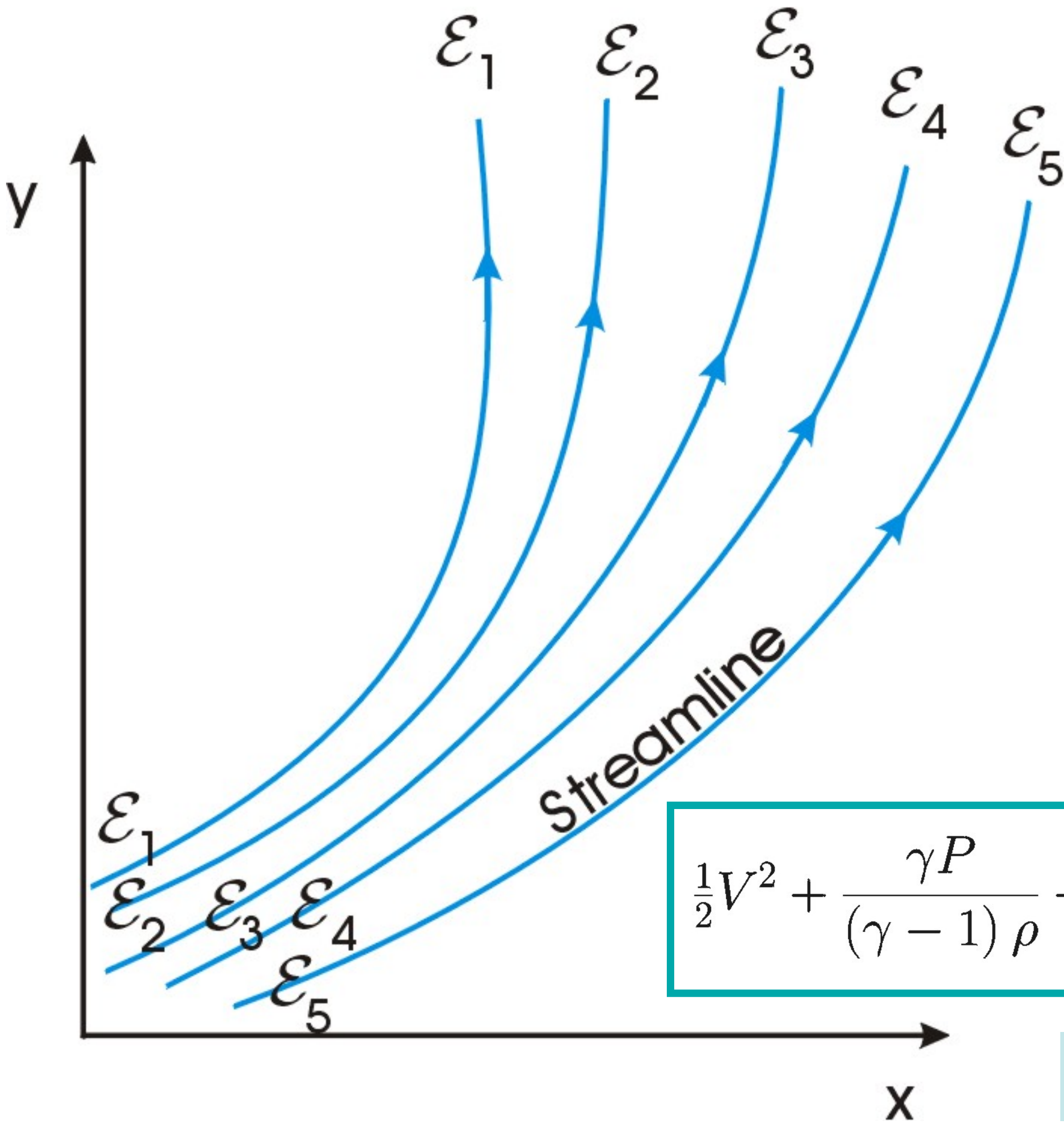
Bernouilli's Law for steady flows:

$$(\boldsymbol{V} \cdot \boldsymbol{\nabla})\left(\tfrac{1}{2}V^2 + h + \Phi\right) = 0 \longrightarrow \tfrac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

NB: bernoulli law dont say anything about the variation of E_{spec} across the flow lines

In general the constant may differ from flowline to flowline → it is not a global constant over all space

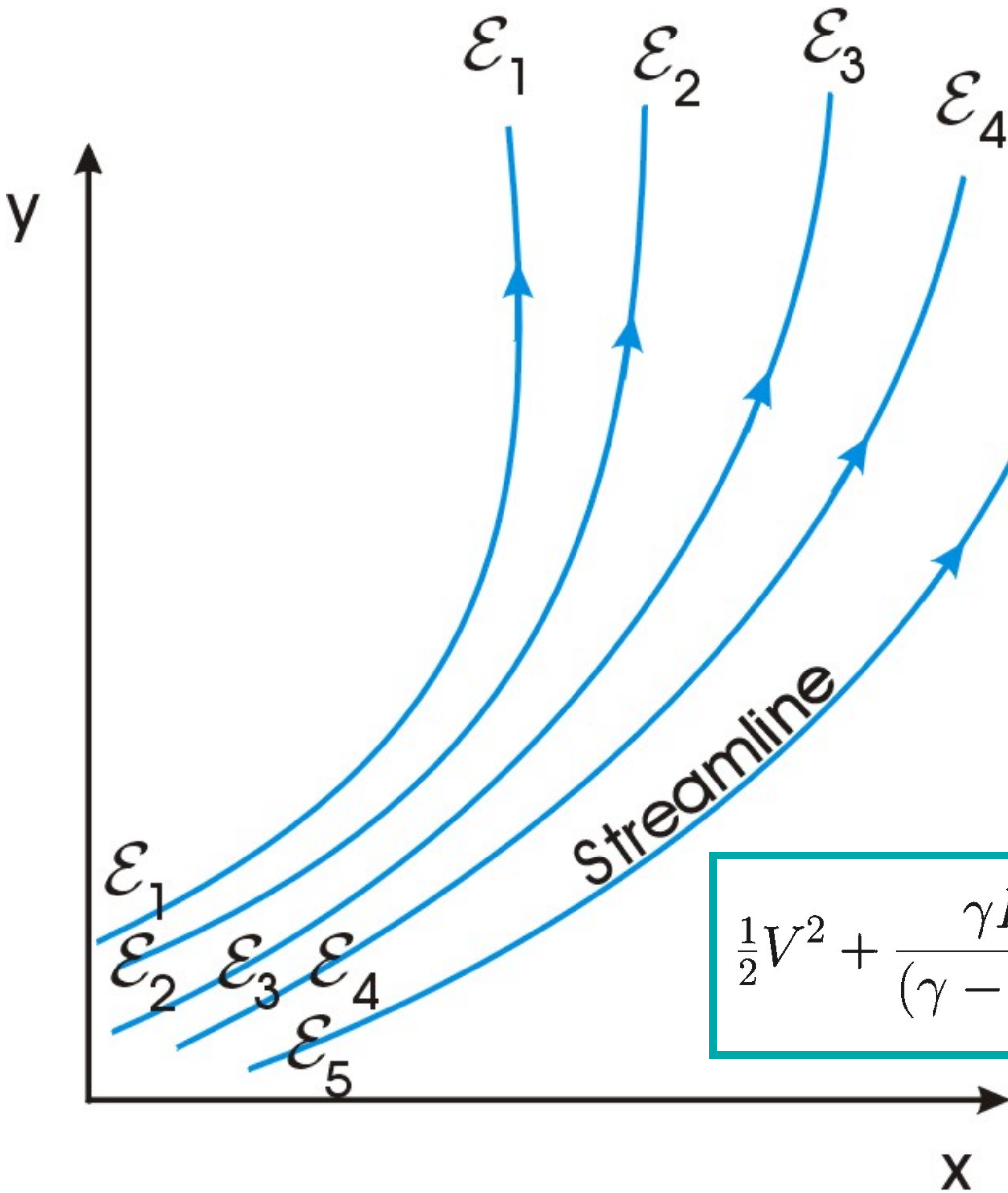




Bernoulli's Law

$$\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

(Energy conservation)



Adiabatic fluid

$$P\rho^{-\gamma} = \text{constant along flowlines}$$

(Entropy conservation)

Bernoulli's Law

$$\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

(Energy conservation)

Stevino's law in astrophysics

Static case: V=0

A short digression: isothermal sphere and globular clusters

Isothermal sphere

The isothermal sphere is a spherically symmetric, self-gravitating system

It is a crude model for a globular cluster, for the quasi-spherical region ("bulge") of a disk galaxy or for the nucleus of an elliptical galaxy

Consider a large number of star with number density distribution n=n(r) only, r is the distance form the center of the sphere and with a mass density $\rho=m_* n(r)$, where m_* is the mass of the stars (supposed to be the same)

If the number of stars is large enough we can describe it as a "gas" of stars with a "temperature" T determined by the velocity dispersion (i.e. energy equipartition)

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 \equiv \sigma^2 = \frac{kT}{m_*}$$

In the isothermal sphere model, the cluster is treated as a self-gravitating ball of gas→ the pressure is then $p(r) = n(r)kT = \rho(r)\sigma^2$

Typically a globular cluster contains 100.000 stars with a mass between $10^4 – 10^6$ solar masses and an average of $10^5 M_{sun}$

Governing Equations:

Equation of Motion: <u>no</u> bulk motion, only pressure!
→Hydrostatic Equilibrium!

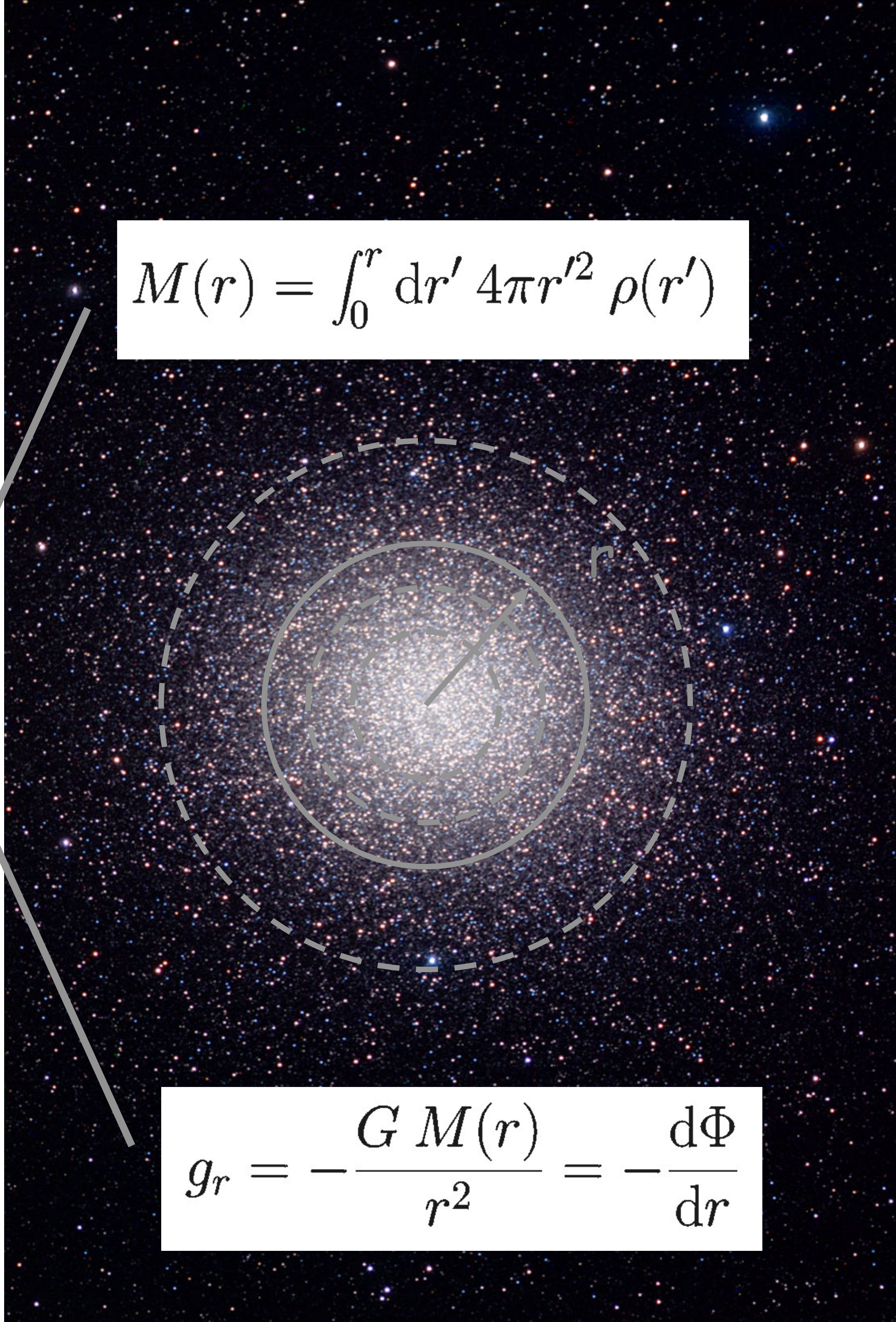
Isothermal sphere means that the velocity dispersion does not depend on the radius r

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \tilde{\sigma}^2 \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\rho\,\frac{G\,M(r)}{r^2}$$

↓

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Fiandrini Cosmic Rays



$$M(r) = \int_0^r \mathrm{d}r'\,4\pi r'^2\,\rho(r')$$

$$g_r = -\frac{G\,M(r)}{r^2} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Where ρ_o is the mass density at r=0, assuming $\Phi(0)=0$

'Down to Earth' Analogy: the Isothermal Atmosphere

Low density & low pressure

Constant temperature

High density & high pressure

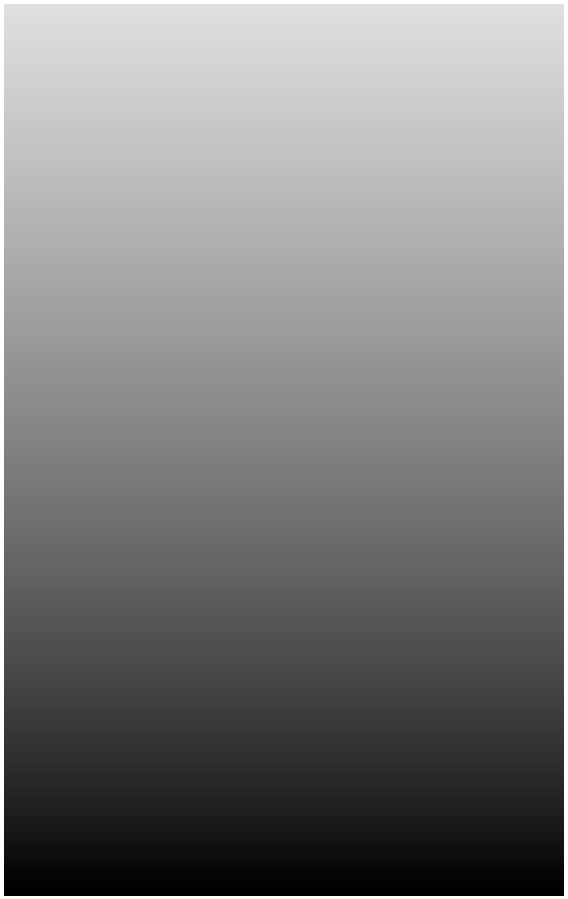
$$\boldsymbol{g} = -\nabla\Phi = -g\hat{\boldsymbol{e}}_z \quad \Leftrightarrow \quad \Phi(z) = gz$$

$$\nabla P = \left(\frac{dP}{dz}\right)\hat{\boldsymbol{e}}_z = \frac{RT}{\mu}\frac{d\rho}{dz}\hat{\boldsymbol{e}}_z$$

Force balance:

$$0 = -\nabla P + \rho\boldsymbol{g} = -\left(\frac{RT}{\mu}\frac{d\rho}{dz} + \rho g\right)\hat{\boldsymbol{e}}_z$$

'Down to Earth' Analogy: the Isothermal Atmosphere



$$0 = -\nabla P + \rho \boldsymbol{g} = -\left(\frac{RT}{\mu} \frac{d\rho}{dz} + \rho g \right) \hat{\boldsymbol{e}}_z$$

'Down to Earth' Analogy: the Isothermal Atmosphere

$$\rho(z) = \rho(0)\exp(-z/H)$$

$$P(z) = P(0)\exp(-z/H)$$

$$0 = -\nabla P + \rho \boldsymbol{g} = -\left(\frac{RT}{\mu}\frac{d\rho}{dz} + \rho g\right)\hat{\boldsymbol{e}}_z$$

Set to zero!

$$\frac{1}{\rho}\frac{d\rho}{dz} = -\frac{\mu g}{RT} \quad \Leftrightarrow$$

$$\rho(z) = \rho(0)\exp(-\frac{\mu g z}{RT}) = \rho(0)\exp(-\frac{\mu\Phi(z)}{RT})$$

$$= \rho(0)\exp(-z/H) \quad , \quad H \equiv \frac{RT}{\mu g}$$

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

The gravitational potential is described by the Poisson's equation

$$\nabla^2\Phi(r) = 4\pi G\rho(r)$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\,\rho(r) = 4\pi G\rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Spherically symmetric Laplace Operator

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\,\rho(r) = 4\pi G\rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Poisson Eqn.

Spherically symmetric Laplace Operator

$$\xi = \frac{r}{r_{\mathrm{K}}} \quad , \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_*\Phi}{k_{\mathrm{b}}T}$$

$$r_{\mathrm{K}} = \left(\frac{\tilde{\sigma}^2}{4\pi G\rho_0}\right)^{1/2} = \left(\frac{k_{\mathrm{b}}T}{4\pi G m_*\rho_0}\right)^{1/2}$$

King radius

Scale Transformation

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Poisson Eqn.

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\,\rho(r) = 4\pi G\rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Spherically symmetric Laplace Operator

Scale Transformation

$$\xi = \frac{r}{r_\mathrm{K}} \quad , \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_*\Phi}{k_\mathrm{b}T}$$

$$r_\mathrm{K} = \left(\frac{\tilde{\sigma}^2}{4\pi G\rho_0}\right)^{1/2} = \left(\frac{k_\mathrm{b}T}{4\pi G m_*\rho_0}\right)^{1/2}$$

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\,\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right) = e^{-\Psi}$$

Density law and Poisson's Equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right) = e^{-\Psi}$$

This dimensionless form displays NO explicit information about the properties of the cluster

$$\xi = \frac{r}{r_{\mathrm{K}}} \quad , \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_*\Phi}{k_{\mathrm{b}}T}$$

$$r_{\mathrm{K}} = \left(\frac{\tilde{\sigma}^2}{4\pi G\rho_0}\right)^{1/2} = \left(\frac{k_{\mathrm{b}}T}{4\pi G m_*\rho_0}\right)^{1/2}$$

Scale Transformation

In particular all the reference to the central density ρ_o and velocity dispersion σ^2 has disappeared

➔ this means that all the isothermal are self-similar

If one plots the density relative to the central value ρ/ρ_o as function of $\xi=r/r_K$, all isothermal spheres have exactly the same density profile

The boundary conditions are: $\Phi(0) = 0$ and $(d\Psi/d\xi)_{\xi=0} = 0$

The 1st is possible because potential is defined up to a costant, while the 2nd is a consequence of the spherical symmetry: at the center the net force is zero

Solution:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Psi}{d\xi}\right) = e^{-\Psi}$$

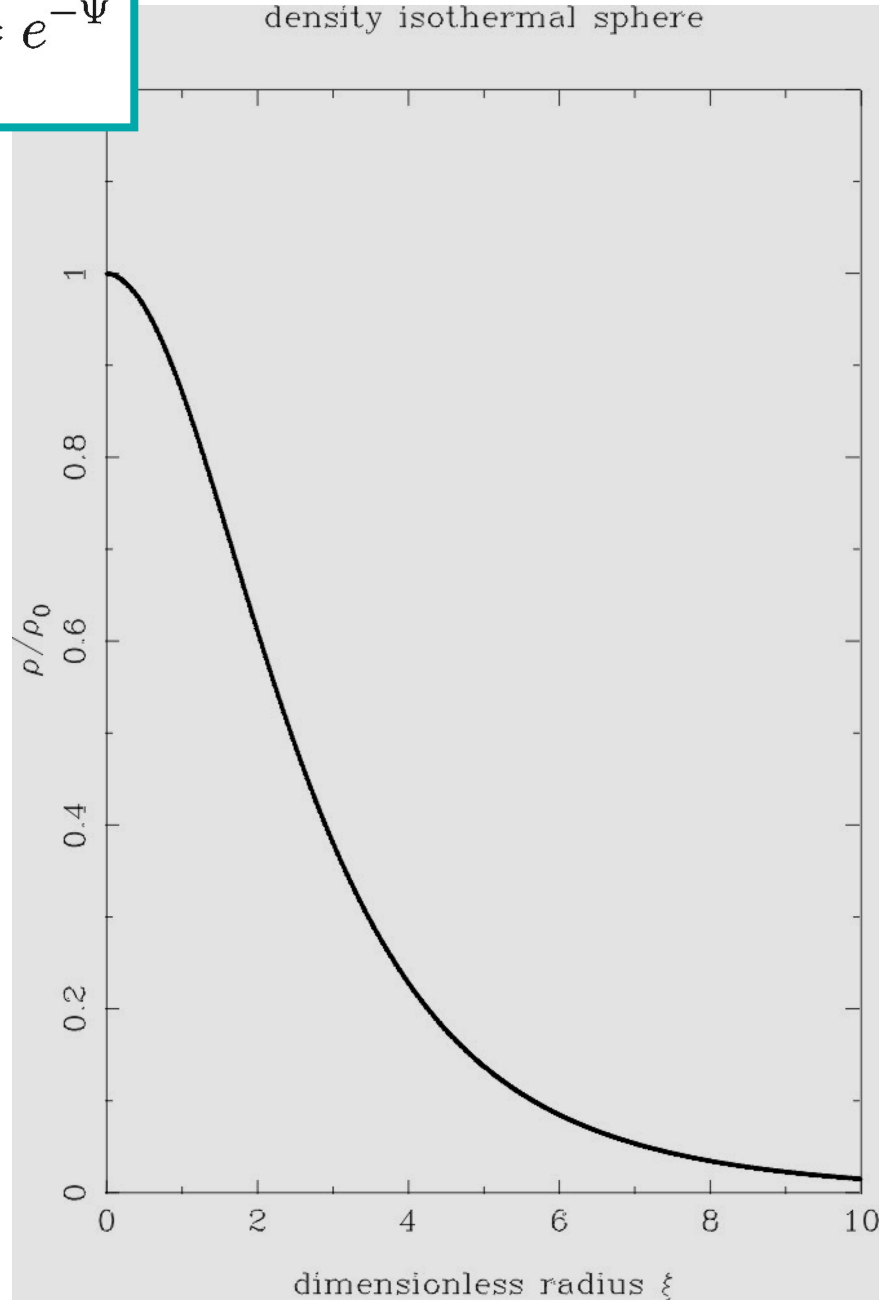
There is no analytical solution

Near ξ=0 one con solve by a power series, using the fact that for Ψ<< 1 so that the exp on RHS can be expanded

$$\text{For } \xi = r/r_K \ll 1: \begin{cases} \rho \approx \rho_0\left(1-\frac{\xi^2}{6}+\frac{\xi^4}{45}\right) \\ \Psi \approx \frac{\xi^2}{6}-\frac{\xi^4}{120} \end{cases}$$

For large ξ, the solution goes asymptotically to $\Psi \sim \log(\xi^2/2)$

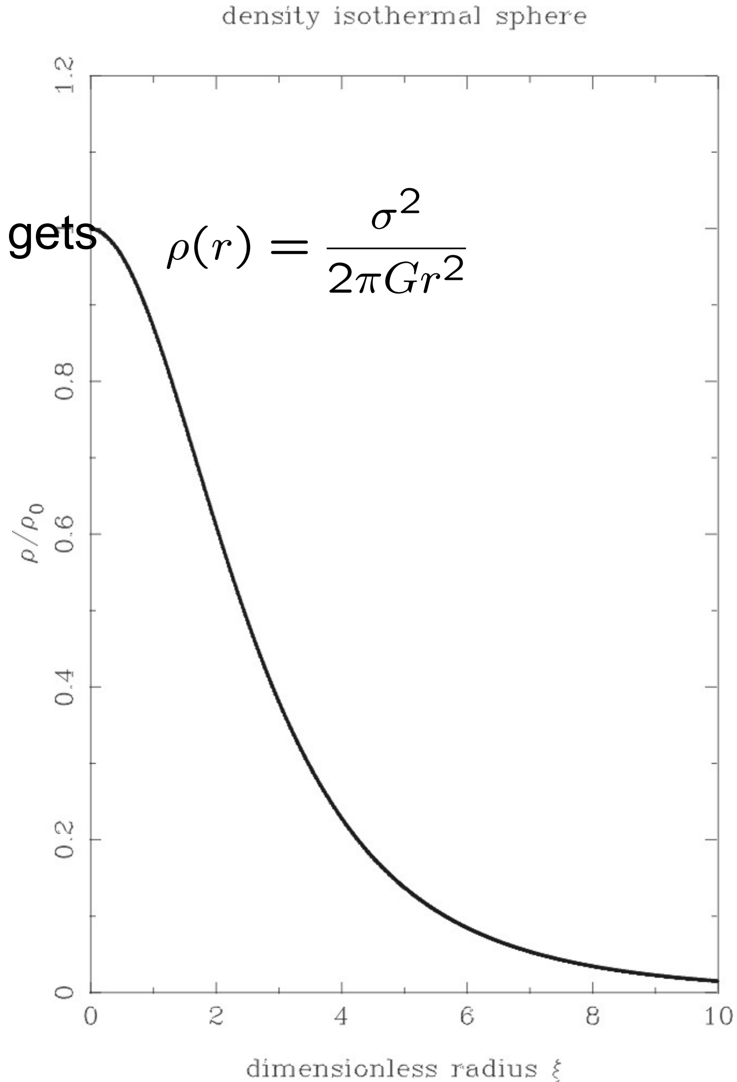
$$\text{For } \xi = r/r_K \gg 1: \begin{cases} \rho \approx \frac{2\rho_0}{\xi^2} = \frac{\widetilde{\sigma^2}}{2\pi G r^2} \\ \Psi \approx \log\left(\frac{\xi^2}{2}\right) \end{cases}$$



Singular Solution

Expressing the density in terms of the radius one gets

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$



Known as the "singular isothermal sphere" solution as the density goes to ∞ as r→0

Infact this is the ONLY analytic solution known to the isothermal sphere equation, as can be checked by substitution

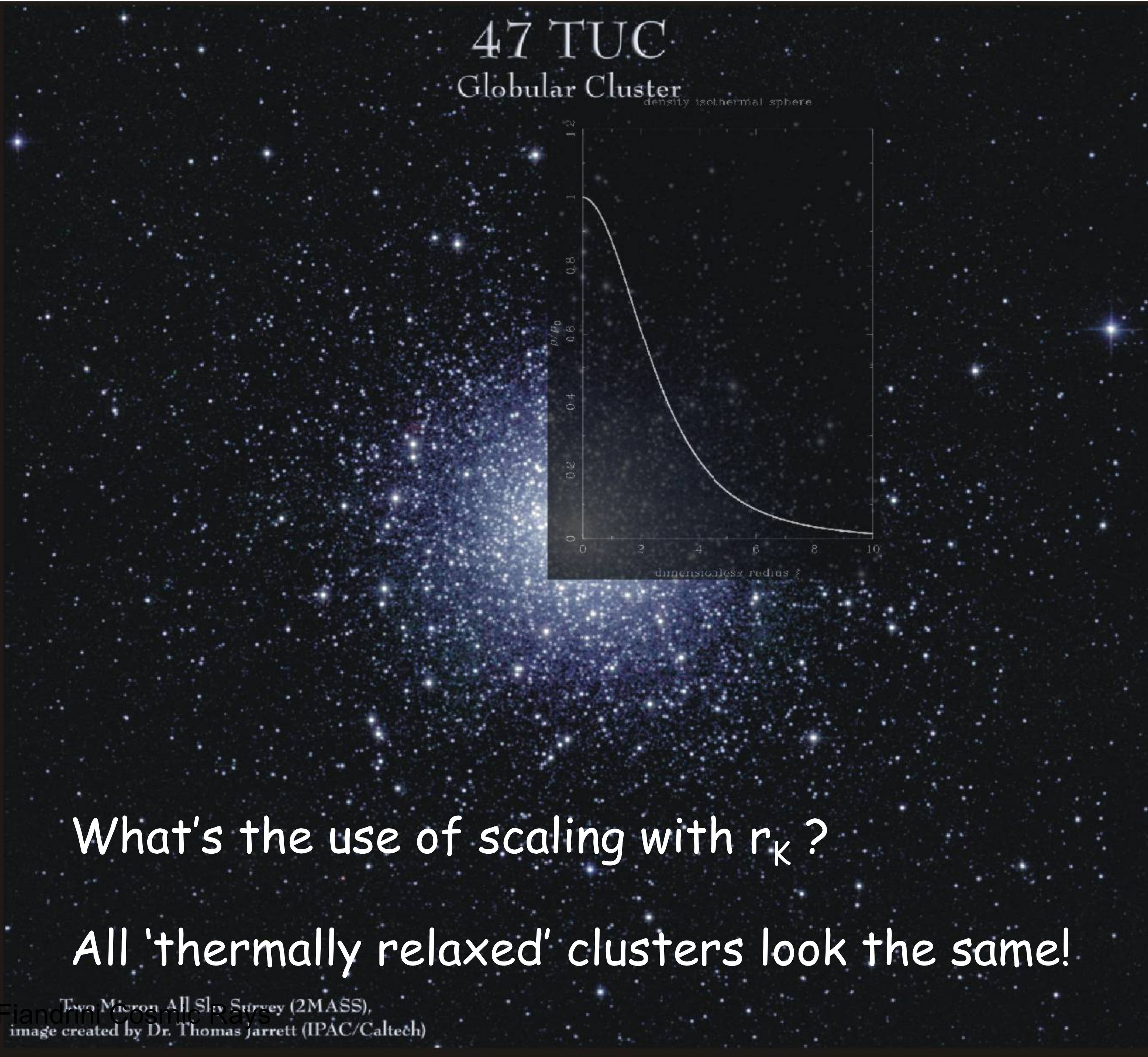
Notice that ρ depends only on dispersion velocity and radius but not on central density ρ_o

For $\xi = r / r_K \gg 1$:

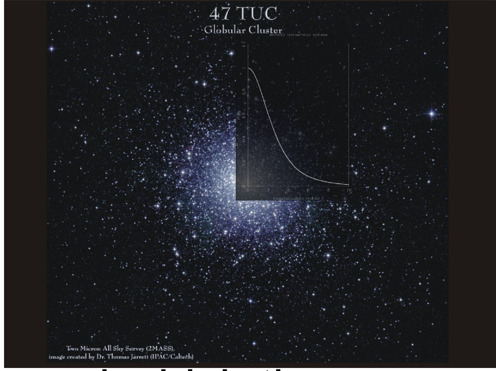
$$M(r) \simeq \int_0^r dr\ 4\pi r^2 \left(\frac{\tilde{\sigma}^2}{2\pi G r^2} \right) = \frac{2\tilde{\sigma}^2 r}{G}$$

$$= 8\pi\rho_0 r_K^2 r$$

Such behavior is clearly unacceptable for a real <globular cluster because m→∞ as r→∞ ➔ isothermal sphere can only be an approximate model which fails at large r


47 TUC
Globular Cluster
density isothermal sphere
ρ/ρ0
0
0.2
0.4
0.6
0.8
1
1.2
0
2
4
6
8
10
dimensionless radius ξ
What's the use of scaling with r_K ?
All 'thermally relaxed' clusters look the same!
Two Micron All Sky Survey (2MASS),
image created by Dr. Thomas Jarrett (IPAC/Caltech)

Tidal radius



Observations show that clusters have a well-defined edge beyond which the stellar density rapidly goes to zero

This can be explained if the tidal forces are taken into account: the variation of the gravitational pull of the galaxy across the globular cluster

If the cluster has a radius r_t and is located at a distance R from galactic center, the typical magnitude of the tidal acceleration is for $r_t \ll R$

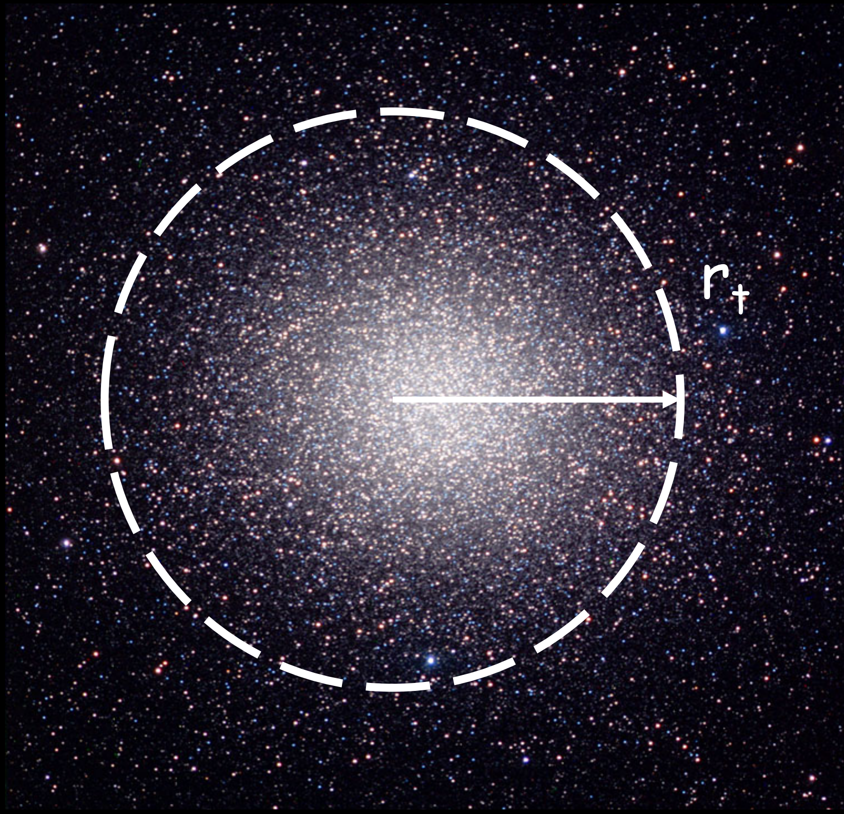
$$g_t \approx r_t \frac{\partial}{\partial r}\left(-\frac{GM_{gal}}{R^2}\right) \quad = \frac{2GM_{gal} r_t}{R^3}$$

This is essentially the difference between the galactic gravitational force at the center and the outer edge of the globular cluster

Tidal Radius

The value of r_t, the so-called tidal radius can be evaluated equating the tidal force to the self-gravitational force of cluster

This defines the maximum size of the cluster where stars in the clusters are still marginally bound by the gravitational pull of the cluster mass



$$\frac{GM_{cl}}{r_t^2} \approx r_t \frac{\partial}{\partial R}\left(-\frac{GM_{gal}}{R^2}\right) = \frac{2GM_{gal} r_t}{R^3}$$

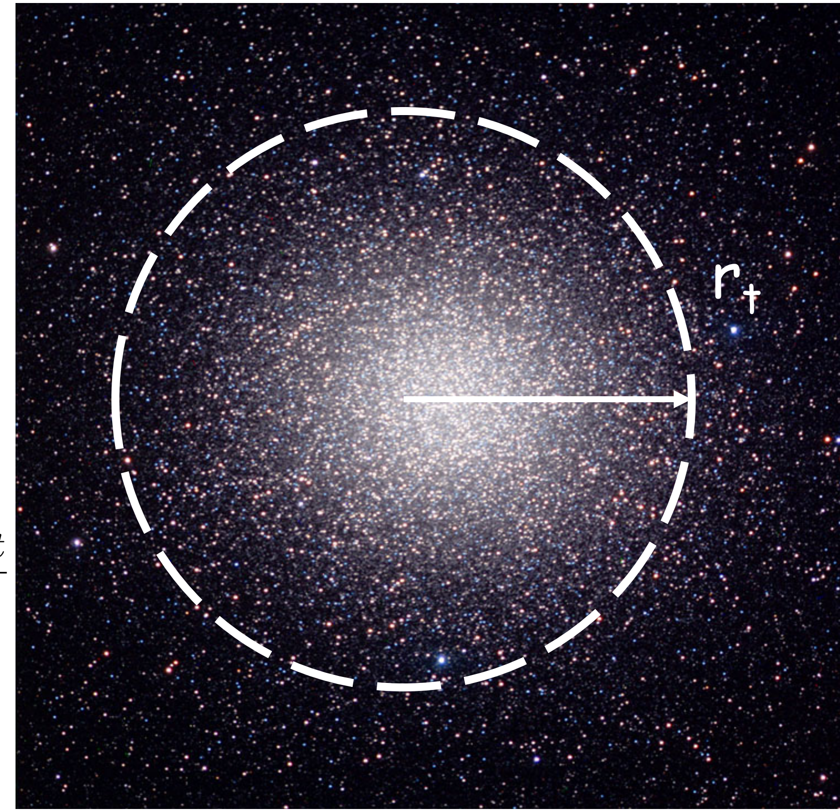
$$\Leftrightarrow$$

$$r_t \approx \left(\frac{M_{cl}}{2M_{gal}}\right)^{1/3} R$$

$$M_{cl} \approx 2.5\times10^6 \left(\frac{\tilde{\sigma}}{5\ \mathrm{km/s}}\right)^3 \left(\frac{R}{10\ \mathrm{kpc}}\right)^{3/2} M_\odot$$

$$M_{cl} \approx 8\pi\rho_0 r_K^2 r_t$$

Tidal Radius



If one uses the relation $M=8\pi\rho_o r^2_K r_t$ for the mass contained within r_t we obtain

$$M_C \approx 8\pi\rho_o r_K^2 r_t$$

And from $$g_t \approx r_t \frac{\partial}{\partial r}(-\frac{GM_{gal}}{R^2}) = \frac{2GM_{gal}r_t}{R^3}$$

$$r_t = (\frac{4\pi\rho_o R^3}{M_{gal}})^{1/2} \qquad r_K = (\frac{\sigma^2 R^3}{GM_{gal}})^{1/2}$$

Using typical values for distances, observed velocity dispersion and central mass of glubular clusters and for the mass of our galaxy

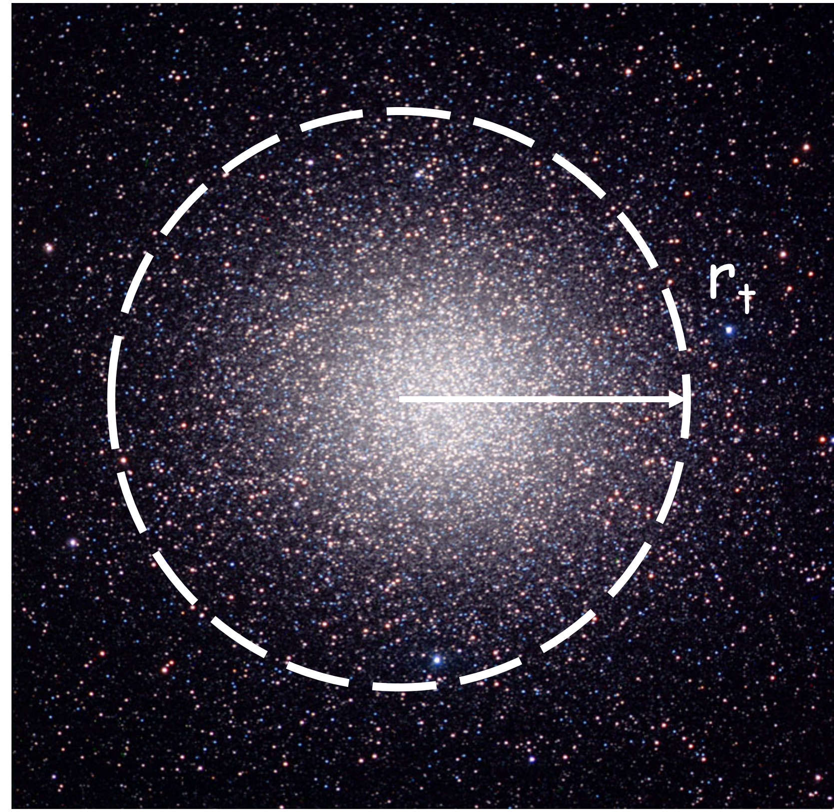
σ ~ 5km/s, ρ_o ~ 10^4 M_o pc^{-3}, R~ 10 kpc, $M_{gal}=10^{11}$ M–

The tidal radius is $$r_t = 200(\frac{\sigma}{5\ km/s})(\frac{R}{10\ jps})^{3/2}$$

It is much larger than the King radius $$r_K \approx 0.2(\frac{\sigma}{5\ km/s})(\frac{\rho_o}{10^4 M_s\ pc^{-3}})^{-1/2}\ pc$$

$$r_K = (\frac{\sigma^2 R^3}{G M_{gal}})^{1/2}$$

$$r_K \approx 0.2(\frac{\sigma}{5\ km/s})(\frac{\rho_o}{10^4 M_s\ pc^{-3}})^{-1/2}\ pc$$



The King radius yields a good estimate for the size of the dense central core of the cluster: the density in an isothermal sphere drops to $\rho_0/2$ at $r \sim 3r_K \sim 1$pc

From these estimates, using

$$M_C \approx 8\pi \rho_o r_K^2 r_t$$

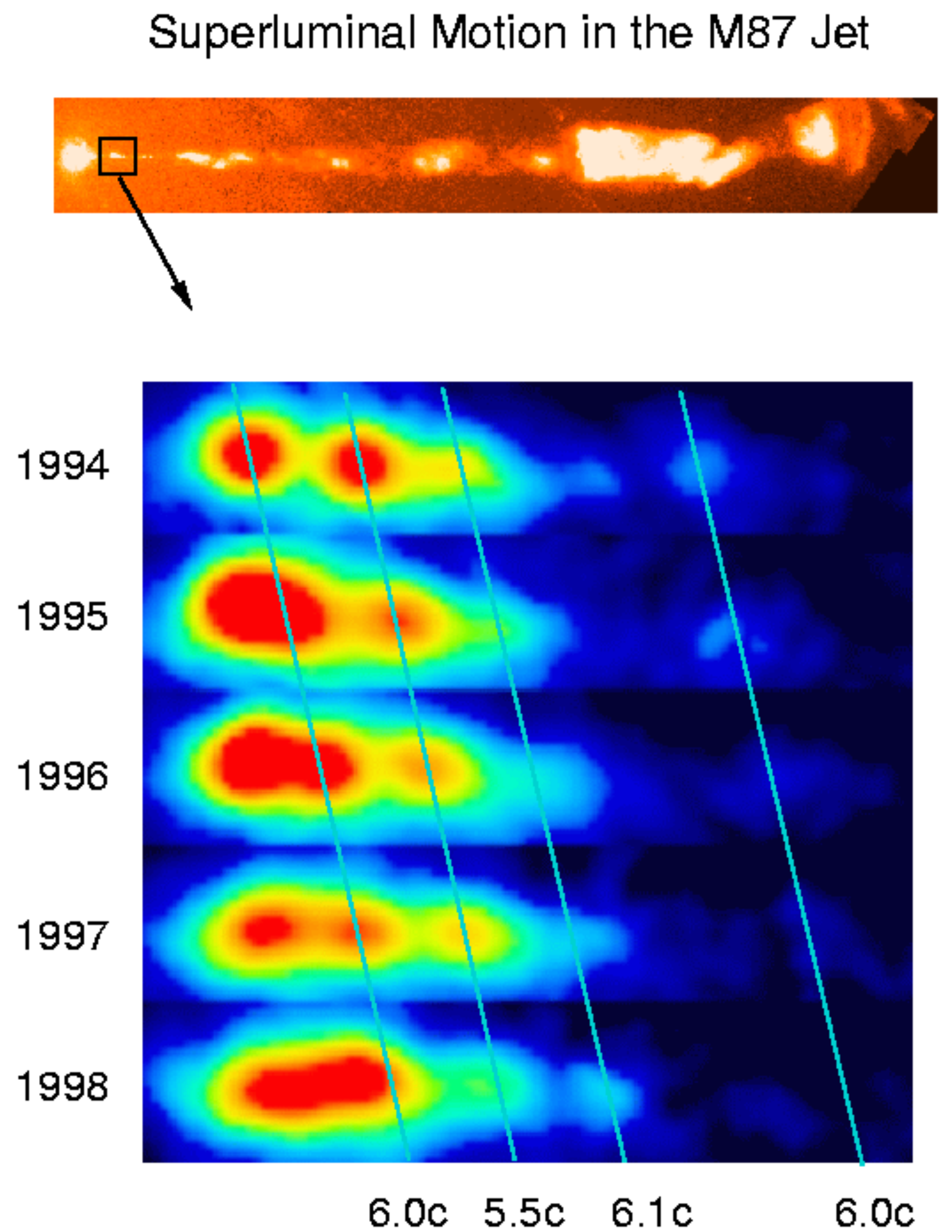
We can obtain the typical mass of a globular cluster

$$M_C \sim \frac{2\sigma^2}{G}\frac{\sigma^2 R^3}{G M_{gal}} \approx 2.5 \times 10^6 (\frac{\sigma}{5\ km/s})(\frac{R}{10\ kpc})^{3/2}\ M_s$$

Which compares well with the masses of globular clusters inferred by observations

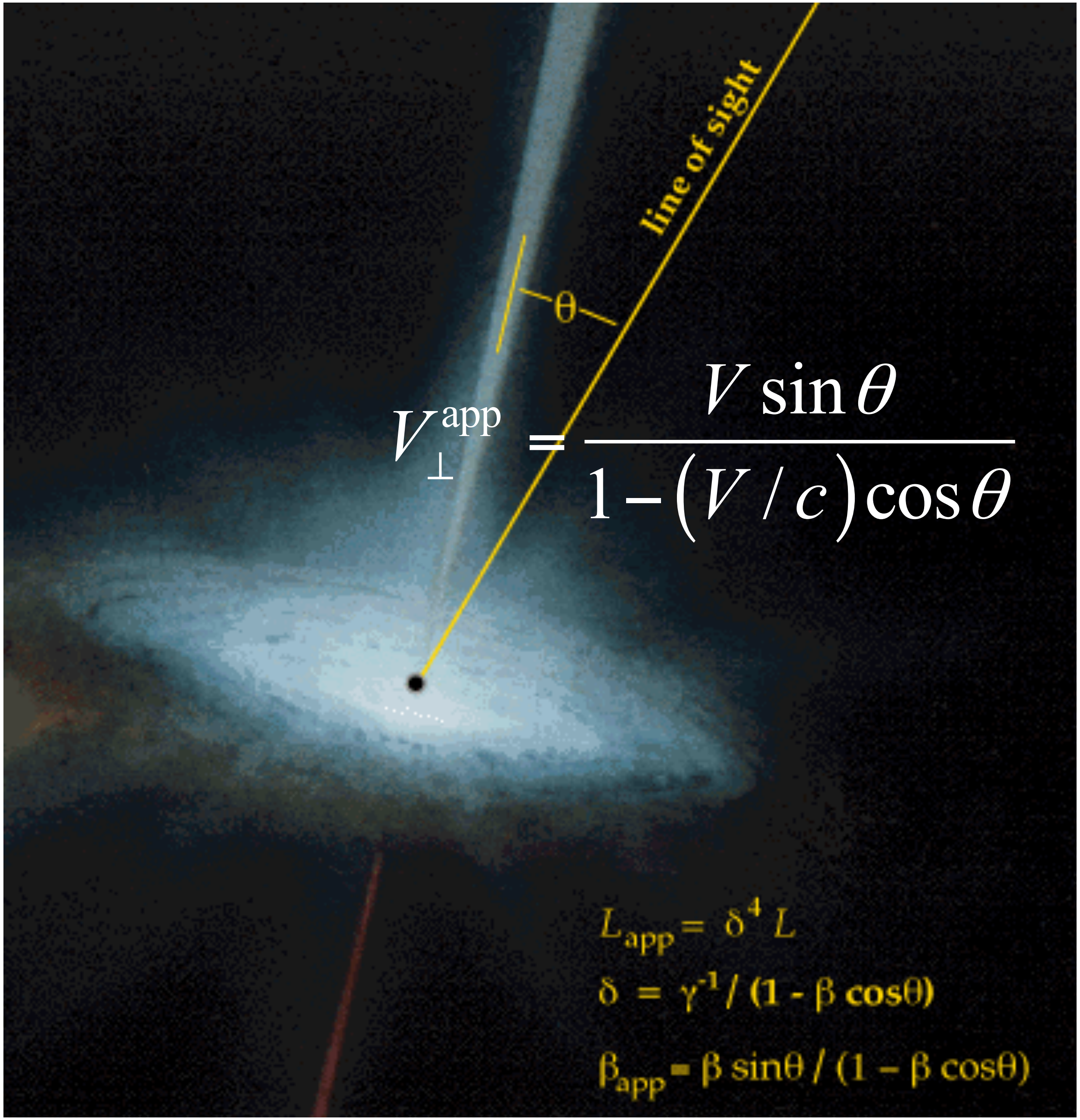
In the case of micro-quasars and powerful radio galaxies, the flow speeds are estimated to be close to the light speed

The consequence is that the apparent speed on the celestial sphere can be greater than c!

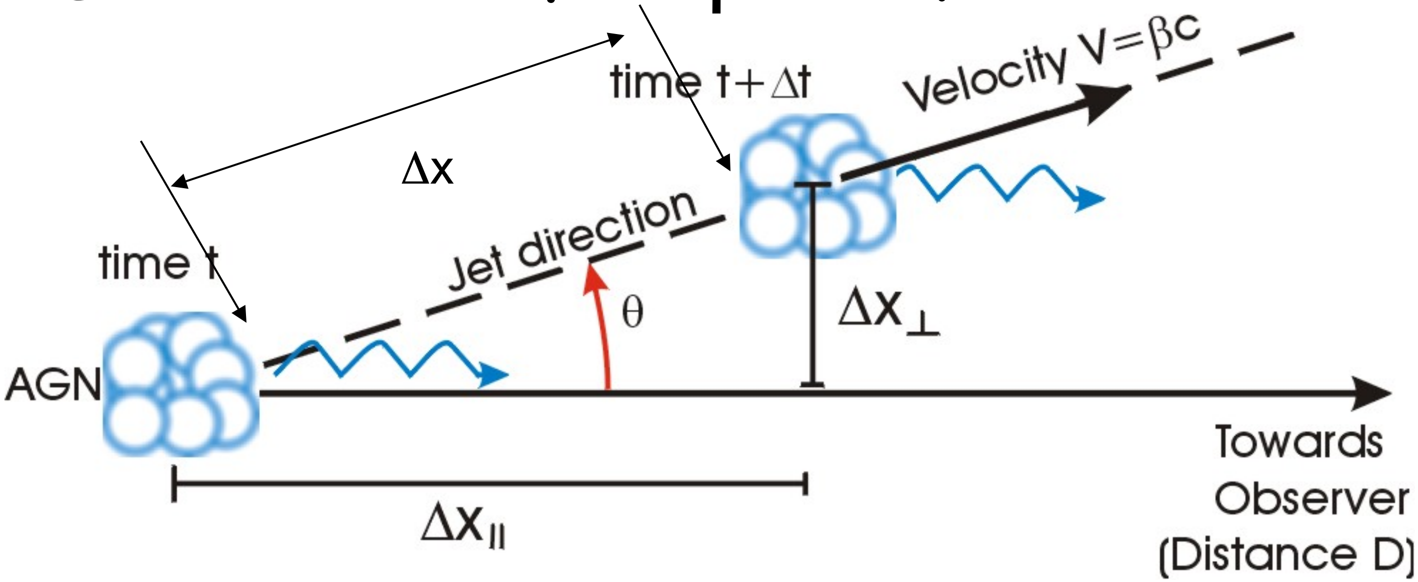


Observational clue:

Superluminal Motion: a relativistic illusion



Derivation of Superluminal Motion

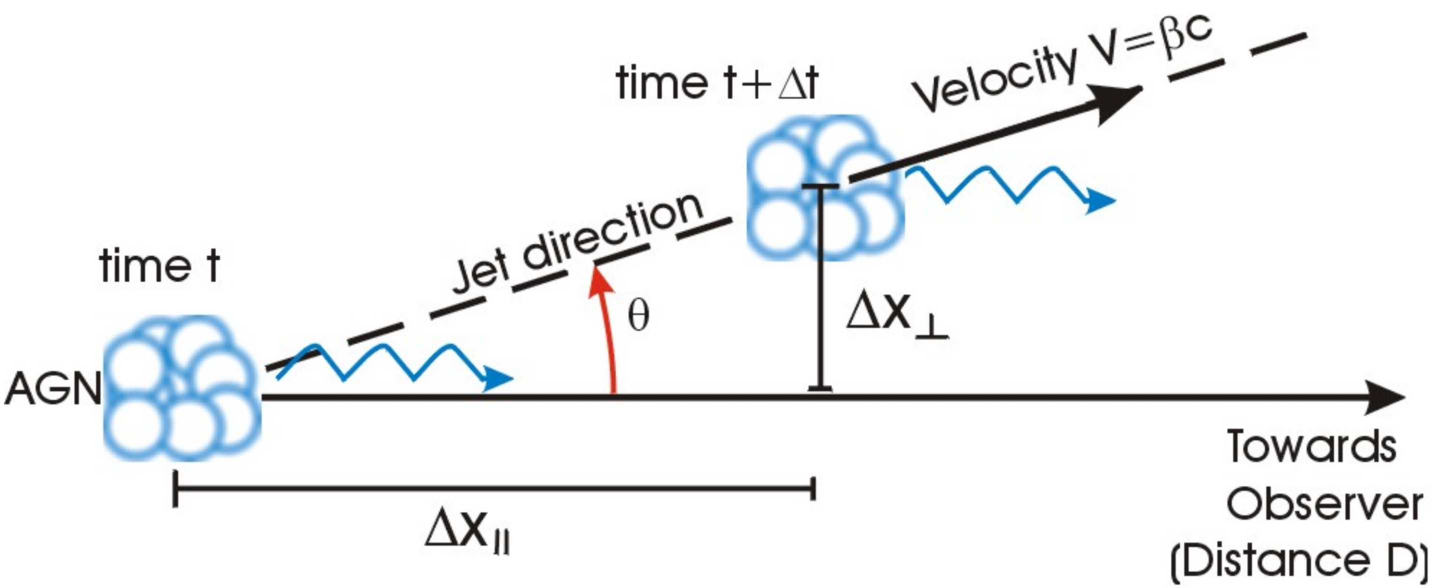


Let the source starts to emit at t → an observer on Earth receives the wave packet after a time $t_1 = t + D/c$

Let the source stop the emission after a time Δt, as measured at the source→ the observer receives the photon after a time $t_2 = t + \Delta t + (D - \Delta x)/c$, being Δx the distance covered in Δt by the emitting blob

The observer at Earth measures a time duration of the emission of

$$t_2 - t_1 = \Delta t - \Delta x_{\parallel}/c$$

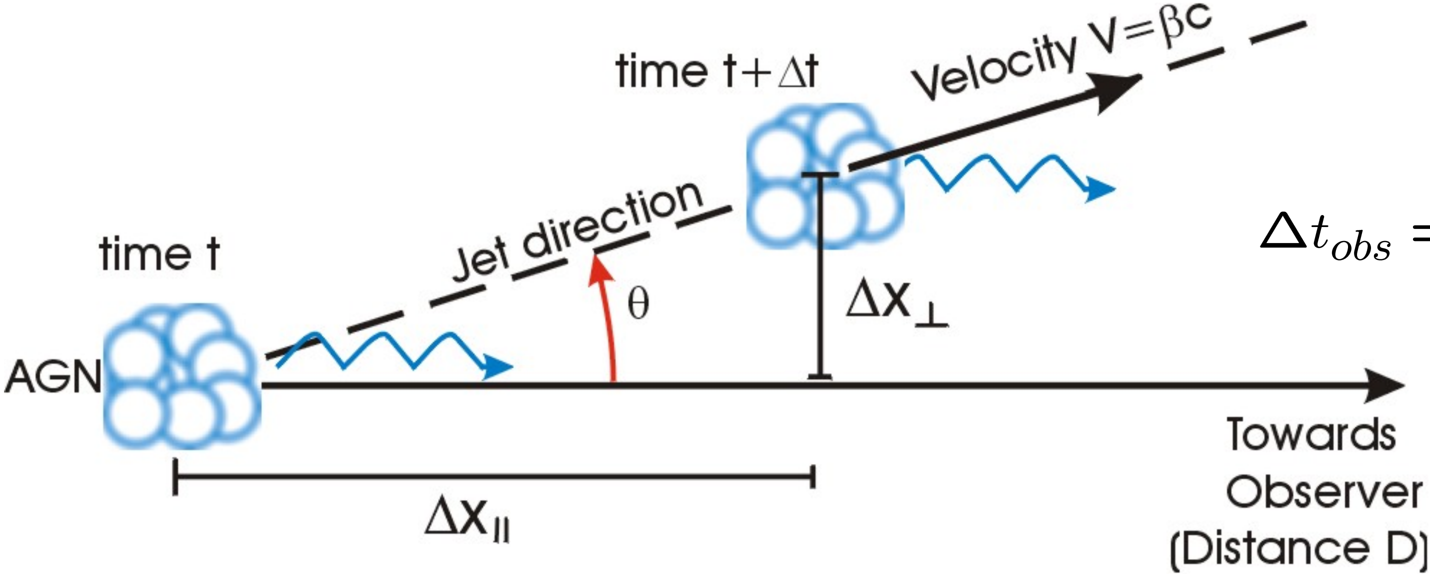


$$\Delta t_{obs} = t_2 - t_1 = \Delta t - \Delta x_{||}/c \qquad \Delta x_{||} = v\Delta t cos\theta = \beta c \Delta t cos\theta$$

$$\Delta t_{obs} = \Delta t(1 - \beta cos\theta)$$

If β =v/c ~ 1, the source "almost" catches up the emitted light, as a consequence the duration of emission measured at Earth is shorter than the duration at source (this is a consequence of the relativity of simultaneity due to the fact that the 2 observers are in different places→ the observer at rest in the source and at earth measure different durations)

$\Delta t_{obs} = \Delta t_{source}$ only if c = inif (as in newtonian mechanics)



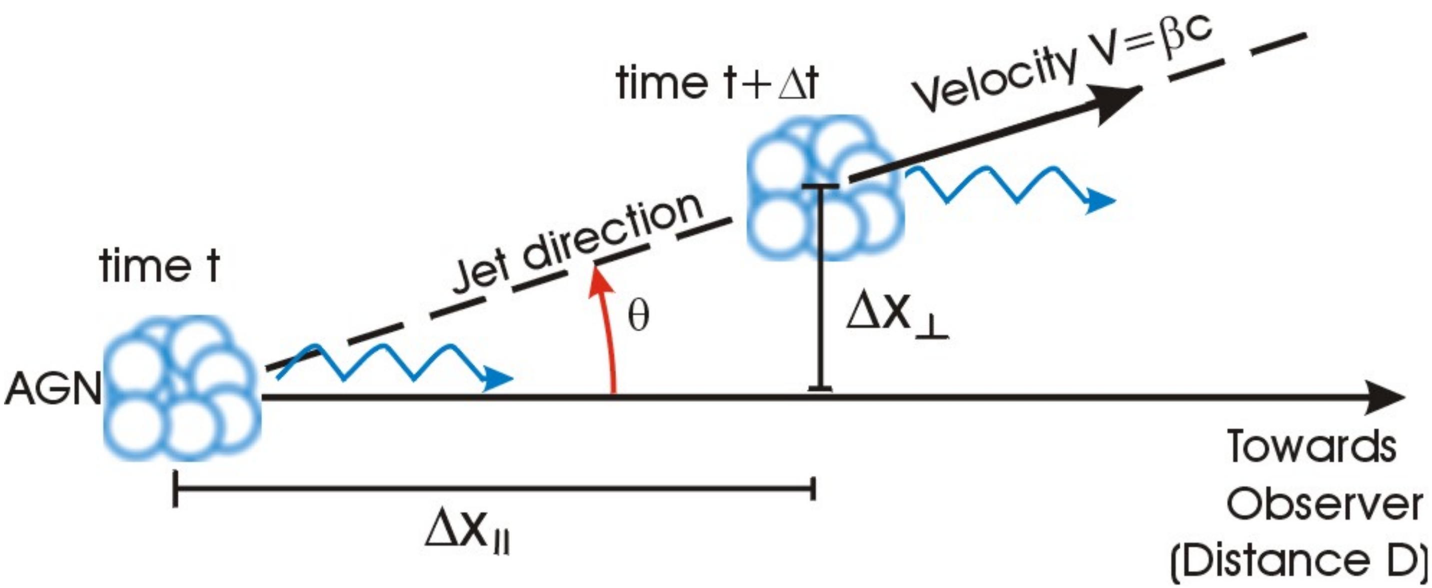
$$\Delta t_{obs} = \Delta t(1 - \beta cos\theta)$$

What we measure is the projection on the celestial sphere of the source motion, or more precisely the motion component orthogonal to the sight line, Δx_n

$$\Delta x_{\perp} = v\Delta t sin\theta = \beta c \Delta t sin\theta$$

The measured apparent speed from Earth is then

$$v_{app} = \Delta x_{\perp}/\Delta t_{obs} = \beta c \Delta t sin\theta/\Delta t(1 - \beta cos\theta) = \beta c sin\theta/(1 - \beta cos\theta)$$



$$v_{app} = \beta c sin\theta/(1 - \beta cos\theta)$$

It is easy to show that v_{app} has a maximum when

$$dv_{app}/d\theta = (cos\theta - \beta)/(1 - \beta cos\theta)^2 = 0$$

This occurs when cosθ = β

At maximum the apparent speed is $v_{app}^{max} = \beta c(1 - \beta^2)^{1/2}/(1 - \beta^2)$

$$v_{app}^{max} = \beta c \gamma \qquad \gamma = (1-\beta^2)^{1/2}$$

It is clear that for β~1 (that is relativistic source motion) γ>>1 and therefore v_{app} > c