Bremsstrahlung, Synchrotron Radiation, and Compton Scattering of High-Energy Electrons Traversing Dilute Gases

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Expressions are derived for the total energy loss and photon-production spectrum by the processes of Compton scattering, bremsstrahlung, and synchrotron radiation from highly relativistic electrons. For Compton scattering, the general case, the Thomson limit, and the extreme Klein–Nishina limit are considered. Bremsstrahlung is treated for the cases where the electron is scattered by a pure Coulomb field and by an atom. For the latter case the effects of shielding are discussed extensively. The synchrotron spectrum is derived for an electron moving in a circular orbit perpendicular to the magnetic field and also for the general case where the electron's motion is helical. The total photon-production spectrum is derived for each process when there is a power-law distribution of electron energies. The problems of the effects of the three processes on the electron distribution itself are considered. It is shown that if the electron loses a small fraction of its energy in a single occurrence of a process, the electron distribution function satisfies a continuity equation which is a differential equation in energy space. For the more general case where the electron can lose energy in discrete amounts (as in bremsstrahlung and extreme Klein–Nishina Compton losses), the electron distribution function satisfies an integro-differential equation. Some approximate solutions to this equation are derived for certain special cases.

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1. INTRODUCTION

In a number of areas of astrophysics, for example in considerations of models of cosmic radio sources, problems involving the interaction of a highly relativistic electron with its surrounding medium are common. This "medium" is usually a low-density partially ionized gas with a cosmic element abundance (consisting mostly of hydrogen and helium) which is permeated by a radiation field and a magnetic field. The electron interacts with this medium by means of essentially four processes: (i) by making elastic and inelastic collisions with the atoms and ions of the gas, (ii) by emitting a bremsstrahlung photon during these same scatterings, (iii) by undergoing Compton scatterings with the photons of the radiation field, (iv) by being deflected by the magnetic field, emitting synchrotron radiation or "magnetic bremsstrahlung" in the process. The first process (i) is important only at low energies $\gamma_e = E_e/mc^2 \lesssim 1000$ (cf. GB67) and will not be considered in this review. There are two reasons for treating the other three processes together in a single review. First, all three are photon-producing processes[1] and can therefore be directly responsible for gaining information about the interaction of the electrons with the medium through the detection of these photons. Second, each process is essentially a special case of one basic process; this process is Compton scattering. Bremsstrahlung [process (ii)] can be considered as Compton scattering of the virtual photons of the Coulomb fields of the particles in the scattering system;

[1] The pure Compton-scattering process (iii) does not produce a new photon. However, in collision with a highly relativistic electron, a low-energy photon from the radiation field has its energy increased by a large factor, so that a new *high-energy* photon is produced.

synchrotron radiation [process (iv)] can be viewed as Compton scattering of the virtual photons of the static magnetic field.

In this review we shall confine ourselves mainly to the *physics* of the three processes. However, in working out the details we have tried to present the final results in a form useful in applications, especially in applications to the general area of high-energy astrophysics (including cosmic-ray physics). As we have mentioned above, an obvious example of a type of problem where these processes come into play is the cosmic radio source. It is generally accepted that the radiation from these objects is produced by the synchrotron process. Since we know that there must be high-energy electrons in these sources, the other two processes, Compton scattering and bremsstrahlung, must also come into play due to the presence of a photon gas and a matter gas (consisting mostly of partially ionized hydrogen and helium) in the same volume. Recently, much of the effort in radio and optical astronomy has been devoted to the quasistellar sources. These compact objects may, in fact, be the best example of a natural configuration wherein all three processes are important. Another area of recent development in astrophysics where the processes are believed to be important is x-ray and γ-ray astronomy. For example, one popular idea for the origin of the diffuse isotropic background of cosmic x rays is that they are due to Compton scattering of high-energy electrons by the recently discovered cosmic blackbody radiation. Also, models of discrete x-ray sources are often very similar to those for radio sources and, in fact, some x-ray sources, for example, the Crab Nebula and M87, are also strong radio sources.

For the reader interested in applications of the results derived here to problems in astrophysics we can recommend several books in particular. In the field of general radio astronomy the fine work by Shklovsky (S60) is still very useful. For the very special quasistellar objects, we are fortunate to have the excellent, fairly up-to-date (at this time) work by Burbidge and Burbidge (BB67). A useful reference for cosmic-ray phenomena is the book by Ginzburg and Syrovatsky (GS64a). Two recent reviews on x-ray astronomy are by Gould (G67) and Morrison (M67b).

Other reviews have been written on the basic physics of bremsstrahlung, synchrotron radiation, and Compton scattering, but with a different point of view. An oft-cited review treating (among other subjects) bremsstrahlung is that of Bethe and Ashkin (BA53) which, however, is concerned primarily with bremsstrahlung when the target scatterer is a heavy (*high*-Z) atom. Our review treats only the case of *low*-Z atoms, namely the cosmically abundant species H and He. Many reviews of synchrotron radiation have been written, and for this reason our treatment has been fairly brief, focusing mainly on some special difficulties which we have tried to clarify. A good recent review is that of Ginzburg and Syrovatsky (GS69). No review of Compton scattering resembling our treatment has been presented before; however, Felten and Morrison's paper (FM66) is a common reference giving a special application. Finally, we should like to give some references here for the basic cross sections derived from quantum electrodynamics. For these the reader is referred to the books by Heitler (H54) and Jauch and Rohrlich (JR55) and the article by Olsen (O68). Throughout this review we shall refer again to many of the articles we have just cited above.

Although we have tried to produce *practical* results in this review, we have also made every attempt to increase *understanding* of the basic physical phenomena. Thus, in this review we have outlined a number of alternative derivations of particular basic results. Often, to simplify the derivations, use is made of the invariance of certain factors or the covariance of certain equations. Simple arguments of symmetry are also frequently employed. The three processes of bremsstrahlung, Compton scattering, and synchrotron radiation are really excellent examples of applications of classical, semiclassical, and quantum electrodynamics.

In the last major section of this paper, we discuss the energy distribution function of the collection of high-energy electrons emitting radiation by these three mechanisms. It will be seen that not only is this function crucial in determining the total radiation spectrum, but also that this distribution function is in turn dependent upon the processes causing the electrons to lose energy. This distribution function satisfies an integro-differential equation in the independent variable, the particle energy. Although some articles (FM66) use a differential equation to obtain an approximate solution, an integro-differential equation is really necessary essentially because for the bremsstrahlung process and the Compton process at high energy the electron can lose a large fraction of its energy in one occurrence of the process. Because of the nature of the problem, this last section is somewhat mathematical. We indicate some approximate solutions to this equation which are valid in certain special cases.

2. COMPTON SCATTERING

The effects of Compton scattering during the passage of a high-energy electron through a photon gas have been treated by a number of authors with a view toward astrophysical applications (FP48, D51, HOTY64, GS64, FM66, GB67, G67, BSL67). The general problem is the following. We have a photon gas with a differential number density $dn = n(\epsilon, \mathbf{i}_\Omega)\, d\epsilon d\Omega$ = number of photons per cm^3 with energies within $d\epsilon$ moving in the direction defined by the unit vector $\mathbf{i}_\Omega$ and the element of solid angle $d\Omega$. An electron of energy γmc^2 moves through the gas in some direction and undergoes Compton scattering, its energy being reduced in the process. We ask: what is the distribution in energy (designated ϵ_1) and solid angle (Ω_1) of the *scattered* photons? In a more general case we might have a distribution of electrons $dn_e = n_e(E_e, \mathbf{i}_{\Omega_e})\, dE_e d\Omega_e$ passing through the photon gas and could ask for the total spectrum of Compton photons, scattered per unit volume and time $dN_1/dtdVd\epsilon_1 d\Omega_1$. The solution for this problem in the completely general case is very complicated and has not even been attempted. However, a number of simplifications result in certain limiting cases which, in fact, correspond to conditions in some problems in astrophysics. In particular, we shall consider the interactions of highly relativistic electrons, $\gamma \gg 1$. A further simplification results when the energy of the photon before scattering in the electron rest frame (ϵ_0') is much less than mc^2; this corresponds to the Thomson limit in which the Compton cross section is independent of the energy of the incoming photon. The opposite case ($\epsilon_0' \gg mc^2$) corresponds to the extreme Klein–Nishina limit in which the Compton cross section can again be approximated by a convenient expression.

The simplest problem is the calculation of the electron total energy-loss rate which, in the Thomson limit, is related in a simple way to the total energy density of the photon gas. We treat this problem first in this section and then outline the derivation of the expressions for the Compton-scattered photon spectra in the general case and in the various limiting cases.

2.1 A Useful Invariant

The ratio, dn/ϵ, where dn represents a differential photon number density[2] can very easily be shown to be an invariant. In terms of the differential *number* of particles dN (an invariant), the three-dimensional spatial volume element dV, and the four-dimensional invariant volume $dX = dx_0 dx_1 dx_2 dx_3 = dx_0 dV$,

$$dn = dN/dV = (dN/dX)\, dx_0. \tag{2.1}$$

Thus dn transforms as the time component (x_0) of the photon position four-vector. Since the photon four-momentum p_μ and position x_μ are "parallel" four-vectors in that their spatial components are related to their time components in the same way (that is, $x_i/x_0 = p_i/p_0$), the ratio $dx_0/p_0 = (\sum a_{0\mu}' dx_\mu')/(\sum a_{0\mu}' p_\mu') = dx_0'/p_0'$. Then, since dN/dX is invariant, we have in terms of photon energy $p_0 = \epsilon$,

$$dn/\epsilon = \text{invariant}. \tag{2.2}$$

This result has been indicated in other papers[3] (FP48, FM66), but the above derivation seems simpler.

[2] The differential number density may represent, for example, the total number density within $d\epsilon$, or the number density within $d\epsilon$ and within the solid angle $d\Omega$ defining the direction of the photon momenta, or the total number of particles moving within $d\Omega$.

[3] The other method of deriving (2.2) is to consider the transformation properties of $c\epsilon dn$ which is the differential energy flux or Poynting vector dS, and express $dS(\propto A^2)$ in terms of the amplitude A of the associated electric and magnetic fields. One finds $A/A' = \epsilon/\epsilon'$ for a plane wave moving in any arbitrary direction making an angle θ with the axis of relative motion.

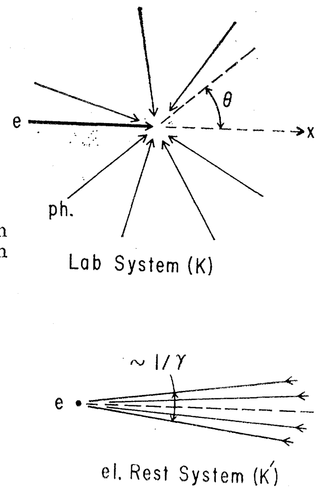

FIG. 1. Electron–photon collisions in the lab system and electron rest system.

A further result can be given, since the ratio $d^3\mathbf{p}/\epsilon$ is an invariant ($d^3\mathbf{p}$ is the three-dimensional momentum volume element). Taking the ratio of (2.2) and this invariant yields

$$dn/d^3\mathbf{p} = \text{invariant}. \tag{2.3}$$

2.2 Relativistic Kinematics in Compton Scattering

Consider a highly relativistic electron moving through a photon gas in the direction of, say, the x axis of a coordinate frame in the lab system (K). The electron suffers Compton collisions with photons moving at various angles θ with respect to the x axis (see Fig. 1). In the electron's rest[4] frame the corresponding angle θ' is given by

$$\tan\theta' = \sin\theta/\gamma(\cos\theta - \beta), \tag{2.4}$$

where γ and βc are the Lorentz factor and velocity of the electron in the lab frame. When $\gamma \gg 1$, then $\beta \to \approx 1 - \frac{1}{2}\gamma^{-2}$, and for all except those photons moving practically along the x axis in K, θ' is very small. In fact, as $\beta \to 1$ in (2.4)

$$\tan\theta' \to -\gamma^{-1} \cot(\theta/2). \tag{2.5}$$

Thus in the electron rest frame K' the photons are incident in a narrow cone in the direction of the negative x' axis. Moreover, the photon energy in K' is

$$\epsilon' = \gamma\epsilon(1 - \beta\cos\theta) \tag{2.6}$$

and so varies (for given ϵ) from $\epsilon_{\min}' \approx \epsilon/2\gamma$ for $\theta = 0$ to $\epsilon_{\max}' \approx 2\gamma\epsilon$ for $\theta = \pi$. Thus, the photons with θ near 0 are, in K', *soft* photons which produce only very small recoils of the electron in the Compton scattering, and are therefore unimportant.

In scattering off the electron in K' the photon goes off at an energy ϵ_1' and scattering angle θ_1' (see Fig. 2). The energy ϵ_1' after scattering is given by the well-

[4] At rest before an individual photon gives it a recoil during the scattering.

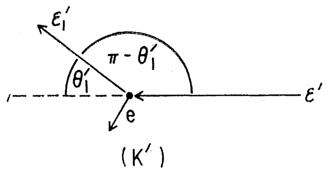

FIG. 2. Scattering angle in the electron rest system.

known relation

$$\epsilon_1' = \frac{\epsilon'}{1+(\epsilon'/mc^2)(1-\cos\theta_1')}. \quad (2.7)$$

In the lab system this energy is

$$\epsilon_1 = \gamma\epsilon_1'[1+\beta\cos(\pi-\theta_1')] \approx \gamma\epsilon_1'(1-\cos\theta_1'), \quad (2.8)$$

and we have $\epsilon_{1\,\max} \approx 2\gamma\epsilon_1'$. Now *in the Thomson limit* $\epsilon' \ll mc^2$, and so by (2.7), $\epsilon_1' \approx \epsilon'$; in this case the electron is given a very small recoil in the scattering. Then

$$\epsilon_{1\,\max} \approx 2\gamma\epsilon_{1\,\max}' \approx 4\gamma^2\epsilon, \quad (2.9)$$

so that the maximum energy of the scattered photon is greater than the initial energy by the large factor $4\gamma^2$. The maximum corresponds to a head-on collision of the electron and photon. Instead of deriving the result (2.9) by considering two Lorentz transformations, one could also proceed by more elementary means, applying momentum and energy conservation to the head-on collision of a photon and an electron of energy $\gg mc^2$.

Although in the Thomson limit the characteristic energy ($\sim\gamma^2\epsilon$) of the scattered photon is very large, it is still small compared with the electron energy, so the electron loses a small fraction of its energy in each Compton scattering. This is *not* true in the extreme Klein–Nishina limit where the scattered photon carries away a large fraction of the electron energy. Thus in the K–N limit the electron does not lose its energy continuously. We shall treat this limiting case and the effects of the discrete energy losses later on in this review.

2.3 Total Scattering Rate

Expressions for the number of Compton scatterings per unit time per electron can be gotten quite readily in the general case (including the Thomson or Klein–Nishina limits), since here only the total cross section is involved. This simple case also illustrates the use the relativistic invariant dn/ϵ and transformations to the electron's rest frame (K'). The expression for the total scattering rate is most readily written down in terms of the rate in K'. In K' immediately before a scattering the electron is at rest; time intervals are related by $dt = \gamma dt'$, and the scattering rate is

$$dN/dt = \gamma^{-1}\, dN'/dt' = \gamma^{-1} c \int \sigma\, dn'. \quad (2.10)$$

The integration is over the number density of photons in K'; σ is the total Compton cross section. However, using the invariant (2.2) and the energy transformation (2.6), we have

$$dN/dt = c\int \sigma(1-\beta\cos\theta)\, dn. \quad (2.11)$$

At this point we might remark on the meaning of the factor $c(1-\beta\cos\theta)$; it is just the relative velocity of the photon and electron along the direction of the latter's motion. Since the electron is the particle for which we compute the collision rate, it is clear that such a factor must come in if we consider the problem only in the lab frame.[5]

In the Thomson limit and where the photon distribution is isotropic, the $\cos\theta$ term in (2.11) integrates to zero and we have, very simply

$$dN/dt = \sigma_T c n \quad (\text{for } dn \text{ isotropic}). \quad (2.12)$$

In many astrophysical problems, the photon gas *is* isotropic, but in a number of cases already considered it is not. Compton scattering has been computed in models of discrete sources such as the Crab Nebula (G65, M67, RW69) which do have anisotropic spectra, and in these treatments the anisotropy effects (such as the factor $1-\beta\cos\theta$) have been ignored. For the case where the electron distribution is isotropic even though the photon distribution is not, averaging over electron directions eliminates the $\cos\theta$ term in (2.11). However, usually the Compton *spectrum* of scattered photons rather than the total scattering rate is computed, and in this case anisotropy effects do indeed come in.

2.4 Total Energy-Loss Rate—Thomson Limit

In the Thomson limit we can make use of the fact that the energy of the scattered photon in the lab frame is much larger than its energy before scattering. Then we can write for the electron energy-loss rate

$$-dE_e/dt = dE_1/dt, \quad (2.13)$$

where E_1 is the energy of the scattered radiation. But dE_1/dt is an invariant since it is the ratio of the same components of two parallel four-vectors. Since in the Thomson limit

$$\epsilon_1' = \epsilon' = \text{photon energy in } K' \text{ before scattering},$$

we have

$$-dE_e/dt = dE_1'/dt' = \int \sigma_T c\epsilon'\, dn' = \sigma_T c\mathcal{E}', \quad (2.14)$$

where $\sigma_T = (8\pi/3)r_0^2$ is the Thomson cross section and $\mathcal{E}'$ is the total photon energy density in the electron's rest frame K'. Here $\mathcal{E}'$ is related to the energy density in K; by (2.2) and (2.6) with $\beta \rightarrow 1$

$$\mathcal{E}' = \int \epsilon'^2 (dn'/\epsilon') = \gamma^2 \int (1-\cos\theta)^2 \epsilon\, dn. \quad (2.15)$$

For an isotropic distribution for dn, averaging over

[5] In collisions with two massless particles, for example the collision between two photons, one cannot, of course, go to a particle rest frame [see, for example, (GS67)].

angles,

$$\langle(1-\cos\theta)^2\rangle_{\rm iso}=\tfrac{4}{3}, \tag{2.16}$$

and

$$\mathcal{E}'=\tfrac{4}{3}\gamma^2\int\epsilon\, dn_{\rm iso}=\tfrac{4}{3}\gamma^2\mathcal{E}_{\rm iso}, \tag{2.17}$$

$$-dE_e/dt=\tfrac{4}{3}\sigma_T c\gamma^2\mathcal{E}_{\rm iso}. \tag{2.18}$$

Thus the total energy loss in the Thomson limit for an isotropic photon gas can be derived very easily, in fact without evaluating any integrals. One might also note that only the total cross section comes in; the angular dependence of the scattering does not enter. Actually the result has another application. As we have mentioned earlier, synchrotron radiation can be considered as Compton scattering of the virtual photons of the static magnetic field. We can then carry over the results (2.14) and (2.18) to the synchrotron problem. However we must verify that the Thomson limit is applicable. The characteristic virtual photon energy in the electron rest frame is $\sim\hbar\omega_c'$, where $\omega_c'=eB'/mc\sim\gamma eB/mc$ is the cyclotron frequency. This is the frequency of variation of the fields in the electron's frame (which is not an inertial frame). In an inertial frame in which the electron is instantaneously at rest or moving with a nonrelativistic velocity, ω_c' would be the frequency of its cycloidal motion. Then the energy of the synchrotron photon in the lab frame would be, as in Compton scattering of "real" photons, [see (2.8, 2.9)]

$$\epsilon_s\sim\gamma\hbar\omega_c'\sim\gamma^2 eB/mc. \tag{2.19}$$

For the validity of the Thomson-scattering approximation we must have

$$\gamma\hbar\omega_c\ll mc^2. \tag{2.20}$$

For all but extremely high-energy electrons this relation is satisfied. In the derivation of (2.18) an assumption of *isotropy* was made. For the synchrotron application the angle averaging analogous to (2.16) would be an averaging over random directions of the electron motion with respect to the direction of the magnetic field **B**. Then, having already performed this averaging, we can write in analogy with (2.18) for Compton scattering in the Thomson limit,

$$-(dE_e/dt)_s=\tfrac{4}{3}\sigma_T c\gamma^2\mathcal{E}_B, \tag{2.21}$$

where the magnetic-field energy density is

$$\mathcal{E}_B=\langle B^2\rangle/8\pi, \tag{2.22}$$

and the average is of the mean-squared magnitude of **B**. Note that (2.21) would fail (and, in fact, be too large) at very high energies when (2.20) would fail to be satisfied.

The mean energy of the Compton-scattered photon $\langle\epsilon_1\rangle$ in the Thomson limit can be found readily by combining the results (2.12) and (2.18) since

$$-dE_e/dt=\langle\epsilon_1\rangle dN/dt. \tag{2.23}$$

This yields, for an isotropic photon gas,

$$\langle\epsilon_1\rangle=\tfrac{4}{3}\gamma^2\langle\epsilon\rangle, \tag{2.24}$$

where $\langle\epsilon\rangle$ is the mean energy of the photon gas (before scattering). For example, for a gas with a blackbody spectrum

$$\langle\epsilon\rangle=[3\zeta(4)/\zeta(3)]kT=2.70kT, \tag{2.25}$$

where the ζ's are Riemann ζ functions.

2.5 Corrections in the Thomson Limit

Two basic approximations have been made in deriving the result (2.18): (i) the scattered photon energy is assumed much larger than the energy before scattering; (ii) the Thomson limit of the Compton cross section has been employed. Approximation (i) is better than (ii), since, as we have seen, $\epsilon_1\sim\gamma^2\epsilon$, while the relative correction to the Thomson cross section is of a lower order. Another approximation made [implied essentially by (ii)] is that the energy ϵ' before scattering in the electron's rest frame is equal to the energy of the scattered photon (ϵ_1'); the relative error made here is *first order* in $\gamma\epsilon/mc^2$. Therefore the correction to the Thomson-limit energy loss cannot be computed as easily as the limiting expression (2.18). In fact, it is convenient to write the basic expression for the energy loss in terms of lab system quantities, although the (invariant) differential cross section is most conveniently expressed in terms of electron-rest-system variables. Neglecting ϵ compared with ϵ_1, our basic energy-loss expression would be

$$-dE_e/dt=\int\int\epsilon_1 c(1-\cos\theta)\, dn d\sigma; \tag{2.26}$$

here the factor $c(1-\cos\theta)$ is just the relative velocity of photon and electron along the direction of the latter (see Sec. 2.3). The exact Compton cross section (Klein–Nishina formula, cf. JR55) and its approximate form in the Thomson limit are given by[6]

$$\frac{d\sigma_{\rm exact}}{d\Omega_1'd\epsilon_1'}=\tfrac{1}{2}r_0^2\left(\frac{\epsilon_1'}{\epsilon'}\right)^2\left(\frac{\epsilon'}{\epsilon_1'}+\frac{\epsilon_1'}{\epsilon'}-\sin^2\theta_1'\right)\delta\left(\epsilon_1'-\frac{\epsilon'}{1+(\epsilon'/mc^2)(1-\cos\theta_1')}\right)$$

$$\approx\tfrac{1}{2}r_0^2(1+\cos^2\theta_1')\left(1-\frac{2\epsilon'}{mc^2}(1-\cos\theta_1')\right)\delta\left[\epsilon_1'-\epsilon'\left(1-\frac{\epsilon'}{mc^2}(1-\cos\theta_1')\right)\right]$$

$$\sigma_{\rm tot}\to\sigma_T\left(1-\frac{2\epsilon'}{mc^2}+\cdots\right). \tag{2.27}$$

[6] An average over initial photon polarizations, and a sum over final photon polarizations has been performed.

Substituting $\epsilon_1=\epsilon_1'\gamma(1-\cos\theta_1')$ and the approximate form of $d\sigma$ from (2.27) into (2.26), integrating over $d\Omega_1'$, and averaging over angles for the initial photon distribution (assumed isotropic), we get

$$-dE_e/dt=\tfrac{4}{3}\sigma_T c\gamma^2\mathcal{E}[1-\tfrac{63}{10}(\gamma\langle\epsilon^2\rangle/mc^2\langle\epsilon\rangle)+\cdots] \quad (2.28)$$

in terms of the mean and mean-squared photon densities. For a blackbody photon gas the correction term in the parentheses of (2.28) has a large coefficient; the term is $24.15\gamma kT/mc^2$.

2.6 Compton Spectrum in the Thomson Limit

Some of the results derived in this section have been obtained previously by a number of authors. We shall follow most closely the treatment of Jones (J68). The final results will be given for the case where the photon distribution before scattering is isotropic. However, it will be clear how the formulation should be generalized to include the anisotropic case. We derive in detail the results for the spectrum of Compton-scattered photons in the Thomson limit. The derivation in the more general case follows along precisely the same lines except that the Klein–Nishina cross section must be used. We proceed by computing the Compton spectrum produced by a high-energy electron of energy γmc^2 scattering off a *segment* of the initial photon distribution having energies within $d\epsilon$. The total Compton spectrum would then be obtained by integrating over ϵ and over the distribution of electron energies.

Again, it is convenient to consider the process in the rest system of the electron. As we have already seen (Sec. 2.2), in this system the photons isotropic in the lab system are incident on the electron essentially in a parallel beam. The basic problem is the determination of the spectrum of these photons in the beam. Let $x\equiv\cos\theta$, where θ is the angle the photon's velocity makes with x axis along which the electron moves (see Fig. 1); then $-1\leq x\leq+1$, and the differential photon density in the lab system K is, for an isotropic distribution,

$$dn=n(\epsilon,x)\,d\epsilon dx=\tfrac{1}{2}n(\epsilon)\,d\epsilon dx. \quad (2.29)$$

Here $n(\epsilon)\,d\epsilon$ is the total differential density (integrated over x). Then, by the invariance of dn/ϵ,

$$\tfrac{1}{2}\epsilon^{-1}n(\epsilon)\,d\epsilon dx=\epsilon'^{-1}dn'(\epsilon';\epsilon)\,d\epsilon', \quad (2.30)$$

where $dn'(\epsilon';\epsilon)d\epsilon'$ represents the total differential photon density in the beam in K' (that is, integrated over all the small angles in the beam) within $d\epsilon'$ which are due to photons within $d\epsilon$ in K. By (2.6), $|d\epsilon'/dx|=\gamma\beta\epsilon\rightarrow\gamma\epsilon$, so that

$$dn'(\epsilon';\epsilon)=n(\epsilon)\,(\epsilon'/2\epsilon^2\gamma)\,S(\epsilon';\epsilon/2\gamma,2\gamma\epsilon)\,d\epsilon, \quad (2.31)$$

where we have inserted a *step function* to designate the range of ϵ':

$$S(z;a,b)=1 \qquad a<z<b$$
$$=0 \qquad \text{otherwise.} \quad (2.32)$$

The distribution for the photon density is linear[7] in ϵ'.

In K' the distribution in energy and angle of the scattered photons is, per electron per interval of ϵ'

$$dN_{\gamma,\epsilon}/dt'd\epsilon'd\Omega_1'd\epsilon_1'=dn'(\epsilon';\epsilon)\,c(d\sigma/d\Omega_1'd\epsilon_1'), \quad (2.33)$$

where

$$d\sigma/d\Omega_1'd\epsilon_1'\rightarrow\tfrac{1}{2}r_0^2(1+\cos^2\theta_1')\,\delta(\epsilon_1'-\epsilon') \quad (2.34)$$

in the Thomson limit[8] [compare the exact formula (2.27)]. We are interested in the energy distribution of the scattered photons in the lab frame. This is obtained from (2.33) by

$$\frac{dN_{\gamma,\epsilon}}{dtd\epsilon_1}=\iint_{(\epsilon,'\Omega_1')}\frac{dN_{\gamma,\epsilon}}{dt'd\epsilon'd\Omega_1'd\epsilon_1'}\frac{dt'}{dt}\frac{d\epsilon'd\Omega_1'd\epsilon_1'}{d\epsilon_1}, \quad (2.35)$$

and the variables which are eventually integrated over are indicated. However, it is convenient to integrate over, instead of Ω_1', the variable

$$\eta_1'=1-\beta\cos\theta_1'\approx1-\cos\theta_1', \quad (2.36)$$

since, as a result of (2.8)

$$\epsilon_1=\gamma\epsilon_1'\eta_1'. \quad (2.37)$$

Then, since $dt'/dt=1/\gamma$, $d\epsilon_1'/d\epsilon_1=1/\gamma\eta_1'$, and $d\Omega_1'=2\pi d\eta_1'$, the result (2.35) can, by substituting (2.33) and (2.34), be cast into the form

$$\frac{dN_{\gamma,\epsilon}}{dtd\epsilon_1}=\frac{\pi r_0^2c}{2\gamma^3}\frac{n(\epsilon)\,d\epsilon}{\epsilon^2}$$
$$\times\iint\frac{\epsilon'}{\eta_1'}(2-2\eta_1'+\eta_1'^2)S\,\delta\left(\frac{\epsilon_1}{\gamma\eta_1'}-\epsilon'\right)d\epsilon'd\eta_1'. \quad (2.38)$$

The integration over ϵ' is then performed using the δ function. Before integrating over η_1', the step function must be expressed in terms of η_1' instead of ϵ'. We then have

$$S(\epsilon';\epsilon/2\gamma,2\gamma\epsilon)\rightarrow S(\eta_1';\epsilon_1/2\gamma^2\epsilon,2\epsilon_1/\epsilon), \quad (2.39)$$

and the integral over η_1' is

$$\int\eta_1'^{-2}(2-2\eta_1'+\eta_1'^2)\,S\,d\eta_1'=(-2/\eta_1'-2\ln\eta_1'+\eta_1')_L{}^U, \quad (2.40)$$

where the limits are

$$U=\min(2,2\epsilon_1/\epsilon)=2; \qquad L=\max(0,\epsilon_1/2\gamma^2\epsilon). \quad (2.41)$$

[7] It is interesting to note that while this distribution is linear, if we were considering the transformed distribution of a *given fixed number* of photons isotropic in K, that distribution would be *flat*.

[8] This formula can be derived by purely classical electrodynamics (cf. J62).

Finally we have

$$\frac{dN_{\gamma,\epsilon}}{dtd\epsilon_1}=\frac{\pi r_0^2 c}{2\gamma^4}\frac{n(\epsilon)d\epsilon}{\epsilon^2}\left(2\epsilon_1\ln\frac{\epsilon_1}{4\gamma^2\epsilon}+\epsilon_1+4\gamma^2\epsilon-\frac{\epsilon_1^2}{2\gamma^2\epsilon}\right). \tag{2.42}$$

This result is perhaps better written by expressing the scattered photon energy in terms of its maximum value

$$\epsilon_1\equiv 4\epsilon\gamma^2\hat{\epsilon}_1. \tag{2.43}$$

Then

$$dN_{\gamma,\epsilon}/dtd\hat{\epsilon}_1=8\pi r_0^2 cn(\epsilon)d\epsilon f(\hat{\epsilon}_1), \tag{2.44}$$

where the distribution function is

$$f(\hat{\epsilon}_1)=2\hat{\epsilon}_1\ln\hat{\epsilon}_1+\hat{\epsilon}_1+1-2\hat{\epsilon}_1^2. \tag{2.45}$$

This distribution is plotted in Fig. 3; it has no peak and in fact has its maximum value at $\hat{\epsilon}_1=0$. Thus, the distribution is quite broad and favors the *low-energy* end. The moments of the distribution are

$$\int f(\hat{\epsilon}_1)\,d\hat{\epsilon}_1=\tfrac{1}{3},$$
$$\int \hat{\epsilon}_1 f(\hat{\epsilon}_1)\,d\hat{\epsilon}_1=\tfrac{1}{9}, \tag{2.46}$$

so that (2.12), (2.18), and (2.24) are checked.

2.7 General Case and Extreme Klein–Nishina Limit

The derivation of the spectrum of photons scattered by a high-energy electron from a segment of an isotropic photon gas of differential density $dn=n(\epsilon)d\epsilon$, for the general case of arbitrary $\gamma\epsilon$, follows along the lines of Sec. 2.6. The essential difference is that the exact Klein–Nishina formula (2.27) must be used for the Compton cross section. Here we shall merely quote the result, first obtained by Jones (J68).

In this general case the electron recoil is more important in that a large fraction of the electron energy can be lost in one Compton scattering. It is convenient to express the energy of the scattered photon in units of the initial electron energy, that is

$$\epsilon_1=\gamma mc^2 E_1. \tag{2.47}$$

Then the general result for the scattered photon spectrum per electron is

$$\frac{dN_{\gamma,\epsilon}}{dtdE_1}=\frac{2\pi r_0^2 mc^3}{\gamma}\frac{n(\epsilon)d\epsilon}{\epsilon}\times\left[2q\ln q+(1+2q)(1-q)+\frac{1}{2}\frac{(\Gamma_\epsilon q)^2}{1+\Gamma_\epsilon q}(1-q)\right], \tag{2.48}$$

where

$$\Gamma_\epsilon=4\epsilon\gamma/mc^2,\qquad q=E_1/\Gamma_\epsilon(1-E_1). \tag{2.49}$$

The dimensionless parameter Γ_ϵ determines the *domain* of the scattering; the Thomson limit corresponds to $\Gamma_\epsilon\ll 1$. In the Thomson limit also, $E_1\ll 1$ and the last

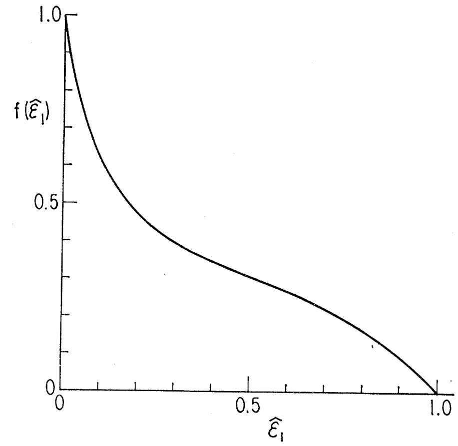

FIG. 3. Scattered photon distribution function in the Thomson limit.

term in the brackets in (2.48) is negligible, the whole expression reducing to (2.42). However, (2.48) is exact for any Γ_ϵ; the only assumption or restriction made in its derivation is that $\gamma\gg 1$.

The range of values of E_1, which follows purely from the kinematics of the problem, is

$$1\gg\epsilon/\gamma mc^2\leq E_1\leq\Gamma_\epsilon/(1+\Gamma_\epsilon). \tag{2.50}$$

The corresponding range for q is

$$1\gg 1/4\gamma^2\leq q\leq 1. \tag{2.51}$$

Further, as we did in the Thomson limit, we can express the scattered photon energy in terms of its maximum value

$$E_1=\Gamma_\epsilon(1+\Gamma_\epsilon)^{-1}\hat{E}_1, \tag{2.52}$$

and the range of $\hat{E}_1$ is essentially from 0 to 1. The spectral distribution of the scattered photons is contained in the expression in brackets in (2.48). Here Γ_ϵ is a parameter in this distribution which we denote as $F(\hat{E}_1;\Gamma_\epsilon)$. We also normalize the distribution

$$\int_0^1 F(\hat{E}_1;\Gamma_\epsilon)\,d\hat{E}_1=1. \tag{2.53}$$

The function $F(\hat{E}_1;\Gamma_\epsilon)$ is plotted in Fig. 4 for several values of the parameter Γ_ϵ. For all Γ_ϵ the distribution goes to zero at the maximum, $F(1,\Gamma_\epsilon)=0$; however, it has quite a different form for different values of Γ_ϵ. For $\Gamma_\epsilon\ll 1$ the distribution approaches the Thomson-limit curve of Fig. 3 (normalized) which is peaked at the low-energy end. In the extreme Klein–Nishina limit corresponding to $\Gamma_\epsilon\gg 1$ the distribution has a peak near the high-energy end; thus in this limit large energy losses in individual Compton scatterings are dominant. In fact,

$$F(\hat{E}_1;\Gamma_\epsilon)\xrightarrow[\Gamma_\epsilon\gg 1]{}(\ln\Gamma_\epsilon)^{-1}\left(1+\frac{1}{2}\frac{(\Gamma_\epsilon q)^2}{1+\Gamma_\epsilon q}(1-q)\right), \tag{2.54}$$

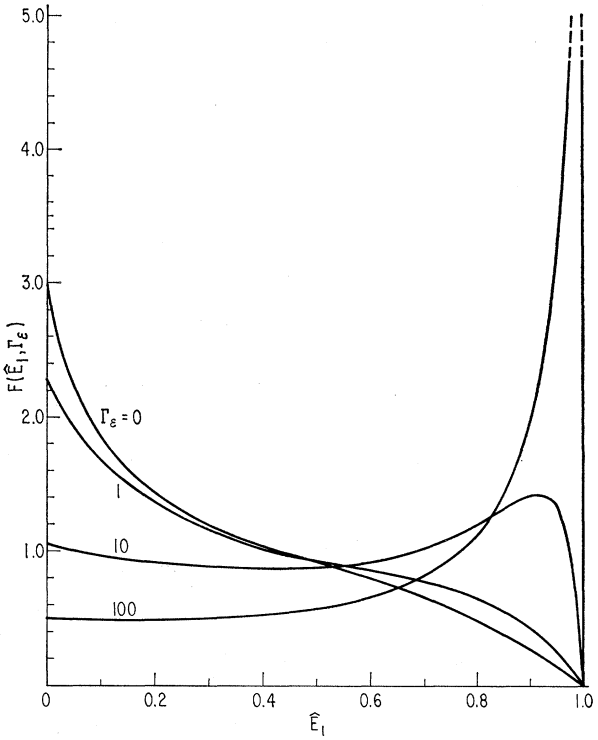

FIG. 4. Normalized scattered photon distribution function in the general case for $\Gamma_\epsilon=0, 1, 10, 100$.

and since $q\ll1$ except for $\hat{E}_1$ near 1, we have

$$F(\hat{E}_1 \text{ not near } 1;\, \Gamma_\epsilon\gg1)\rightarrow(\ln\Gamma_\epsilon)^{-1}[1-\hat{E}_1+(1-\hat{E}_1)^{-1}]. \tag{2.55}$$

In the above expressions the factor $\ln\Gamma_\epsilon$ arises from the normalization.

2.8 Total Energy-Loss Rate—Extreme Klein–Nishina Limit

We have already derived (Sec. 2.4) the simple expression for the total energy loss in the Thomson limit; in that case the total loss rate is proportional to γ^2 and to the total energy density of the photon gas. In the general case the total loss rate per electron would be computed from

$$-dE/dt=\int(\epsilon_1-\epsilon)(dN/dtd\epsilon_1)\,d\epsilon_1. \tag{2.56}$$

In (2.56), for all the cases of interest, ϵ can be neglected in comparison to ϵ_1. The distribution $dN/dtd\epsilon_1$ should be taken from (2.48) in the general case, and an integration over ϵ should also be performed. Jones (J68) has obtained an expression for dE/dt from (2.56) by integrating over ϵ_1; however, the formula is a little complicated, and, more important, no integration over ϵ (the initial photon spectrum) had been performed.

In the extreme Klein–Nishina limit a simple expression for dE/dt can be found by using the distribution (2.54) in the factor $dN/dtd\epsilon_1$ in (2.56). Integrating over ϵ_1, one readily finds

$$-\frac{dE}{dt}\xrightarrow[\Gamma_\epsilon\gg1]{}\pi r_0^2m^2c^5\int\frac{n(\epsilon)}{\epsilon}\left(\ln\frac{4\epsilon\gamma}{mc^2}-\frac{11}{6}\right)d\epsilon. \tag{2.57}$$

For an electron passing through a blackbody distribution for which

$$n(\epsilon)=[\pi^2(\hbar c)^3]^{-1}[\epsilon^2/(e^{\epsilon/kT}-1)], \tag{2.58}$$

integration over ϵ in (2.57) gives

$$-\frac{dE}{dt}\rightarrow\tfrac{1}{6}\pi r_0^2\frac{(mckT)^2}{\hbar^3}\left(\ln\frac{4\gamma kT}{mc^2}-\frac{5}{6}-C_E-C_l\right). \tag{2.59}$$

In (2.57), $C_E=0.5772$ (Euler's constant) and

$$C_l=\frac{6}{\pi^2}\sum_{k=2}^{\infty}\frac{\ln k}{k^2}=0.5700. \tag{2.60}$$

Note that in the extreme Klein–Nishina limit $-dE/dt$ increases only logarithmically with E (or γ) and is essentially proportional to T^2 (or $\langle\mathcal{E}\rangle/\langle\epsilon\rangle^2$), while in the Thomson limit, $-dE/dt\propto E^2T^4$. However, in the extreme Klein–Nishina limit the total energy loss does not have the same meaning as in the Thomson limit, where in each Compton collision the electron loses a small fraction of its energy. In the extreme Klein–Nishina limit, that is at very high energies, the electron loses its energy in discrete amounts which are a sizeable fraction of its initial energy. The energy of an electron as a function of time might be as in Fig. 5. We shall consider this problem again in Sec. 5.3.

2.9 Total Compton Spectrum—Integration over Electron and Initial Photon Spectra

We have derived the spectrum of Compton-scattered photons $dN_{\gamma,\epsilon}/dtd\epsilon_1$ resulting from the interaction of electrons of energy γmc^2 with an isotropic density segment, $dn=n(\epsilon)d\epsilon$, of photons of energy within $d\epsilon$. The total Compton spectrum results from an integration over γ and ϵ. If the differential number of electrons were $dN_e=N_e(\gamma)d\gamma$, the total Compton spectrum would be

$$dN_{\text{tot}}/dtd\epsilon_1=\int\int N_e(\gamma)\,d\gamma(dN_{\gamma,\epsilon}/dtd\epsilon_1), \tag{2.61}$$

where the last factor would be taken from (2.48) in the general case, and the integration would be over γ and ϵ. These integrations can be performed to give useful formulas for the case where the electron energy distribution is a *power law*:

$$N_e(\gamma)=K_e\gamma^{-p}, \qquad \gamma_0<\gamma<\gamma_m$$
$$=0, \qquad \text{otherwise}; \tag{2.62}$$

γ_0 and γ_m are the cutoffs in the distribution. For the domain where $dN_{\gamma,\epsilon}/dtd\epsilon_1$ may be approximated by the Thomson-limit expression (2.42), the lower limit

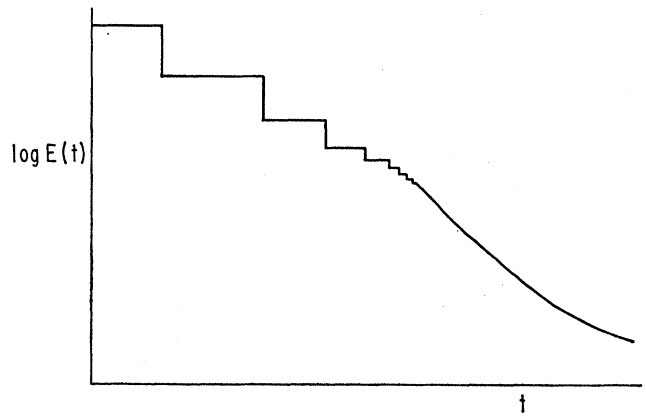

FIG. 5. Sketch of a typical time evolution of an electron's energy due to losses by Compton scattering.

on the γ integration in (2.61) would be[9]

$$\gamma_{\min}=\max\left[\tfrac{1}{2}(\epsilon_1/\epsilon)^{1/2},\ \gamma_0\right]. \tag{2.63}$$

We shall assume that we are always away from the *low-energy* end of the Compton spectrum and are considering photons energetic enough so that $\frac{1}{2}(\epsilon_1/\epsilon)^{1/2}>\gamma_0$. Moreover, we assume we are away from the *high-energy* end so that $\frac{1}{2}(\epsilon_1/\epsilon)^{1/2}\ll\gamma_m$ and the upper part of the γ integration in (2.61) does not contribute. We then obtain *a power law in* ϵ_1,

$$\frac{dN_{\text{tot}}}{dtd\epsilon_1}=\pi r_0^2cK_e2^{p+3}\frac{p^2+4p+11}{(p+3)^2(p+1)(p+5)}\epsilon_1^{-(p+1)/2}\times\int\epsilon^{(p-1)/2}n(\epsilon)\,d\epsilon. \tag{2.64}$$

When $n(\epsilon)$ is the blackbody distribution (2.58),

$$dN_{\text{tot}}/dtd\epsilon_1=\pi^{-1}(r_0^2/\hbar^3c^2)K_e(kT)^{(p+5)/2}F(p)\epsilon_1^{-(p+1)/2}, \tag{2.65}$$

TABLE I. The function $F(p)$ in Eq. (2.66).

p	$F(p)$
0	3.48
0.5	3.00
1.0	3.20
1.5	3.91
2.0	5.25
2.5	7.57
3.0	11.54
3.5	18.44
4.0	30.62
4.5	52.57
5.0	92.90

[9] Jones (J68) incorrectly took $\gamma_{\min}=1$ in his treatment.

where the parameter

$$F(p)=2^{p+3}[(p^2+4p+11)/(p+3)^2(p+1)(p+5)]\times\Gamma[\tfrac{1}{2}(p+5)]\zeta[\tfrac{1}{2}(p+5)]. \tag{2.66}$$

We have evaluated the Γ function and Riemann ζ function for several values of p and give the values of $F(p)$ in Table I.

The result (2.65), first derived by Ginzburg and Syrovatsky (GS64), has an important application because it is thought that the cosmic x-ray spectrum, which has a power-law form (cf. G67), is due to Compton scattering of high-energy electrons by the cosmic blackbody photons (FM66). Unfortunately, in the considerations of this effect, an approximate expression has been employed [instead of (2.65)] which essentially results from making a delta-function approximation to $dN_{\gamma,\epsilon}/dtd\epsilon_1$. However, (2.65) is *exact* as long as ϵ_1 is not near the end points where the effects of the cutoffs γ_0 and γ_m are important. In fact, the range of validity of (2.65) is

$$\gamma_0^2kT\ll\epsilon_1\ll\gamma_m^2kT. \tag{2.67}$$

Of course, the Thomson-limit criterion is also imposed:

$$\epsilon_1\sim\gamma^2kT\ll\gamma mc^2 \tag{2.68}$$

or

$$(\epsilon_1kT)^{1/2}\ll mc^2. \tag{2.69}$$

In the general case, when the Thomson-limit condition need not be satisfied, it becomes necessary to use the general result (2.48) for $dN_{\gamma,\epsilon}/dtd\epsilon_1$ in (2.61) for the total Compton spectrum. In this spectrum, which again results from an integration over electron and initial photon energies, we shall ignore "cutoff effects." That is, since we are assumed away from the Thomson limit,

$$\gamma_0\epsilon>mc^2, \tag{2.70}$$

and the characteristic energies of the scattered photon is

$$\epsilon_1\sim\gamma mc^2. \tag{2.71}$$

It is assumed that we are away from the endpoints of the Compton spectrum, or that

$$\gamma_0 \ll \epsilon_1/mc^2 \ll \gamma_m. \qquad (2.72)$$

This assumption allows us to ignore the mutual restrictions on ϵ and γ in their integration to give $dN_{\rm tot}/dtd\epsilon_1$. In this integration it is convenient to introduce the dimensionless quantity

$$s = \epsilon\epsilon_1/m^2c^4, \qquad (2.73)$$

and to transform the γ integration to an integration over the dimensionless q introduced in (2.49). The parameter s, like Γ_ϵ, determines the domain of scattering, with $s^{1/2} \ll 1$ corresponding to the Thomson limit, and $s \gg 1$ corresponding to the extreme Klein–Nishina limit. In terms of these quantities, (2.49) gives

$$\gamma = (mc^2/2\epsilon)s(1+[(1+sq)/sq]^{1/2}). \qquad (2.74)$$

This procedure is convenient since q varies between the limits 0 and 1 (2.51). We then have for the general total Compton spectrum, for the power-law electron spectrum (2.62):

$$\frac{dN_{\rm tot}}{dtd\epsilon_1} = \pi r_0^2 cK_e 2^{p+1}\epsilon_1^{-(p+1)/2} \int d\epsilon\epsilon^{(p-1)/2}n(\epsilon) \int_0^1 dq q^{(p-1)/2} \times \frac{[2q\ln q+1+q-2q^2+2sq(1-q)]}{\{1+[sq/(1+sq)]^{1/2}\}^{p+2}(1+sq)^{(p+3)/2}}. \qquad (2.75)$$

This expression[10] is exact; the restriction $s \gg 1$ has not yet been made. Unfortunately, the q integration cannot be performed in the general case. We shall, however, make two applications of (2.75): (i) the Thomson-limit correction and (ii) the extreme Klein–Nishina limit formula.

In the Thomson limit, when $s^{1/2} \ll 1$, the integral over q is independent of s, and the scattered photon spectrum reduces to (2.64). It is now possible, however, to calculate the first-order correction to the spectrum in the Thomson limit. This is done by expanding the integrand in (2.75), keeping only the first-order term in $(qs)^{1/2}$. We then obtain the spectrum

$$\frac{dN_{\rm tot}}{dtd\epsilon_1} = \pi r_0^2 cK_e 2^{p+3} \frac{p^2+4p+11}{(p+1)(p+3)^2(p+5)} \epsilon_1^{-(p+1)/2} \times \int d\epsilon\epsilon^{(p-1)/2}n(\epsilon)\left(1-G(p)\frac{(\epsilon\epsilon_1)^{1/2}}{mc^2}\right), \qquad (2.76)$$

where

$$G(p) = \frac{(p^2+6p+16)(p+1)(p+3)^2(p+5)}{(p^2+4p+11)(p+4)^2(p+6)}. \qquad (2.77)$$

For a blackbody photon distribution, integration over ϵ gives the result (2.65) multiplied by the first-order correction factor

$$f_{\rm corr} = 1-G(p)\frac{\Gamma[(p+6)/2]\zeta[(p+6)/2]}{\Gamma[(p+5)/2]\zeta[(p+5)/2]}\frac{(\epsilon_1 kT)^{1/2}}{mc^2}. \qquad (2.78)$$

The total Compton photon spectrum deviates from a pure power law at high energies when (2.69) is no longer strictly valid.[11] The result (2.78) gives the form of this correction and shows that the spectrum will begin to *steepen* from the $\epsilon_1^{-(p+1)/2}$ power law. Of course, deviation from the restriction (2.67) $\epsilon_1 \ll \gamma_m^2 kT$ would also produce a steepening.

The procedure in obtaining a formula for the total Compton spectrum in the extreme Klein–Nishina limit ($s \gg 1$) is more complicated because of the difficulty in finding an expansion valid for the whole range of q in (2.75). In this limit the q integrand in (2.75) is zero at the endpoints and has its maximum around $q \sim 1/s \ll 1$. Over *most* of the range of q, $qs \gg 1$, and the integrand is essentially of the form $2^{-(p+1)}s^{-(p+1)/2}q^{-1}$. Therefore the integral over q must be of the form

$$I_q(s;p) \xrightarrow[s\gg 1]{} 2^{-(p+1)}s^{-(p+1)/2}[\ln s + C(p)], \qquad (2.79)$$

where $C(p)$ is a parameter of order unity. Note that this result differs from that of Jones (J68) whose expression is too large by a factor of 2. To evaluate this expression (2.79), that is, to determine $C(p)$, we proceed by separating the integral over q into two parts:

$$\int_0^1 = \int_0^{1/s} + \int_{1/s}^1. \qquad (2.80)$$

Now, these integrals are of the form (see 2.75)

$$\int dq q^{(p-1)/2}[N(q,s)/D(q,s;p)], \qquad (2.81)$$

where N is the expression in brackets in (2.75) (the numerator of the last term), and D is the complicated denominator. For $q \gg 1/s \ll 1$, D approaches the asymptotic value

$$D_a = 2^{p+2}(sq)^{(p+3)/2}. \qquad (2.82)$$

Thus in the integral from $1/s$ to 1 in (2.80) we write

$$1/D = 1/D_a + (1/D - 1/D_a); \qquad (2.83)$$

the second term in parentheses approaches zero for $q \gg 1/s$. In integrating the term $1/D_a$ over q in (2.80) we can approximate, $N \approx 1+2sq(1-q)$. For $s \gg 1$, we then get,

$$\int_{1/s}^1 dq q^{(p-1)/2}\left(\frac{N}{D_a}\right) = 2^{-(p+1)}s^{-(p+1)/2}(\ln s - \tfrac{1}{2}). \qquad (2.84)$$

In the integral from 0 to $1/s$, $N \approx 1+2sq$. In the integration of the term $(1/D - 1/D_a)$ over $1/s$ to 1, only the lower end contributes, so again $N \approx 1+2sq$. Changing

[10] Jones (J68) derived a similar expression but with an error of a factor of 2 in the last term in the brackets. This term is most important in the extreme Klein–Nishina limit.

[11] This means that $(\epsilon_1 kT)^{1/2}$ begins to approach the order of magnitude of mc^2, although it must still be strictly less than mc^2.

the integration variable to $x=sq$, and collecting terms, we see that $C(p)$ in (2.79) is given by[12]

$$2C(p)=-1+\int_0^1 \frac{dx}{x^2}\frac{1+2x}{D/D_a}+\int_1^\infty \frac{dx}{x^2}(1+2x)\left(\frac{D_a}{D}-1\right), \qquad (2.85)$$

where

$$D/D_a=\{\tfrac{1}{2}+\tfrac{1}{2}[x/(1+x)]^{1/2}\}^{p+2}(1+1/x)^{(p+3)/2}. \qquad (2.86)$$

The parameter $C(p)$ is plotted in Fig. 6.

The total Compton spectrum in the extreme Klein–Nishina limit is then given by

$$\frac{dN_{\text{tot}}}{dtd\epsilon_1}=\pi r_0^2 cK_e(mc^2)^{p+1}\epsilon_1^{-(p+1)} \times \int \frac{d\epsilon}{\epsilon}n(\epsilon)\left(\ln\frac{\epsilon\epsilon_1}{m^2c^4}+C(p)\right). \qquad (2.87)$$

Thus we see that the spectrum ($\propto \epsilon_1^{-(p+1)}$) is much steeper than in the Thomson limit (2.64) where it is of the form $\epsilon_1^{-(p+1)/2}$. When $n(\epsilon)$ is a blackbody distribution

$$\frac{dN_{\text{tot}}}{dtd\epsilon_1}=\frac{\pi r_0^2}{6\hbar^3c^2}K_e(mc^2)^{p+1}(kT)^2\epsilon_1^{-(p+1)} \times\left(\ln\frac{\epsilon_1 kT}{m^2c^4}+1+C(p)-C_E-C_l\right), \qquad (2.88)$$

where $C_E=0.5772$ and $C_l=0.5700$ as in (2.60). Again, the assumption has been made that the endpoints, γ_0 and γ_m, of the electron distribution do not contribute to the total spectrum. In the Klein–Nishina limit this condition has the form

$$\gamma_0 mc^2 \ll \epsilon_1 \ll \gamma_m mc^2. \qquad (2.89)$$

3. BREMSSTRAHLUNG

Although the exact bremsstrahlung cross section can be derived only by the methods of quantum electrodynamics, limiting formulas such as the expressions when the bremsstrahlung photon energy is small may be arrived at by simple applications of classical or semiclassical electrodynamics. Indeed, some useful insights into the quantum mechanical results may be gained through the semiclassical approach to the problem. For this reason we begin this section by deriving the low-frequency limits to the bremsstrahlung cross section by these elementary semiclassical methods. As in our treatment of Compton scattering we consider only incident electrons which are highly relativistic ($\gamma\gg1$). We give two derivations of the low-frequency bremsstrahlung cross section. In the first method, the result is obtained by computing the probability of emission of a soft photon during the Coulomb scattering of a high-energy electron. In the second method, which can also be used to derive an accurate expression not

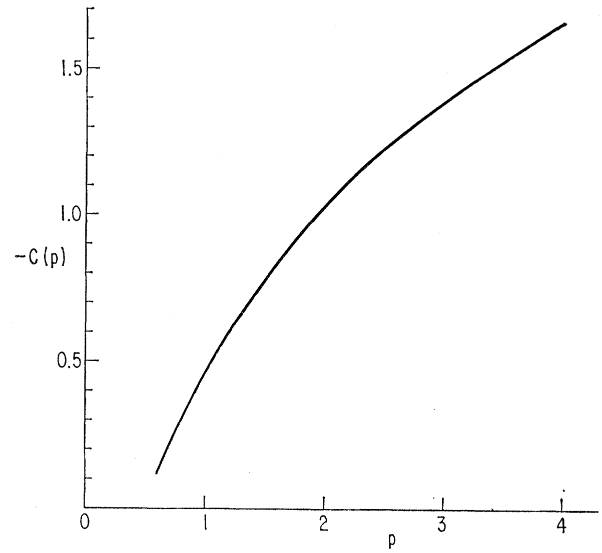

FIG. 6. Graph of the function $-C(p)$.

restricted to the low-frequency limit, we make use of the *Weizsäker–Williams* method whereby the bremsstrahlung process is treated as Compton scattering of the virtual photons of the Coulomb field of the scattering charge. Since the basic formulation for these methods is probably not too familiar to the general reader, a brief derivation of the fundamental formulas is given in an appendix at the end of this paper.

3.1 Low-Frequency Limit

Perhaps first of all we might quote the *exact* formula for the bremsstrahlung cross section for high-energy electrons incident on an unshielded static charge Ze. This formula is based on the Born approximation which is valid at high energies where the effects of the Coulomb field of the scatterer on the incoming and outgoing electron are negligible.[13] Then the differential cross section for emitting a bremsstrahlung photon of energy within $\hbar d\omega$ in the scattering of an electron of initial energy E_i and final energy $E_f=E_i-\hbar\omega$ is (cf. JR55)

$$d\sigma=4Z^2\alpha r_0^2\frac{d\omega}{\omega}\frac{1}{E_i^2}(E_i^2+E_f^2-\tfrac{2}{3}E_iE_f)\left(\ln\frac{2E_iE_f}{mc^2\hbar\omega}-\frac{1}{2}\right). \qquad (3.1)$$

In the low-frequency limit this expression reduces to

$$d\sigma \xrightarrow[\text{(sm }\omega)]{} \tfrac{16}{3}Z^2\alpha r_0^2(d\omega/\omega)[\ln(2E_i^2/mc^2\hbar\omega)-\tfrac{1}{2}]. \qquad (3.2)$$

This formula can be derived quite easily, uncertain to within a factor $\sim$1 in the (large) argument of the logarithm, as we show in the following. The procedure is to compute the probability that in a Coulomb scattering a single soft photon is emitted. The basic expression for the low-frequency bremsstrahlung cross

[12] The upper limit in the second integral is really s which is a large number ($\rightarrow\infty$).

[13] Effects of the deviation from Born approximation have been computed by Davies, Bethe, and Maximon (DBM54).

section is then

$$d\sigma = d\sigma_{sc} dw_\omega, \tag{3.3}$$

where $d\sigma_{sc}$ is the differential elastic Coulomb-scattering cross section and dw_ω is the probability of emitting a soft photon of frequency within $d\omega$ in the Coulomb scattering. Both $d\sigma_{sc}$ and dw_ω are functions of, say, the scattering angle θ_{sc} which would eventually be integrated over in (3.3) to get $d\sigma$. The elastic Coulomb cross section would be given by the Born limit of the "Mott formula" (cf. MM65)

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{Z^2 e^4}{4m^2 v^4} \frac{1}{\sin^4(\theta/2)_{sc}} \frac{1-\beta^2 \sin^2(\theta/2)_{sc}}{\gamma^2}. \tag{3.4}$$

The formula differs from the nonrelativistic Rutherford formula by the factors γ^{-2} and $1-\beta^2 \sin^2(\theta/2)_{sc}$. This latter factor arises due to effects of the electron's magnetic moment; in the Coulomb scattering of a relativistic spinless particle such a factor would not occur. However, in Coulomb scattering (and in Coulomb bremsstrahlung) *small angle scatterings* are the most important, and for these the spin-dependent factor is unity. The physical reason for this is elementary: the magnitude of the spin or magnetic force on the electron decreases with distance from the scattering center faster than the pure Coulomb force; since small angle scattering corresponds to large impact parameters, the magnetic effects should be negligible for small θ_{sc}. Thus for small scattering angles and highly relativistic energies

$$d\sigma_{sc}/d\theta_{sc} = 8\pi Z^2 r_0^2 \gamma^{-2} \theta_{sc}^{-3}. \tag{3.5}$$

The factor γ^{-2} can also be easily understood, since (3.5) can be derived from a simple classical–Born or impact approximation method; γ^{-2} appears essentially because the scattered particle momentum is $\gamma mv (\rightarrow \gamma mc)$ instead of mv in the nonrelativistic derivation, and $d\sigma_{sc}/d\Omega \propto (pv)^{-2}$. The probability factor dw_ω in (3.3) can be taken from the expression (A.15) derived in the Appendix which is, in terms of θ_{sc},

$$dw_\omega(\theta_{sc}) = (2\alpha/3\pi) \gamma^2 \theta_{sc}^2 (d\omega/\omega). \tag{3.6}$$

Then by (3.3), (3.5), and (3.6) we get

$$d\sigma = \tfrac{16}{3} Z^2 \alpha r_0^2 (d\omega/\omega) \int (d\theta_{sc}/\theta_{sc}). \tag{3.7}$$

The logarithmic integral in (3.7) can more conveniently be written in terms of the minimum and maximum impact parameters

$$\int (d\theta_{sc}/\theta_{sc}) = \ln (b_{max}/b_{min}). \tag{3.8}$$

For the determination of the ratio of the maximum and minimum impact parameters it is convenient to consider the process in the electron's initial rest frame. Since the impact parameters are *transverse* distances, $b'=b$. The minimum impact parameter is $b_{min}' \sim \hbar/mc$, a result due to quantum-mechanical effects which can be understood in terms of the uncertainty principle. The electron cannot be localized to smaller distances without introducing enough energy to produce a surrounding cloud of electron–positron pairs. The (bremsstrahlung) emission from such a configuration would be *reduced*, since the total charge (e) would no longer be distributed in a coherent manner as in a single particle of charge e and mass m. The maximum impact parameter can be understood in terms of purely classical effects, essentially based on the material within Eqs. (A18)–(A21) in the Appendix; in these equations quantities should really be primed, since they refer to the electrons rest frame. The result is $b_{max}' \sim \gamma c/\omega'$; then since $\omega = \gamma\omega'(1+\cos\theta')$ and the emission is over a wide range of angles θ' in the frame where the electron motion is nonrelativistic, $\omega' \sim \omega/\gamma$. Thus we have

$$b_{max}/b_{min} = b_{max}'/b_{min}' \sim \gamma^2 mc^2/\hbar\omega, \tag{3.9}$$

which is to be substituted in (3.8) and (3.7) to give the bremsstrahlung cross section. The result (3.9) obtained is, of course, uncertain to within a factor ~ 1; in fact, comparing with the exact expression (3.1), we see that (3.9) should be multiplied by $2/e^{1/2}$. However, since (3.9) is a large number and appears in a logarithmic factor, the simple derivation giving the results (3.7)–(3.9) yields a fairly accurate expression for the cross section.

While the developments in this section are limited to the domain of low-frequency bremsstrahlung, there is a method of getting essentially the general expression (3.1) valid not just for small ω. This procedure, the Weizsäker–Williams (W–W) method, does, however, require the use of the Klein–Nishina formula (2.27) which can be derived only by detailed methods of quantum electrodynamics. Nevertheless, the W–W method is quite an interesting little trick, and is of some help in at least providing a better understanding of the exact result of quantum electrodynamics. Moreover, it is a *general* method for problems of this type, and its application to the bremsstrahlung problem is instructive.

3.2 Weizsäker–Williams Approach

The basic idea here is that bremsstrahlung is considered as Compton scattering, by the incoming electron, of the virtual photons of the Coulomb field of the scattering center. The basic relation giving the bremsstrahlung cross section is

$$d\sigma = dN d\sigma_C, \tag{3.10}$$

where dN is the differential number of incident virtual photons in the electron's rest frame and would be given by the expression (A.22) in the Appendix (with, however, b and ω primed), and $d\sigma_C$ is the differential Compton cross section. Let us first derive the low-frequency limit expression (3.7). This limit corresponds to Thomson scattering with $\hbar\omega' \ll mc^2$, ω' being the photon frequency before scattering in the electron initial rest frame K'. Then $\omega_1'=\omega'$, where ω_1' is the scattered frequency. In the lab frame, $\omega_1 \sim \gamma\omega_1'$, (we now drop this subscript of ω) so "low-frequency" means $\hbar\omega \ll \gamma mc^2$. Then we can take the total Thomson cross section $\sigma_T=(8\pi/3)r_0^2$ for $d\sigma_C$ in (3.10) and get

$$d\sigma=\tfrac{16}{3}\alpha Z^2 r_0^2 (d\omega/\omega)\int (db/b), \tag{3.11}$$

that is, the same expression as (3.7), but derived in a different manner. The consideration of the logarithmic integral in (3.11) is the same, giving the result (3.9).

In the derivation of the more general expression not restricted to low frequencies it is convenient to express photon energies in the following units:

lab system (K)—units of γmc^2;

electron rest system (K')—units of mc^2.

We also drop the subscript 1 on the scattered photon energy ϵ_1 in K. Then the kinematic and transformation relations are

$$\begin{aligned}\epsilon_1'&=\epsilon'/[1+\epsilon'(1-\cos\theta_1')],\\ \epsilon&=\epsilon_1=\epsilon_1'(1-\cos\theta_1'),\\ \epsilon'&=[\epsilon/(1-\epsilon)](1-\cos\theta_1')^{-1}=\epsilon_1'/(1-\epsilon).\end{aligned} \tag{3.12}$$

For fixed ϵ, the minimum photon energy in K' before scattering is

$$\epsilon_{\min}'=\epsilon/2(1-\epsilon). \tag{3.13}$$

The maximum energy is gotten from $\omega_{\max}' \sim \gamma c/b_{\min} \sim \gamma mc^2/\hbar$ [see (A21) and Sec. 3.1], so

$$\epsilon_{\max}' \sim \gamma \gg \epsilon_{\min}'. \tag{3.14}$$

Now with the help of the kinematic relations (3.12) the Klein–Nishina formula (2.27) can be written as a function of ϵ and ϵ', with the differential solid angle transformed in terms of a differential $d\epsilon$. Then

$$d\Omega_1'=2\pi d(1-\cos\theta_1')=(2\pi/\epsilon')[d\epsilon/(1-\epsilon)^2], \tag{3.15}$$

and the differential Klein–Nishina cross section becomes

$$d\sigma_C=\pi r_0^2\frac{1}{\epsilon'}\left[1-\epsilon+\frac{1}{1-\epsilon}-\frac{2}{\epsilon'}\frac{\epsilon}{1-\epsilon}+\frac{1}{\epsilon'^2}\left(\frac{\epsilon}{1-\epsilon}\right)^2\right]d\epsilon. \tag{3.16}$$

Substituting

$$dN=\frac{2\alpha Z^2}{\pi}\frac{db'}{b'}\frac{d\epsilon'}{\epsilon'}, \tag{3.17}$$

and (3.16) into (3.10), an integration over ϵ' and then b' can be performed. By virtue of (3.13) and (3.14) only the lower limit (3.13) contributes[14] in the integration over ϵ' and we get

$$d\sigma=4\alpha r_0^2 Z^2(\tfrac{4}{3}(1-\epsilon)+\epsilon^2)(d\epsilon/\epsilon)\int(db'/b'). \tag{3.18}$$

[14] More precisely, in the ϵ' integration $\sim\gamma(b'_{\min}/b')$ should be taken for $\epsilon_{\max}'$; however, this is still $\gg\epsilon_{\min}'$ for all except the very low end of the integration over b'. Since the b' integration is logarithmic, this would give a result essentially the same as that taking $\epsilon_{\max}'\sim\gamma$ (independent of b').

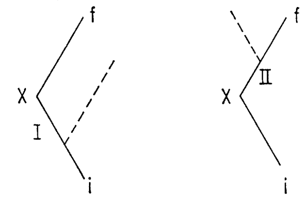

FIG. 7. Feynman diagrams for bremsstrahlung in scattering by an external potential (X).

Now, again $b_{\min}' \sim \hbar/mc$, while

$$b_{\max}' \sim \frac{\gamma c}{\omega_{\min}'}=\left(\frac{\hbar}{mc}\right)\left(\frac{\gamma}{\epsilon_{\min}'}\right)=\frac{2\gamma(\hbar/mc)(1-\epsilon)}{\epsilon}. \tag{3.19}$$

We have, finally, to within a factor ~ 1 in the argument of the logarithm,

$$d\sigma=4\alpha r_0^2 z^2(\tfrac{4}{3}(1-\epsilon)+\epsilon^2)(d\epsilon/\epsilon)\ln[2\gamma(1-\epsilon)/\epsilon], \tag{3.20}$$

which is essentially the same as the exact expression (3.1).

The bremsstrahlung cross section was first derived by these methods by von Weizsäker (W34). It should be emphasized, however, why the W–W method *works* for this problem. Basically it is because the main contribution to the total cross section comes from the *soft* virtual photons. Had this not been so, this semiclassical method would not have given the right answer. The W–W method has been applied to a number of other problems with a lesser degree of success in some cases. Nevertheless, it always provides helpful insight into the problem.

3.3 Pure Coulomb Bremsstrahlung—Momentum Transfer Distribution

We have already quoted the exact (in the highly relativistic limit) expression (3.1) for the cross section for emitting a bremsstrahlung photon of energy within $\hbar d\omega$, in the scattering of an electron of initial energy $E_i(\gg mc^2)$ by a static unshielded charge Ze. We shall not derive this expression here; it is derived by the old-fashioned perturbation theory methods in Heitler's book (H54) and by the modern covariant perturbation theory in the book by Jauch and Rohrlich (JR55). Only a rough outline of the general procedure will be given to facilitate discussion of the associated problems of electron–electron and electron–atom bremsstrahlung in the following sections.

The problem is closely related to the simpler one of radiationless Coulomb scattering; that problem is one of first-order perturbation theory while bremsstrahlung involves second-order perturbation theory. In the static Coulomb approximation, the scattering center (a heavy nucleus) plays only the role of providing the field which scatters the incoming electron; the small recoil it receives has a negligible kinematic effect on the photon emission, although it is necessary for the process to occur. The scattering Coulomb field is considered an *external potential* and is represented by an X in the Feynman diagrams for the process (Fig. 7). The electron makes a transition from an initial state i to a final state f. In the process it is scattered by the Coulomb field and a photon is emitted (as in Fig. 7, "before" or "after" the scattering). The total transition amplitude is given by an expression of the form (in noncovariant perturbation theory)

$$K_{fi}=\sum\left(\frac{V_{(\mathrm{sc})fI}H_{(\mathrm{em})Ii}}{E_i-E_I}+\frac{V_{(\mathrm{sc})IIi}H_{(\mathrm{em})fII}}{E_i-E_{II}}\right), \quad (3.21)$$

where the sum is over spin and energy states of the electron in intermediate states (I and II). The matrix elements of the scattering potential $V_{(\mathrm{sc})}$ are given in terms of the recoil momentum $\mathbf{q}=\mathbf{k}_f-\mathbf{k}_I=\mathbf{k}_{II}-\mathbf{k}_i$; since the initial, final, and intermediate electron states are plane waves in the Born approximation,

$$V_{(\mathrm{sc})fI}=V_{(\mathrm{sc})IIi}=\int dVe^{i\mathbf{q}\cdot\mathbf{r}}(Q/r)=4\pi Q/q^2; \quad (3.22)$$

here Q is the charge of the scattering center. Thus, a common factor appearing in the amplitude K_{fi} is the Fourier transform[15] of the scattering potential; it is a simple function of only the momentum transfer $\mathbf{q}$. In terms of the initial and final electron momenta $\mathbf{k}_i$ and $\mathbf{k}_f$ and the photon momentum $\mathbf{k}$

$$\mathbf{q}=\mathbf{k}_i-\mathbf{k}_f-\mathbf{k}. \quad (3.23)$$

The other factors in (3.21) are the matrix elements of the part of the interaction Hamiltonian $H_{(\mathrm{em})}$ corresponding to the "photon emission from the electron lines" in Fig. 7.

The differential cross section is gotten essentially from

$$d\sigma\propto\tfrac{1}{2}\sum_i\sum_f\sum_{\mathrm{pol}}|K_{fi}|^2\,d\mathbf{k}_fd\mathbf{k}, \quad (3.24)$$

that is, an average over initial electron spin states (i), a sum over final spin states (f), a sum over polarizations of the emitted photon, and eventually a sum over the phase space of the outgoing electron and photon. Actually, instead of integrating over the total photon phase space, an integration only over photon angles is performed, yielding the cross section differential in photon energy (3.1). Bethe (B34) has shown how this integration can be transformed and integrated so that all that remains is an integration over the magnitude of the momentum transfer q. This development by Bethe is especially important for considerations of electron–electron and electron–atom bremsstrahlung, which we discuss in the following sections.

From now on in our discussion of bremsstrahlung it is convenient to adopt mc^2 as energy unit and mc as momentum unit for both photon and electron. Then, in these units, and in our high-energy limit, E_i, $E_f\gg1$; also, significantly, we consider only bremsstrahlung photons of high energy, $k\gg1$. For given E_i, E_f, and k one can readily convince oneself that the minimum momentum transfer $q_{\min}$ corresponds to the case where $\mathbf{k}_f$ and $\mathbf{k}$ in (3.23) are both along the direction of $\mathbf{k}_i$. This minimum momentum transfer is very small and thus k is very close to E_i-E_f; the actual value of $q_{\min}$ is easily found to be

$$q_{\min}=\delta=k/2E_iE_f. \quad (3.25)$$

More important is the *effective maximum* value of q. The *actual* maximum value is of the order of E_i, but values of q larger than one do not contribute much to the bremsstrahlung cross section. This was shown originally by Bethe (B34) and more recently (and in greater generality) by Suh and Bethe (SB59). The physical reason for this result can be seen quite readily (G69). Since E_i, E_f, $k\gg1$, we can have $q\ll E_i$, E_f, k even for values of q appreciably larger than one for which $k\cong E_i-E_f$. Since the momenta $\mathbf{k}_f$ and $\mathbf{k}$ are (on the whole) at small angles to $\mathbf{k}_i$ for all except the very minimum value (3.25), $\mathbf{q}$ must be approximately perpendicular to $\mathbf{k}_i$: $q=q_\perp$. Now consider the process in the (primed) reference frame of the incident electron and from the point of view of the Weizsäker–Williams approach. Since $q_\perp'=q_\perp=q$, values of q greater than 1 *necessarily* involve, for a large fraction of the solid angles of the outgoing electron and photon, Compton scatterings in the Klein–Nishina domain, where the Compton cross section is reduced below its large value in the Thomson limit. This means that we can break up the momentum transfer distribution into the domains (i) $q=\delta$ to 1 and (ii) $q=1$ to $\sim E_i$. Then in (ii) only the lower end will contribute since in (i) over all except the upper end (where it begins to drop off) the distribution goes as $q^{-1}dq$ (B34, SB59).

3.4 Electron–Electron Bremsstrahlung

A number of authors (B47, V48, AB53, JR55, JR58, SB59, BFK66, M67) have investigated this problem or the associated one of pair production in the field of a free electron. The exact solution to the general problem, including the necessary integrations over the phase space of the outgoing particles, has not been carried through analytically due purely to the mathematical complexity. Eight Feynman diagrams are involved, the four in Fig. 8 plus the four exchange diagrams. However, a considerable simplification results for the case where one of the electrons (for example, electron 1 in Fig. 8) is initially at rest in the lab frame and the other particle (electron 2 and the outgoing photon) energies E_i, E_f, k are $\gg1$. For this case only Diagrams (a) and (b) contribute; the reason is basically the arguments given in the previous section. As we have seen, in bremsstrahlung involving the scattering of an electron by the Coulomb field of the scattering center, the recoil momentum given to the scatterer has an effective

[15] In (3.22), $\hbar=1$, and $\mathbf{r}$ represents the coordinate of the scattered electron. Also, it might be remarked that the integral (3.22) is not convergent as it stands. To get the meaningful result $4\pi Q/q^2$, one must modify the long-range Coulomb potential by a factor $\exp(-\alpha r)$, evaluate the integral, and then take the limit $\alpha=0$.

maximum ~1. But since we are considering photon energies $k \gg 1$, these photons cannot be emitted by the system with low momentum. Therefore diagrams (c) and (d) should not be expected to contribute appreciably.[16] This argument is really based on the self-consistency of the result, since the reasoning leading to an effective maximum $q \sim 1$ involved consideration of an (virtually static) external scattering potential. However, Heitler (H54) has shown by direct application of the Weizsäker–Williams method, that the contribution to the bremsstrahlung cross section from Compton scattering of the virtual photons of electron 2 by electron 1 [corresponding to Diagrams (c) and (d)] is negligible compared to the contribution from scattering of 1's photons by 2 [Diagrams (a) and (b)]. Exchange effects are negligible for the circumstances (E_i, E_f, $k \gg 1$) considered; because of the low recoil momentum of electron 1, the two electrons are essentially distinguishable by means of their vastly different momenta.[17]

Thus only two (a and b) diagrams contribute for electron–electron bremsstrahlung. In fact, due to the low recoil momentum of electron 1, its field (which scatters electron 2) can be considered essentially static, that is, that of an external potential (Fig. 7). We conclude that, away from the low photon energy end of the spectrum, the cross section for *e–e* bremsstrahlung where one electron is initially at rest should be identical to that for electron–proton bremsstrahlung (*p* at rest). Actually, several detailed calculations (B47, SB59, BFK66, M67) have yielded this result. A discussion has been given here because a number of authors (V48, AB53, H54, JR55, JR58) have suggested slightly but significantly different (and incorrect) results for the *e–e* problem. Our elementary discussion is taken from G69. Experimentally, the equivalence of the *e–e* and *e–p* bremsstrahlung cross sections can be considered to have been essentially demonstrated by measurements (HCCS59, M59, J60) of pair production in the fields of electrons and protons. Pair production is closely related to bremsstrahlung, having essentially the same Feynman diagrams and associated matrix elements. The important theoretical work of Mork (M67a) [see also discussion by Olsen, (O68)] on pair production in the field of an electron should also be mentioned. Mork calculated the cross section numerically by using Monte Carlo methods to integrate over the phase space volume. He included all eight Feynman diagrams and did indeed find that at high (photon) energy the cross section approached that corresponding to pair production in the field of a proton.

Finally, on the basis of the elementary discussion given here, it is clear that for the conditions considered (E_i, E_f, $k \gg 1$) bremsstrahlung from incident positrons would also be equivalent. That is, for any combination of signs, bremsstrahlung from $e^{\pm}-e^{\pm \text{or} \mp}$ and $e^{\pm}-p^{\pm \text{or} \mp}$ are all equivalent.[18] Bremsstrahlung in collisions of electrons and positrons and (anti-) atomic species would also be equivalent.

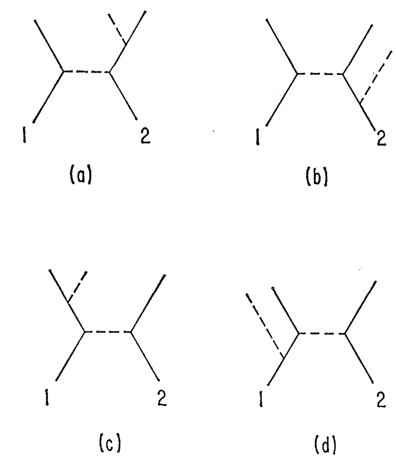

FIG. 8. Feynman diagrams for electron–electron bremsstrahlung; there are four additional "exchange" diagrams.

3.5 Electron–Atom Bremsstrahlung

The results of the discussion in the previous section, although they were for free electrons, are very relevant to problems involving bound electrons. Since we found that the scattering-electron recoil and exchange effects are negligible, the basic formulation of the problem of electron–atom bremsstrahlung is a straightforward generalization of that for pure Coulomb and electron–electron bremsstrahlung. The scatterer, that is the atom, can still be considered as producing an external potential. However, now the total potential is

$$V_{(\text{sc})} = Ze/r - e\sum_j |\mathbf{r} - \mathbf{r}_j|^{-1}, \qquad (3.26)$$

where $\mathbf{r}$ is the position of the scattered electron, $\mathbf{r}_j$ the position of the rth atomic electron, and the nucleus (Z) is at the origin. With the atom as scatterer, the effects of atomic transitions during the scattering process must be included. The matrix elements of $V_{(\text{sc})}$ must now be between the plane-wave states of the scattered electron and also between the electronic states of the atom. Denoting the initial and final atomic states by 0 and n respectively, this total matrix element is

$$\langle n \mid \int dV \exp(i\mathbf{q}\cdot\mathbf{r})(Ze/r - e\sum_j |\mathbf{r}-\mathbf{r}_j|^{-1}) \mid 0\rangle = 4\pi e q^{-2}\langle n \mid \mathrm{Z} \mid 0\rangle, \qquad (3.27)$$

[16] For bremsstrahlung photon energies at the low-energy end ($k \sim 1$ or smaller) this conclusion would not be valid, and Diagrams (c) and (d) would contribute (see BFK66).

[17] The exchange corrections given in (JR58) are erroneous.

[18] However, at nonrelativistic incident energies, e^+-e^- bremsstrahlung $\gg e^{\pm}-e^{\pm}$ bremsstrahlung, since the e^+-e^- system has a dipole moment. Although this is very clear from considerations of classical electrodynamics, a good deal of confusion has arisen concerning this question. Some references to the literature on this subject may be found in the paper by Stabler (S65).

where we have defined

$$\mathcal{Z}=Z-\sum_j \exp(i\mathbf{q}\cdot\mathbf{r}_j). \quad (3.28)$$

The bremsstrahlung total cross section will be proportional to the squared absolute value of the matrix element (3.27) then summed over final atomic states n. In this summation we can make use of the closure relation ($\sum | n\rangle\langle n | =1$) based on the completeness of the atomic eigenfunctions. This summation must include the continuum (ionization). Then the purely atomic factor occuring in the differential cross section is (WL39)

$$\zeta_{sc}(q)=\sum_n |\langle n | \mathcal{Z} | 0\rangle|^2=\sum_n\langle 0 | \mathcal{Z}^* | n\rangle\langle n | \mathcal{Z} | 0\rangle$$
$$=\langle 0 | \mathcal{Z}^*\mathcal{Z} | 0\rangle. \quad (3.29)$$

Thus the scattering factor is just the expectation value of $| \mathcal{Z} |^2$ for the initial atomic state. Further, we can write

$$\zeta_{sc}(q)=Z^2-2Z\sum_j\langle 0 | \exp(i\mathbf{q}\cdot\mathbf{r}_j) | 0\rangle$$
$$+\sum_{j,k}\langle 0 | \exp[i\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_k)] | 0\rangle. \quad (3.30)$$

The expectation values in this expression vary from 1 to 0 as q varies from 0 to ∞. The characteristic change-over in the units (mc) we use is $q_{char}\sim\alpha Z_{eff}$, where α is the fine structure constant and Z_{eff} is the effective nuclear charge. The second term on the right of (3.30) is just $2Z$ times the "atomic form factor" $F(q)$.

The above result, that the atomic-scattering factor is a function of q, is essentially the reason why Bethe (B34) worked out the transformation of the phase space integration in (3.24) into an integration over q. The result of this work is that the differential bremsstrahlung cross section can be written in the form

$$d\sigma=\Phi(k)\,dk=\alpha r_0^2(dk/k)(E_i^2)^{-1}$$
$$\times[(E_i^2+E_f^2)\phi_1-\tfrac{2}{3}E_iE_f\phi_2]; \quad (3.31)$$

ϕ_1 and ϕ_2 are functions of E_i, E_f, and $k(=E_i-E_f)$. When the scattering system is an unshielded charge Ze, then $\phi_1=\phi_2=Z^2\phi_u$ where [see (3.1)]

$$\phi_u=4[\ln(2E_iE_f/k)-\tfrac{1}{2}]. \quad (3.32)$$

For a general atom the ϕ's are gotten from (B34, G69)

$$\phi_i=4\int f_i(q;E_i,E_f,k)\zeta_{sc}(q)\,dq. \quad (3.33)$$

For the whole range of q, there are no simple expressions for the f_i's. However, one can proceed in the following manner. The expression (3.30) for $\zeta_{sc}(q)$ can be broken up into terms which are of the form of a constant, a term in $1-F$, and perhaps a term in $1-F^2$. These latter terms approach 0 and 1 for $q\ll$ and $\gg\alpha Z_{eff}$, respectively. The constant term in $\zeta_{sc}(q)$, say c_0, simply gives a term $c_0\phi_u$ to ϕ_i. The $1-F$ and $1-F^2$ terms can be integrated in (3.33), since analytic expressions can be found for the f_i in the limits $q\ll1$ and $q\gg\delta$, αZ_{eff}. Moreover, a value q_0 can be found where both domains overlap. In the domain of small q, the f_i (designated f_s) are simple expressions, while for large q the expressions (designated f_c) are complicated. However for large q, $F\rightarrow0$ and we can then write, for the terms in $1-F$ (or $1-F^2$),

$$\int_\delta^{q_{max}} f(q)(1-F)\,dq$$
$$=\int_\delta^{q_0} f_s(q)(1-F)\,dq+\int_{q_0}^{q_{max}} f_c(q)\,dq$$
$$=\int_\delta^{q_{max}} f_s(q)(1-F)\,dq+\int_{q_0}^{q_{max}}[f_c(q)-f_s(q)]\,dq. \quad (3.34)$$

This last integral can be evaluated, using the f's derived by Bethe (B34), to be $1-\ln q_{max}$ (for f_1), and $\tfrac{5}{6}-\ln q_{max}$ (for f_2); that is, q_0 does not appear. Also the integral from δ to q_{max} can be broken up into an integral from δ to 1 plus an integral (over which $F\rightarrow0$) from 1 to q_{max}. The latter integral is, for both f_1 and f_2, just $\ln q_{max}$ which cancels[19] the logarithm from the last integral in (3.34). We can now express the basic result in the following manner. Suppose $\zeta_{sc}(q)$ can be expressed as a sum of terms of the form

$$\zeta_{sc}(q)=c_0+\sum_p c_p(1-F^p), \quad (3.35)$$

that is, with the form factor appearing in powers. Then by (3.33) and (3.34),

$$\phi_i=c_0\phi_u+4\sum_p c_p\mathrm{r}\left(\alpha_i+\int_\delta^1 f_i(q)(1-F^p)\,dq\right), \quad (3.36)$$

where $\alpha_1=1$, $\alpha_2=\tfrac{5}{6}$, and the f_i are the simple fucntions[20] of q and δ:

$$f_1=q^{-3}(q-\delta)^2,$$
$$f_2=q^{-4}[q^3-6\delta^2q\ln(q/\delta)+3\delta^2q-4\delta^3]. \quad (3.37)$$

For a hydrogenic atom or ion (nucleus Z, one electron), ζ_{sc} derived from (3.30) is very simple:

$$\zeta_{sc(1)}=(Z-1)^2+2Z(1-F_Z), \quad (3.38)$$

where the atomic form factor is

$$F_Z(q)=(1+a_Z^2q^2)^{-2}, \quad (3.39)$$

and

$$a_Z=(2\alpha Z)^{-1}\equiv a/Z=68.5194/Z. \quad (3.40)$$

The ϕ_i computed from (3.36) and (3.37) are slowly varying functions of δ (and a_Z). In Fig. 9 ϕ_1 and ϕ_2 for

[19] We have gone through all this in some detail (see also G69) because apparently some confusion has arisen on the q integration. Bethe (B34) did not indicate all the steps explicitly and some authors seem to have gotten the false impression that the integration was *cut off* at a maximum value $q=1$. That was not the case, as we have shown above.

[20] These functions are such that if $F=0$ in the integrand in (3.36), the resulting term multiplying each c_p is just ϕ_u, aside from small terms of order δ.

atomic hydrogen ($Z=1$), and $2\phi_u$ [twice the unshielded expression (3.32)] are plotted as a function of the dimensionless [see (3.25) and (3.40)]

$$\Delta = a\delta. \tag{3.41}$$

The curves illustrate the transition from complete screening ($\Delta \ll 1$, corresponding to *high*-energy incident electrons) to weak screening ($\Delta \gg 1$ corresponding to *low*, but still relativistic, electron energies). Essentially Fig. 9 compares the bremsstrahlung cross section for an electron incident on a hydrogen atom, with the total cross section for an electron incident on an unshielded free proton and an unshielded free electron. We see that the hydrogen cross section is reduced for small Δ, essentially because of the reduced contribution of small momentum transfers below $q \sim 1/a$. In fact, the asymptotic values are[21]

$$\Delta \ll 1: \quad \phi_1 \to 8(\ln a + \tfrac{3}{2}) = 45.79,$$
$$\phi_2 \to 8(\ln a + \tfrac{4}{3}) = 44.46; \tag{3.42}$$
$$\Delta \gg 1: \quad \phi_1 \to \phi_2 \to 2\phi_u = 8[\ln (a/\Delta) - \tfrac{1}{2}].$$

Probably the most interesting result for general one-electron ions is contained in the expression (3.38) for $\zeta_{sc(1)}$. It shows that one term in the bremsstrahlung cross section is $(Z-1)^2$ times the cross section $d\sigma_p$ for an electron incident on an unshielded proton (or electron). For ultrahigh incident energies ($E_i \to \infty$, $\Delta \ll 1$) this term increases logarithmically and dominates. For a more detailed discussion and results of calculations for the general one-electron atom ($Z \neq 1$) the reader may be referred to the paper G69.

Finally, it is of interest to compare the expressions which determine the total cross section with the result where summation over n in (3.29) had *not* been performed. The result, which would then correspond to bremsstrahlung with the atom left in the initial (ground) state only, is

$$|\langle 0 | Z | 0 \rangle|^2 = [\zeta \text{ in } (3.38)] - (1 - F_Z{}^2) \tag{3.43}$$

so that the last term represents the contribution of excited states. The effects of excited states (and, in general, atomic electrons) are therefore largest for low-Z atoms. In fact, for hydrogen they contribute about half the total cross section. In the original Bethe–Heitler treatment (B34, BH34) of bremsstrahlung and electron shielding effects, this summation over final atomic states was not performed, and it remained for Wheeler and Lamb (WL39) to emphasize the importance of excited states. However in their initial

[21] For $\Delta \gg 1$, the expressions for ϕ_i computed from (3.36) actually do not approach the unshielded expressions due to approximations made in deriving the f_i in (3.37). This discrepancy is due to the "small terms" mentioned in Footnote 20, and comes in only for $\Delta \gtrsim 3$, making the ϕ_i calculated from (3.36) a little too large (see also Table I). Thus once the asymptotic unshielded expression is approached (around $\Delta \sim 1$), for larger Δ the unshielded asymptotic expression should be used.

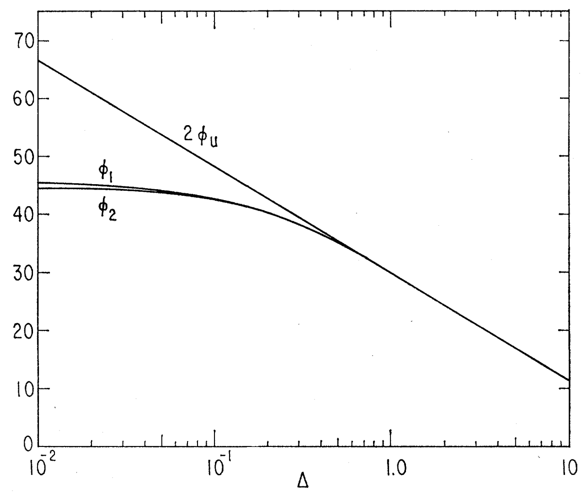

FIG. 9. The functions ϕ_1 and ϕ_2 for atomic hydrogen, as well as the unshielded function $2\phi_u$.

treatment, Bethe and Heitler were interested in applications to stopping power of electrons traversing high-Z material like Pb. For that problem it did not matter much how the atomic electrons were treated (in fact, they employed a Thomas–Fermi model for the atom). The total bremsstrahlung cross section is roughly proportional to $Z^2 + Z_{el}$, where Z is the nuclear charge and Z_{el} the number of atomic electrons. Thus the relative contribution of electrons is about $1/Z$ and is especially important for low-Z atoms. In this paper we have in mind applications to astrophysics. Since the cosmic abundances of hydrogen and helium are much larger than that of all the heavy elements combined (even when the abundances are multiplied by Z^2 factors), it is only these elements and their ions which we treat.

For two-electron or heliumlike atoms with the electronic wave function approximated by a product $\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)$, as with a one-parameter Hylleraas or with a product Hartree wave function, the atomic-scattering parameter (3.30) reduces to

$$\zeta_{sc(2)} = (Z-2)^2 + 4Z(1-F_1) - 2(1-F_1{}^2). \tag{3.44}$$

The result is valid for any Z, including $Z=1$ (H^- ion), but would be more accurate for large Z where a product expression is a better approximation to $\psi(\mathbf{r}_1, \mathbf{r}_2)$. In (3.44), F_1 is the single-electron form factor $\langle 0 | \exp(i\mathbf{q} \cdot \mathbf{r}_1) | 0 \rangle$, which for a one-parameter Hylleraas function is the same as the hydrogenic expression (3.39) with a_Z now replaced by

$$a_{Z_e} = a/Z_e = 68.5194/(Z - \tfrac{5}{16}). \tag{3.45}$$

The functions ϕ_1 and ϕ_2 to be used in (3.31) are again given by the general expression (3.36) which for heliumlike atoms contains terms with $p=1$ and 2, in addition to the constant term. In this case four integrals

TABLE II. Cross-section factors[a] when scattering system is p (or e), H, He^+, He.

Δ	p or e ϕ_u	H ϕ_1	H ϕ_2	He^+ ϕ_1	He^+ ϕ_2	He: Hylleraas (one parameter) ϕ_1	He: Hylleraas (one parameter) ϕ_2	He: Hartree–Fock ϕ_1	He: Hartree–Fock ϕ_2
0	∞	45.79	44.46	∞	∞	121.54	117.54	134.60	131.40
0.01	33.33	45.43	44.38	113.50	111.14	120.99	117.47	133.85	130.51
0.02	30.56	45.09	44.24	110.37	108.27	120.46	117.31	133.11	130.33
0.05	26.89	44.11	43.65	105.67	104.13	118.89	116.61	130.86	129.26
0.1	24.12	42.64	42.49	101.28	100.34	116.89	115.51	127.17	126.76
0.2	21.33	40.16	40.19	95.56	95.21	112.00	111.53	120.35	120.80
0.5	17.68	34.97	34.93	84.90	84.87	101.28	101.26	104.60	105.21
1	14.91	29.97	29.78	74.00	73.72	88.86	88.43	89.94	89.46
2	12.14	24.73	24.34	61.45	60.70	74.03	72.90	74.19	73.03
5	8.49	18.09	17.28	44.66	43.04	54.26	51.84	54.26	51.84
10	5.70	13.65	12.41	32.99	30.52	40.94	37.24	40.94	37.24

[a] See text and Footnote 21 concerning the values for $\Delta=5$ and 10.

are needed

$$I_i^{(p)}(\Delta;Z)=\int_\delta^1 (1-F_1^p)f_i(q;\delta)\,dq, \qquad (3.46)$$

where $p=1, 2$; $i=1, 2$. The asymptotic values of these integrals are, with F_1 computed from one-parameter Hylleraas functions,

$$I_1^{(1)}(0;Z)=I_2^{(1)}(0;Z)=\ln a_{Z_e}+\tfrac{1}{2},$$
$$I_1^{(2)}(0;Z)=I_2^{(2)}(0;Z)=\ln a_{Z_e}+\tfrac{11}{12}; \qquad (3.47)$$

$\Delta\gg1$:

$$I_1^{(1)}(\Delta;Z)\rightarrow I_1^{(2)}(\Delta;Z)\rightarrow\tfrac{1}{4}\phi_u-1,$$
$$I_2^{(1)}(\Delta;Z)\rightarrow I_2^{(2)}(\Delta;Z)\rightarrow\tfrac{1}{4}\phi_u-\tfrac{5}{6}, \qquad (3.48)$$

The large-Δ limits are, of course, just the unshielded expressions. For general intermediate Δ, values for the integrals (3.46) may be read from the curves in (G69).

The results have a very weak dependence on the particular approximate wave function for the heliumlike atom, since essentially the wave function only determines the precise effective lower limit $q_{e-\min}$ in the integral over the momentum-transfer distribution. This small-q effective cutoff is the shielding effect and depends on the spatial spread of the electron wave function. However, because of the nature of the integrands in (3.46), these integrals are roughly $-\ln q_{e-\min}$ so that the dependence on $\psi_{\text{He like}}$ is approximately logarithmic. To illustrate this, Table II gives the values of ϕ_1 and ϕ_2 for atomic helium, for a one-parameter Hylleraas function, and for a representation of a Hartree–Fock function consisting of the sum of two exponentials (MM65, G69). The Hartree–Fock values should be the most accurate since the ψ_{HF} gives a more accurate representation of the charge distribution. At low energies (large Δ) the ϕ's approach the unshielded expressions, independent of ψ_{He}. The dependence on ψ_{He} is greatest in the strong-shielding limit ($\Delta\rightarrow0$: high energies).

Also given in Table 2 are the cross-section factors when the scattering system is an unshielded proton or electron, a hydrogen atom, and a singly ionized helium ion. These are the most important species in a gas with a cosmic element abundance. It should be noted, however, that for $\Delta=5$ and 10, rather than the tabulated values, $(Z^2+Z_{e1})\phi_u$ should be used.[21]

3.6 Bremsstrahlung Spectrum and Total Energy Loss from Individual Electrons

For an electron passing through a medium containing various species (atoms, ions, and electrons) with number densities n_s, the bremsstrahlung spectrum *per electron* is

$$dN_i/dtdk=c\sum_s n_s(d\sigma_s/dk). \qquad (3.49)$$

Here dN_i represents the number of photons emitted with energy (or momentum) within dk, by an electron of initial energy E_i, and $d\sigma_s/dk=\Phi_s(k)$ would be taken from (3.31) with the appropriate $\phi_{i(s)}$. The bremsstrahlung spectrum is of the form k^{-1} for small k, exhibiting the well-known "infrared divergence." In this section we should first of all like to show the form of the spectrum for general k and its dependence on both the energy E_i of the incident electron, as well as on whether the target species is ionized or not. These effects can be illustrated readily for the case where the target species is neutral atomic hydrogen and ionized hydrogen. For the latter we consider the combined effect of the free protons and free electrons. Also, as in Sec. 3.2, it is convenient to denote the bremsstrahlung photon energy in terms of the initial electron energy $E_i(mc^2)$

$$E_i-E_f=k=\epsilon E_i. \qquad (3.50)$$

Moreover, to avoid the divergence at $k=0$, we shall

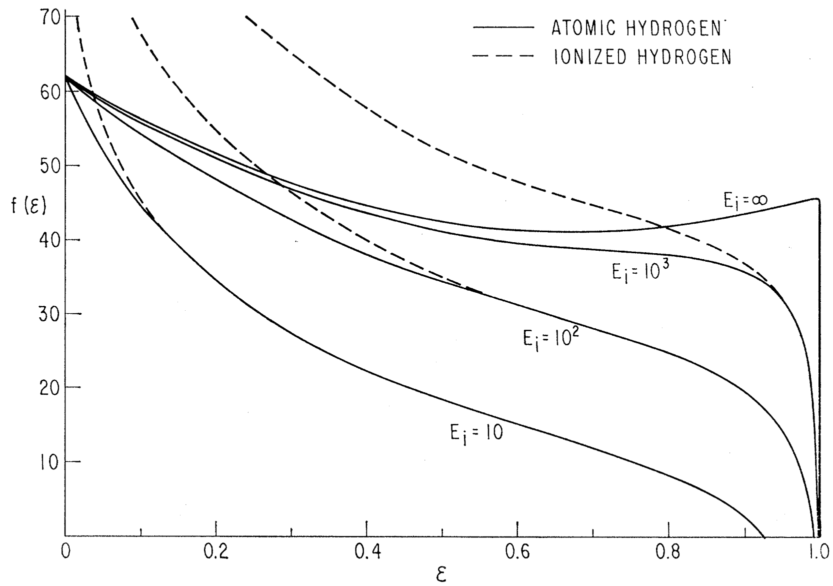

FIG. 10. Bremsstrahlung energy spectrum (3.51) for electrons incident on atomic hydrogen and on ionized hydrogen. For the latter the contribution from the target protons and electrons is summed; that is, $2\phi_u$ is used in (3.51) in place of ϕ_1 and ϕ_2 (see Fig. 9).

plot the *energy* emission spectrum $k(d\sigma/dk)$. This is given essentially by the factor

$$f(\epsilon)=[1+(1-\epsilon)^2]\phi_1-\tfrac{2}{3}(1-\epsilon)\phi_2. \qquad (3.51)$$

With the ϕ's taken from Fig. 9, this expression is plotted in Fig. 10 for $E_i=10$, 10^2, 10^3, and ∞. This figure illustrates several things. It shows how the magnitude and shape of the spectrum depends on whether the hydrogen is ionized or not. It also shows how, in particular, the shape of the spectrum depends on E_i; for more highly relativistic incident electrons the spectrum is more flat-topped. Some remarks must be made concerning the form of the spectrum for $E_i=10$ near the upper end. Figure 10 shows the distribution going to zero around $\epsilon=0.925$, but this should not be taken very seriously. The spectra were computed from Born-approximation expressions where, in addition, the extreme relativistic approximation (E_i, $E_f\gg 1$) was made. Because of this one can expect inaccuracies in the bremsstrahlung formula for ϵ near 1 where $E_f\sim 1$. However, the qualitative form of the $E_i=10$ spectrum for large ϵ should still exhibit the main effect correctly, and hence, the spectrum decrease slowly to zero and does not plunge to zero like the spectra for the more highly relativistic electrons.

Explicit simple expressions for the total energy loss,

$$-dE/dt=\int dk k(dN/dtdk), \qquad (3.52)$$

can be found only in the strong-shielding and weak-shielding limits. The latter case corresponds to low incident energies where $\phi_i\approx(Z^2+Z_{el})\phi_u$. Then for an over-all neutral plasma where $\sum Z_{el}=\sum Z$, one readily finds by integrating (3.52),

$$-(dE/dt)=4\alpha r_0^2c(\sum_Z n_Z Z(Z+1)(\ln 2E_i-\tfrac{1}{3})E_i$$

$$\text{(weak shielding or completely ionized).} \qquad (3.53)$$

In (3.53), the units of E are the same as the units (mc^2) of E_i. The expression would be appropriate for $E_i\lesssim 30/Z$ but would be exact for all E_i for a completey ionized medium. In the strong-shielding limit ($E_i\gtrsim 30/Z$), for the neutral components of the medium (or for the shielded *components*[22] of the bremsstrahlung cross section for ions) the ϕ's are constants (designated $\phi_{i(s)}$) and

$$-(dE/dt)=\alpha r_0^2c\sum_s n_s(\tfrac{4}{3}\phi_{1(s)}-\tfrac{1}{3}\phi_{2(s)})E_i$$

$$\text{(strong shielding).} \qquad (3.54)$$

In the intermediate case where neither the strong- or weak-shielding limit applies, the integration (3.52) must be done numerically for each energy E_i, using the appropriate $\phi_i(\Delta)=\phi_i(k;E_i)$. However, $\phi_1\approx\phi_2$ always, and the integral to be evaluated is essentially

$$\frac{1}{E_i^2}\int_0^{E_i}(\tfrac{4}{3}E_i^2-\tfrac{4}{3}E_ik+k^2)\phi(k;E_i)\,dk.$$

Since ϕ is a slowly varying function of k, it may be taken out of the integral and set equal to its value at the characteristic $k\approx\frac{1}{2}E_i$ (corresponding to $\Delta=a/2E_i$). In this approximation, a simple expression for the energy loss from the neutral species (or neutral components[22] of ions) can be written:

$$-(dE/dt)\approx\alpha r_0^2c[\sum_s n_s\phi_{(s)}(\Delta=a/2E_i)]E_i$$

$$\text{(general case).} \qquad (3.55)$$

Thus the total energy emission rate is essentially proportional to the instantaneous value of the electron energy. This does *not* mean that the electron energy decreases exponentially with time. Instead, in each

[22] For the explicit expression for the components for ϕ_1 and ϕ_2 for the He^+ ion, for example, see G69.

bremsstrahlung event the electron loses (on the whole) a large fraction of its energy. This effect is similar to that for Compton losses in the extreme Klein–Nishina limit (see Fig. 5) and will be discussed in more detail in the last part of this paper.

3.7 Bremsstrahlung from a Distribution of Electrons

As we did for Compton scattering, we now derive the total spectrum of bremsstrahlung photons resulting from a distribution of high-energy electrons. In the notation of this section where E_i (units: mc^2) is the (initial) energy of the incident electron and the differential number is $dN_e=N_e(E_i)dE_i$, the total bremsstrahlung spectrum would be found from

$$dN_{\rm tot}/dtdk=\int dE_iN_e(E_i)(dN_i/dtdk), \quad (3.56)$$

where the last factor in the integral would be taken from (3.49). Again, we take a power law for $N_e(E_i)$:

$$N_e(E_i)=K_eE_i^{-p}, \qquad E_0<E_i<E_m$$
$$=0, \qquad \text{otherwise.} \quad (3.57)$$

To simplify our formulas we shall assume that $k\ll E_m$, so that the upper limit to the integration in (3.56) does not contribute. The lower limit is

$$E_L=\max(k, E_0). \quad (3.58)$$

By (3.49) and (3.31), with k in units of mc^2,

$$\frac{dN_{\rm tot}}{dtdk}=\alpha r_0^2cK_ek^{-1}\sum_s n_s\int_{E_L}dE_iE_i^{-(p+2)}$$
$$\times[(2E_i^2-2E_ik+k^2)\phi_{1(s)}-\tfrac{2}{3}E_i(E_i-k)\phi_{2(s)}]$$
$$\equiv\alpha r_0^2cK_ek^{-1}\sum_s n_sI_s(k, E_L; p), \quad (3.59)$$

where we have defined I_s, the integrals which determine (in addition to the factor k^{-1}) the magnitude and form of the bremsstrahlung spectrum.

For species (s), or components[22] of species, and energies where the strong-shielding approximation applies, and the ϕ's are constants, the integral I_s over E_i can be evaluated exactly, giving

I_s(strong shielding)

$$=\left(\frac{2E_L^{-(p-1)}}{p-1}-\frac{2kE_L^{-p}}{p}+\frac{k^2E_L^{-(p+1)}}{p+1}\right)\phi_{1(s)}$$
$$-\frac{2}{3}\left(\frac{E_L^{-(p-1)}}{p-1}-\frac{kE_L^{-p}}{p}\right)\phi_{2(s)}. \quad (3.60)$$

When $E_L=k$, I_s in (3.60) is proportional to $k^{-(p-1)}$.

In the unshielded case or weak-shielding limit where species (s) consists of a nucleus (Z) and $Z_{\rm el}$ electrons, ϕ_1 and ϕ_2 approach

$$\phi_{\rm weak}=4(Z^2+Z_{\rm el})\{\ln[2E_i(E_i-k)/k]-\tfrac{1}{2}\}. \quad (3.61)$$

With this function for ϕ in I_s the integral cannot be evaluated exactly in terms of simple functions. However a simple approximate formula may be obtained by bringing the slowly varying factor ϕ outside the integrand and setting it equal to some typical value $\bar{\phi}$. That is, we set

$$I_s(\text{weak shielding})\approx I'\bar{\phi}_{\rm weak}, \quad (3.62)$$

where

$$I'=\int_{E_L}dE_iE_i^{-(p+2)}(\tfrac{4}{3}E_i^2-\tfrac{4}{3}E_ik+k^2)$$
$$=\frac{4}{3}\frac{E_L^{-(p-1)}}{p-1}-\frac{4}{3}\frac{kE_L^{-p}}{p}+\frac{k^2E_L^{-(p+1)}}{p+1}. \quad (3.63)$$

For $E_L=E_0$, we can simply set $E_i=E_0$ in (3.61) to get $\bar{\phi}_{\rm weak}$; for $E_L=k$ or for $k\approx E_0$, we can set $E_i\approx 2k$ in (3.61). Thus

$$\bar{\phi}_{\rm weak}\approx4(Z^2+Z_{\rm el})\{\ln[2E_0(E_0-k)/k]-\tfrac{1}{2}\} \qquad (k<E_0)$$
$$\approx4(Z^2+Z_{\rm el})(\ln 4k-\tfrac{1}{2}) \qquad (k> \text{ or} \approx E_0). \quad (3.64)$$

In the case of general or moderate shielding we can do essentially the same thing [approximating, as in (3.55), $\phi_{1(s)}\approx\phi_{2(s)}$]:

$$I_s(\text{moderate shielding})\approx I'\phi_{(s)}(\bar{\Delta}), \quad (3.65)$$

where

$$\bar{\Delta}\approx(a/2E_0)[k/(E_0-k)] \qquad (k<E_0)$$
$$\approx a/4k \qquad (k> \text{ or} \approx E_0). \quad (3.66)$$

4. SYNCHROTRON RADIATION

The problem of synchrotron radiation due to an electron traversing a magnetic field has been treated in numerous articles and reviews (W59, GS65, S49, GS64A, GSS68, GS69). In astrophysical applications, synchrotron radiation from a distribution of electrons may account for the radio, optical, or x-ray flux from an object, or, as believed is the case for the Crab nebula, all three fluxes. In this section we hope to clarify some of the conceptual problems in the derivation of the synchrotron formulas.

An electron of energy $E=\gamma mc^2$ spiraling along a magnetic-field line with pitch angle α (the constant angle between its velocity and **B**) spirals with angular frequency $\Omega=eB/\gamma mc$ independent of α. The spectrum of emitted photons can then be calculated either by considering the classical Liénard–Wiechert potentials seen by a distant observer, or by transforming to the electron's rest frame and calculating the Compton-scattered spectrum due to the virtual photons of the magnetic field. Because of the difficulties involved in transformations to the noninertial rest frame, we shall adopt the former approach here. To obtain the synchrotron spectrum, we first calculate the radiation emitted by an electron always moving perpendicular to the magnetic field; Bessel functions are introduced only near the end of the calculation. Lorentz transformations are later used to generalize this spectrum to arbitrary pitch angle. This method of introducing the pitch angle after the detailed calculation is completed, helps to clarify the difference between the emitted and received synchrotron spectra. In the final portion of this section, the radiation received from a distribution of electrons is calculated. Here the treatment is generally kept brief since synchrotron radiation is discussed in many articles and textbooks. For the same reason, the polarization properties of the spectrum and the effect of the medium through which the electron travels are not treated here; the interested reader is referred instead to (W59) and (GS65).

4.1 Synchrotron Total Energy-Loss Rate

We have already calculated the total synchrotron power from a highly relativistic electron moving in a magnetic field in Sec. 2.4. That derivation was from the Weizsäker–Williams approach and was specialized to the case where the electron's velocity was at a random direction to the magnetic field (a directional average was taken). The more general expression can also be derived very easily, however, with the help of the principle of covariance, and we shall outline its derivation here.

The basic procedure is to start from the well-known Larmor formula *valid in the nonrelativistic limit*:

$$P_{\text{emitted}} = -dE/dt = 2q^2\mathbf{a}^2/3c^3, \qquad (4.1)$$

where $\mathbf{a}$ is the acceleration of the charge $q(=-e)$. We try to find a covariant equation whose component reduces to this expression in the nonrelativistic limit, the idea being that (by the principle of covariance) if a covariant expression is found which is valid in one frame (where the electron is nonrelativistic), it is valid in all frames. A covariant factor which reduces to $\mathbf{a}^2$ in a frame where the velocity, acceleration, and energy loss are small is $m^{-2}(dp_\mu/d\tau)(dp_\mu/d\tau)$. Here $p_\mu=(iE/c,\mathbf{p})$ is the energy momentum four-vector. In terms of the four-vector velocity $v_\mu=\gamma(ic,\mathbf{v})$, $p_\mu=mv_\mu$, and $d\tau$ is the invariant proper time such that $d\tau=\gamma^{-1}dt=(\gamma ic)^{-1}dx_0$. Then (4.1) can be cast into the covariant form[23]

$$dp_\lambda = -\frac{2q^2}{3c^3}\frac{1}{m^2c^2}\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau}dx_\lambda. \qquad (4.2)$$

The covariant equation of motion is, in terms of the four-vector force K_μ and electromagnetic field tensor $F_{\mu\nu}$,

$$dp_\mu/d\tau = K_\mu = (q/c)F_{\mu\nu}v_\nu. \qquad (4.3)$$

For motion in a magnetic field with the z axis of a coordinate system lined up along $\mathbf{B}$, the only nonvanishing components of $F_{\mu\nu}(\mu,\nu=0,\ 1,\ 2,\ 3)$ are $F_{23}=-F_{32}=B$. Evaluating (4.3) and (4.2) we then

[23] The radiated power is thus, for synchrotron losses, an invariant. This is because for this process the energy is lost in the form of *radiation*, and for photons ΔE and Δt are parallel four-vectors (see Sec. 2.1).

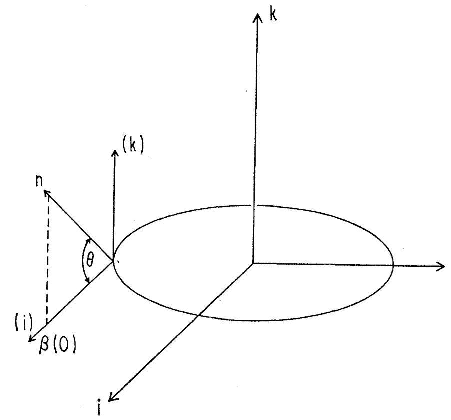

FIG. 11. Trajectory of electron in synchrotron calculation when the pitch angle is $\pi/2$.

get $(r_0=e^2/mc^2)$

$$dE/dt = -(2r_0^2/3c)\gamma^2B^2(v_x^2+v_y^2). \qquad (4.4)$$

Or, introducing the angle α between $\mathbf{v}$ and $\mathbf{B}$,

$$dE/dt = -(2r_0^2/3c)\gamma^2B^2v^2\sin^2\alpha. \qquad (4.5)$$

For an electron moving through a randomly oriented magnetic field, the average over angles in (4.5) yields (2.21).

Before proceeding to the problem of the synchrotron spectrum, we should like to make some remarks concerning the radiation reaction force due to synchrotron radiation. It should be noted, for example, that in (4.3) we include on the right-hand side only the force due to the *external* field and do not include a term Γ_μ corresponding to radiation reaction. Such a term would, in fact, be essential at very high energies if one were interested in determining the time evolution of the electron's orbit. For the appropriate form of Γ_μ in the extreme relativistic case the reader should be referred to the book by Landau and Lifshitz (LL62). The reaction force is, of course, due to the energy loss (4.2) itself and is opposite to the direction of motion. However, for the problem of the total energy loss this term in (4.3) can be ignored.[24] This is evident when one considers the process from the Weizsäker–Williams approach. Then, clearly all that comes in is the flux of virtual photons incident on the electron in its instantaneous rest frame. Thus (4.4), (4.5), and (2.21)

[24] When it is included and substituted into (4.2), the total energy loss becomes proportional to $K_\mu K_\mu+2K_\mu\Gamma_\mu+\Gamma_\mu\Gamma_\mu$, where K_μ is the force due to external fields and Γ_μ is the radiative reaction force. The first term yields just the standard expression (4.5). However, for a particle in a magnetic field, $K_0=0$, and $\mathbf{K}$ is perpendicular to $\mathbf{p}$. Then since $\mathbf{\Gamma}$, the radiative force, is parallel to $\mathbf{p}$, the cross term $K_\mu\Gamma_\mu$ vanishes. To calculate the third term we note that Γ_μ is approximately $c^{-1}(idE/d\tau,\ \mathbf{n}\cos\theta\ dE/d\tau)$, where $\mathbf{n}$ is the direction of the force and $\theta\sim\gamma^{-1}$ is the angular spread of the emitted radiation. Then, $\Gamma_\mu\Gamma_\mu\sim(c\gamma)^{-2}\times(dE/d\tau)^2$. But this term is only significant when $B\gamma\gtrsim e/r_0^2$ which is far above the energy at which quantum mechanical effects become important (see Sec. 2.4).

should be exact up to energies where quantum-mechanical effects come in (see Sec. 2.4). For the determination of the synchrotron *spectrum*, however, radiation reaction can be important, while at the same time not affecting the total energy loss result. This is because for the synchrotron spectrum it is the time evolution of the electron orbit which is *crucial*. Then it is necessary to consider radiation reaction. One can show that these effects come in for $\gamma^2 B \gtrsim e/r_0^2$, or, in terms of the synchrotron frequency ($\omega_s \sim \gamma^2 eB/mc$), for $\omega_s \gtrsim c/r_0$. The corresponding synchrotron photon energy is $\epsilon_s = \hbar\omega_s \gtrsim 70$ MeV. This criterion ($\gamma^2 B \gtrsim e/r_0^2$) corresponds to the situation where the electron loses a large fraction of its energy in one turn of its orbit. Clearly, extremely high-energy (70 MeV) synchrotron photons would be involved, and it is by no means clear that this case ever occurs in nature. In all that follows we shall assume that we are not in this domain and that γ is indeed essentially an adiabatic constant of the motion.

4.2 Radiation from an Electron Moving Perpendicular to the Field

We consider here the situation shown in Fig. 11: an energetic ($\gamma \gg 1$) electron spiraling around a magnetic-field line with $\alpha = \pi/2$. The electron's velocity is then given by

$$\boldsymbol{\beta} = \beta(\mathbf{i} \cos \Omega t + \mathbf{j} \sin \Omega t). \tag{4.6}$$

To find the spectrum of synchrotron radiation, we work in the laboratory frame and apply the expression for the energy emitted by a relativistic electron per unit *observer*'s time ($\bar{t}$) per unit solid angle in the direction $\mathbf{n}$ (see Fig. 11) given by (J62, p. 473),

$$\frac{dP(t)}{d\Omega_n} = \frac{e^2}{4\pi c} \frac{|\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^6}. \tag{4.7}$$

The total power as a function of time is then proportional to the square of a vector. Using the results (A1)–(A5) of the Appendix, the spectrum of radiation becomes,

$$dI_\omega / d\Omega_n = (e^2/4\pi^2 c) \, |\mathbf{f}_\omega|^2, \tag{4.8}$$

where

$$\mathbf{f}_\omega = \int_{-\infty}^{\infty} \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \boldsymbol{\beta}]}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \exp(i\omega\bar{t}) \, d\bar{t}. \tag{4.9}$$

Here, $\bar{t}$ is the observer's time, while t is the time at the electron's position: $\bar{t} = t + c^{-1}R(t)$, where $R(t) = |\mathbf{v}_{\text{obs}} - \mathbf{v}(t)|$, so

$$d\bar{t}/dt = 1 - \mathbf{n} \cdot \boldsymbol{\beta}. \tag{4.10}$$

For very distant observers, $\bar{t}$ may be approximated

$$\bar{t} \approx t - c^{-1} \mathbf{n} \cdot \mathbf{r}(t), \tag{4.11}$$

where a constant term has been ignored as contributing only an over-all phase factor to $\mathbf{f}_\omega$. After a change in variables in (4.9) from $\bar{t}$ to t, $\mathbf{f}_\omega$ may be integrated by parts using the relationship

$$(d/dt)[\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})/(1 - \mathbf{n} \cdot \boldsymbol{\beta})] = \mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]/(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2 \tag{4.12}$$

to obtain

$$\mathbf{f}_\omega = \omega \int_{-\infty}^{\infty} dt \, \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) \exp[i\omega(t - c^{-1}\mathbf{n} \cdot \mathbf{r})]. \tag{4.13}$$

It is well known that the angular distribution of radiation in the rest frame of an accelerated charge is a dipole distribution, which is roughly isotropic. Therefore, using the angle transformation (2.4), in analogy with Compton scattering for $\gamma \gg 1$, (in the lab) nearly all of the emitted synchrotron radiation makes at most an angle $\theta \sim \gamma^{-1}$ with the instantaneous velocity vector. Thus, the electron radiates in a given direction for a time $\Delta t \sim (\Omega\gamma)^{-1}$. For times significantly greater than this, the exponential in (4.13) oscillates very rapidly, essentially making the integrand zero. Therefore, for small t we may use the expansion (see J62, page 482)

$$t - c^{-1}\mathbf{n} \cdot \mathbf{r}(t) = \tfrac{1}{2}[(\theta^2 + \gamma^{-2})t + \Omega^2 t^3/3]. \tag{4.14}$$

The double cross product in (4.13) is most easily evaluated by using a different coordinate system. Letting $\mathbf{e} = \mathbf{n} \times \mathbf{j}$, the velocity (4.6) becomes,

$$\boldsymbol{\beta} = \beta(\mathbf{j} \sin \Omega t + \mathbf{e} \sin \theta \cos \Omega t + \mathbf{n} \cos \theta \cos \Omega t). \tag{4.15}$$

Then, to lowest order in θ and Ωt,

$$\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) = (\mathbf{e}\theta + \mathbf{j}\Omega t). \tag{4.16}$$

Substituting these expressions back into (4.13) and letting $\xi = \Omega t$, we have,

$$\mathbf{f}_\omega = \frac{\omega}{\Omega} \int_{-\infty}^{\infty} d\xi (\mathbf{j}\xi + \mathbf{e}\theta) \exp\left\{(i\omega/2\Omega)\left[(\theta^2 + \gamma^{-2})\xi + \frac{\xi^3}{3}\right]\right\}. \tag{4.17}$$

With $\mu = \omega/2\Omega$ and $\eta^2 = \theta^2 + \gamma^{-2}$, it is clear from (4.17) that $\mathbf{f}_\omega$ will be largest for $\mu \sim \gamma^3 \gg 1$. Using these parameters, the square of $\mathbf{f}_\omega$ becomes

$$|\mathbf{f}_\omega|^2 = \frac{\omega^2}{\Omega^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (xy + \theta^2) \times \exp\left\{i\mu\left[\eta^2(x - y) + \frac{(x^3 - y^3)}{3}\right]\right\}. \tag{4.18}$$

This expression can be simplified by changing variables to $x = u + v$ and $y = u - v$, and noting that the Jacobian is 2:

$$|\mathbf{f}_\omega|^2 = \frac{2\omega^2}{\Omega^2} \int_{-\infty}^{\infty} dv \exp[2i\mu(\eta^2 v + \tfrac{1}{3}v^3)] \times \int_{-\infty}^{\infty} du (u^2 - v^2 + \theta^2) \exp(2i\mu v u^2). \tag{4.19}$$

The u integration can now be performed to yield

$$|\mathbf{f}_\omega|^2 = \frac{2(\pi)^{1/2}\omega^2}{\Omega^2}\int_{-\infty}^{\infty} dv \exp\left[2i\mu(\eta^2 v+\tfrac{1}{3}v^3)\right] \times\left[(2\mu v)^{-1/2}e^{i\pi/4}(\theta^2-v^2)-\tfrac{1}{2}(2\mu v)^{-3/2}e^{-i\pi/4}\right]. \quad (4.20)$$

To calculate I_ω from (4.8) it is necessary to integrate over solid angle. Then, since θ is the colatitude, $d\Omega_n = 2\pi \sin[(\pi/2)-\theta]d\theta \approx 2\pi d\theta$. The factor of 2π arises because I_ω is the energy emitted per revolution, not just that seen by observers in one particular plane. Because the integrand falls off long before $\theta=\pi/2$, the limits on the θ integration may be extended to infinity. Then, substituting (4.20) into (4.8) and using $\eta^2=\theta^2+\gamma^{-2}$, the θ integration becomes identical to the u integration in (4.19) and we obtain,

$$I_\omega = -\frac{e^2\omega i}{\Omega c}\int_{-\infty}^{\infty} dv\left[v-(2\mu i v^2)^{-1}\right]\exp\left[2\mu i(v\gamma^{-2}+\tfrac{1}{3}v^3)\right]. \quad (4.21)$$

The second term in brackets may be integrated by parts to yield,

$$I_\omega = \frac{-e^2\omega i}{\Omega c}\int_{-\infty}^{\infty} dv\left[2v-(\gamma^2 v)^{-1}\right]\exp\left[2\mu i(v\gamma^{-2}+\tfrac{1}{3}v^3)\right]. \quad (4.22)$$

The above expression can be simplified by setting

$$I_\omega = -(e^2\omega i/\Omega c)\left[I_\omega^{(1)}+I_\omega^{(2)}\right], \quad (4.23)$$

where the $I_\omega^{(j)}$ correspond to the integrals of the two terms in brackets in (4.22). In calculating the first integral, we set $v=x/\gamma$ and $\xi=4\gamma^{-3}\mu/3$:

$$I_\omega^{(1)} = 2\gamma^{-2}\int_{-\infty}^{\infty} x\,dx \exp\left[i\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\right]. \quad (4.24)$$

The integrand of this expression can itself be expressed as an integral to yield

$$I_\omega^{(1)} = \frac{-i}{\gamma^2}\int_{2\omega/3\Omega\gamma^3}^{\infty} d\xi \int_{-\infty}^{\infty} dx(3x^2+x^4)\exp\left[i\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\right]. \quad (4.25)$$

To calculate $I_\omega^{(2)}$, we note that

$$\gamma^2 I_\omega^{(2)} = \int_{-\infty}^{\infty} dv v^{-1}\exp\left[2i\mu(v\gamma^{-2}+\tfrac{1}{3}v^3)\right]; \quad (4.26)$$

differentiating with respect to γ^{-2} and then integrating from γ^{-2} to infinity yield

$$I_\omega^{(2)} = \frac{-2i\mu}{\gamma^2}\int_{\gamma^{-2}}^{\infty} dy \int_{-\infty}^{\infty} dv \exp\left[2i\mu(vy+\tfrac{1}{3}v^3)\right]. \quad (4.27)$$

Finally, since $\mu=\omega/2\Omega$, the change of variables $x=vy^{-1/2}$ and $\xi=\frac{4}{3}\mu y^{3/2}$ yields

$$I_\omega^{(2)} = \frac{-i}{\gamma^2}\int_{2\omega/3\Omega\gamma^3}^{\infty} d\xi \int_{-\infty}^{\infty} dx \exp\left[i\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\right]. \quad (4.28)$$

To obtain the total spectrum, we substitute these results into (4.23) and get

$$I_\omega = -\frac{e^2\omega}{\gamma^2\Omega c}\int_{2\omega/3\Omega\gamma^3}^{\infty} d\xi \int_{-\infty}^{\infty} dx(1+3x^2+x^4) \times \exp\left[i\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\right]. \quad (4.29)$$

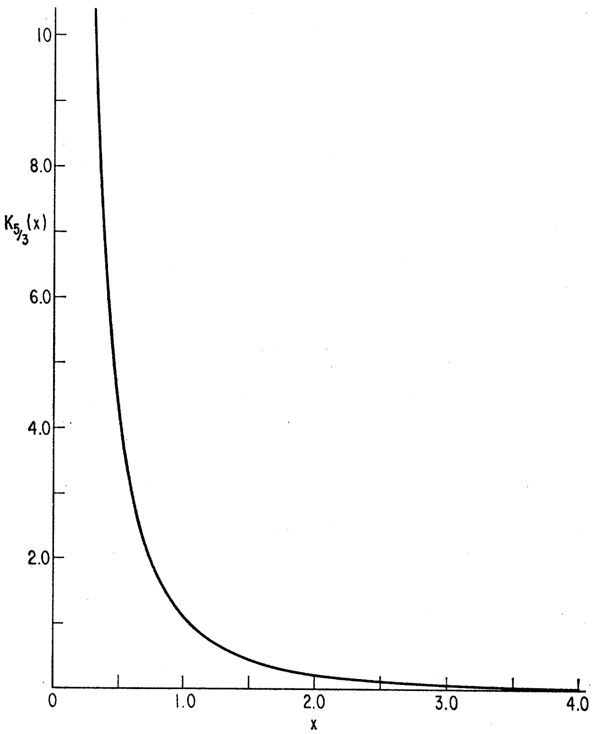

FIG. 12. Graph of the function $K_{5/3}(x)$.

However, the integral over x is just $-(2/\sqrt{3})K_{5/3}(\xi)$, and the spectrum is[25]

$$I_\omega = \frac{2e^2\omega}{\sqrt{3}\gamma^2\Omega c}\int_{2\omega/3\Omega\gamma^3}^{\infty} d\xi K_{5/3}(\xi), \quad (4.30)$$

where $K_{5/3}(\xi)$ is the modified Bessel function of 5/3 order (see Fig. 12). Since (4.30) is the spectral energy per revolution, to obtain the power emitted, (4.30) must be multiplied by $(2\pi)^{-1}\Omega$. Doing this and changing from ω to $\nu=(2\pi)^{-1}\omega$, the spectral power becomes (see Fig. 13)

$$P(\nu) = \frac{\sqrt{3}e^3 B}{mc^2}\frac{\nu}{\nu_c}\int_{\nu/\nu_c}^{\infty} d\xi K_{5/3}(\xi), \quad (4.31)$$

where the critical frequency, ν_c, is defined as

$$\nu_c = 3\Omega\gamma^3/4\pi = 3eB\gamma^2/4\pi mc. \quad (4.32)$$

Equation (4.31) gives the total instantaneous power emitted by an electron spiraling around a B field with pitch angle $\alpha=\pi/2$. The distribution $P(\nu)$ is peaked near $\nu\approx\nu_c$. At high frequencies, $\nu\gg\nu_c$, the spectrum

[25] This can most easily be seen by using the relationship, $K_{5/3}=-K_{1/3}-2K_{2/3}$, and the standard forms for $K_{1/3}$ and $K_{2/3}$ given in (J62).

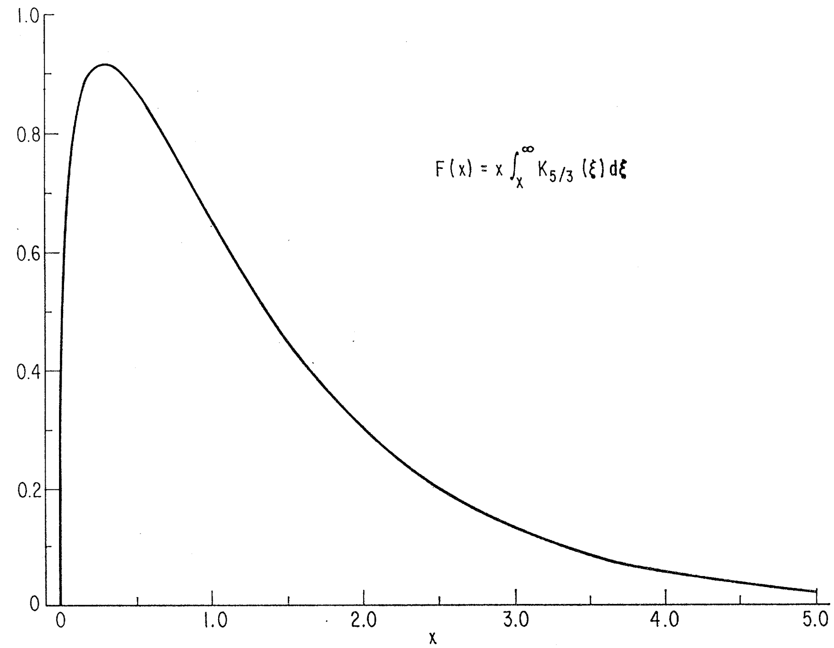

FIG. 13. Synchrotron spectrum from a single electron as a function of $x=\nu/\nu_c$.

approaches

$$P(\nu)\cong(\tfrac{3}{2}\pi)^{1/2}(e^3B/mc^2)\,(\nu/\nu_c)^{1/2}e^{-\nu/\nu_c}. \quad (4.33)$$

Meanwhile, at very low frequencies, $\nu\ll\nu_c$, the spectrum (4.31) may be approximated by

$$P(\nu)\cong[4\pi e^3B/\Gamma(\tfrac{1}{3})mc^2](\nu/2\nu_c)^{1/3}. \quad (4.34)$$

In our derivation of the synchrotron spectrum, (4.31), we failed to take account of the fact that the electron's motion repeats itself with period $2\pi/\Omega$. Therefore, the spectral decomposition should be performed using Fourier *series* rather than integrals. Such a calculation as performed in (W59) and (GS65) shows that the synchrotron spectrum is really discrete with the emitted frequency, ν, being an integral multiple of $\Omega/2\pi$, the gyration frequency. However, since most of the power is emitted at frequencies $\nu\gg\Omega$, it follows that for all but the very low frequency part of the spectrum, the synchrotron radiation distribution may be regarded as continuous, and (4.31) may be used.

4.3 Radiation from an Electron with Arbitrary Pitch Angle

In the previous section we calculated the spectral power emitted by an electron spiraling in a magnetic field with pitch angle $\pi/2$. In that situation, the power emitted by a single electron is equal to the power received by a distant observer. Here, we calculate the power emitted and received from an electron moving at arbitrary pitch angle α; it will be seen that the power emitted does not equal the power received because the average distance between electron and observer changes with time.

Consider now an electron with pitch angle α spiraling along a magnetic field in the lab system K. This electron then has the equation of motion,

$$d(\gamma mc\boldsymbol{\beta})/dt=-e\boldsymbol{\beta}\times\mathbf{B}, \quad (4.35)$$

from which it follows that β, γ, and $\beta_{||}$ (the component of $\boldsymbol{\beta}$ parallel to $\mathbf{B}$) are all constants of the motion. Then,

$$d\boldsymbol{\beta}_\perp/dt=\boldsymbol{\Omega}\times\boldsymbol{\beta}_\perp, \quad (4.36)$$

where,

$$\boldsymbol{\Omega}=e\mathbf{B}/\gamma mc \quad (4.37)$$

as before. Thus the frequency of rotation is independent of pitch angle.

To calculate the synchrotron distribution in the lab system K, we transform to a coordinate system K' moving with velocity $\boldsymbol{\beta}_{||}$ with respect to K. Then, in K', the electron spirals with pitch angle $\alpha'=\pi/2$. Since $\beta_{||}=\beta\cos\alpha$, for $\gamma\gg1$ and $\alpha\gg\gamma^{-1}$, the Lorentz factor Γ connecting K and K' is given by

$$\Gamma=(1-\beta^2\cos^2\alpha)^{-1/2}\approx1/\sin\alpha. \quad (4.38)$$

Since the relative motion between the two frames is parallel to the magnetic field, $B'=B$. Also, transforming $\gamma=E/mc^2$, the zeroth component of a four-vector, yields

$$\gamma'=\Gamma^{-1}\gamma=\gamma\sin\alpha. \quad (4.39)$$

Because the same equation of motion (4.35) is applicable in K' as well as in K, it follows that the frequency of rotation in K' is given by

$$\Omega'=eB'/\gamma'mc=\Omega/\sin\alpha. \quad (4.40)$$

To connect the spectra in K and K', we make use of the

Doppler-shift formula

$$\nu' = \Gamma^{-1}(1+\beta\cos\theta')^{-1}\nu, \qquad (4.41)$$

where θ' is the angle the radiation makes with the direction of $\mathbf{B}'$. However all of the radiation in K' is within a cone of angle $\sim 1/\gamma' \ll 1$ around the direction of the electron motion. Therefore, $\theta' = \pi/2 + O(\gamma'^{-1})$, which means that $\cos\theta' \ll 1$, and (4.41) reduces to

$$\nu' = \nu \sin\alpha. \qquad (4.42)$$

In Sec. 4.1 it was shown that for radiative losses the electron's energy loss rate is a Lorentz invariant. Therefore, $P_{\text{emitted}} = P'$, which implies that

$$P_{\text{emitted}}(\nu) = (d\nu'/d\nu)P'(\nu') = \sin\alpha P'(\nu'). \qquad (4.43)$$

Since $P'(\nu')$ is given by equation (4.31) calculated for $\alpha' = \pi/2$, $P_{\text{emitted}}(\nu)$ is obtained from (4.43) by substituting the correct expressions for ν', γ', and B' obtained above. The result is

$$P_{\text{emitted}}(\nu) = \frac{\sqrt{3}e^3 B\sin\alpha}{mc^2}\frac{\nu}{\nu_c}\int_{\nu/\nu_c}^{\infty} d\xi K_{5/3}(\xi), \qquad (4.44)$$

where ν_c is now redefined as

$$\nu_c = (3eB\gamma^2/4\pi mc)\sin\alpha. \qquad (4.45)$$

This is the standard expression for the synchrotron spectrum.

It has recently been pointed out (EF67, S68, GSS68) that when $\alpha \neq \pi/2$, the power emitted by an electron does not equal the received power. To calculate P_{received}, we note that power received is proportional to the flux at the observer:

$$P_{\text{received}}(\nu)\,d\nu \propto \nu^2[dn(\nu)/\nu]. \qquad (4.46)$$

Then, using the invariance of $\nu^{-1}dn$, we have

$$P_{\text{received}}(\nu) = (\nu^2/\nu'^2)(d\nu'/d\nu)P'(\nu') = (\sin\alpha)^{-1}P'(\nu'). \qquad (4.47)$$

The received spectrum thus becomes

$$P_{\text{received}}(\nu) = P_{\text{emitted}}(\nu)/\sin^2\alpha, \qquad (4.48)$$

where P_{emitted} is given by (4.44). One can easily see that the total power follows the same relationship as (4.48). From (4.5), using $P_{\text{emitted}} = P'$ we have

$$P_{\text{received}} = \tfrac{2}{3}r_0^2 cB^2\gamma^2, \qquad (4.49)$$

and therefore the total received power is independent of pitch angle.

The physical cause for the received power not equaling the emitted power can be seen from Fig. 14. Since the period of emission, $\tau = 2\pi\Omega^{-1}$, does not equal the observed time between pulses,

$$\tau_{\text{received}} = \tau\sin^2\alpha. \qquad (4.50)$$

This can also be seen by applying the Doppler formula (4.41) to (4.40) for Ω_{received}. From energy conservation, the energy emitted in one period must equal that received in one period which gives

$$P_{\text{received}} = P_{\text{emitted}}\tau/\tau_{\text{received}} = P_{\text{emitted}}/\sin^2\alpha. \qquad (4.51)$$

Actually, (4.51) can be obtained by considering a conservation-of-energy equation in which the total emitted power equals the energy flux through a surface at the observer, plus the change in total energy contained within the surface; this change being due to the changing distance between electron and observer (see Fig. 14).

Thus, we see that the equations calculated for an electron spiraling with pitch angle $\pi/2$ are easily generalized to arbitrary α. The complication that the power emitted does not equal the power received is essentially due to the ever-changing average (over period) distance between electron and observer. For a distribution of electrons confined within a given region, however, the emitted and received powers are identical since on the average the distance between the electrons and the observer does not change with time. This problem is treated in the next section.

4.4 Synchrotron Spectrum from a Distribution of Electrons

In the last section we showed that for a given electron, the power emitted differs from the power received by a factor of $\sin^2\alpha$. In this section we calculate the total power received per unit volume per unit frequency, $dW/d\nu dt$, from a distribution of electrons in a magnetic field. It will turn out that for electrons emitting from a fixed region of space, the received spectrum obeys the classical formulas (W59) and equals the emitted spectrum (GSS68).

Let $N_{\text{obs}}(\gamma, \alpha, \mathbf{r}, t)\,d\gamma d\Omega_\alpha r^2 dr d\Omega$ be the total number of *observed* electrons within $r^2 dr d\Omega$ with energy within $d\gamma$ and pitch angle within $d\Omega_\alpha$. Then, using $P_{\text{received}}(\nu)$ from (4.48), the received synchrotron spectrum becomes

$$dW/d\nu dt = \iint P_{\text{received}}(\nu) N_{\text{obs}}(\gamma, \alpha, \mathbf{r}, t)\,d\gamma d\Omega_\alpha. \qquad (4.52)$$

This is the energy received from a fixed unit volume at the position $\mathbf{r}$ at the observer's time t. Since N_{obs} is not a very useful quantity to work with, we define $N(\gamma, \alpha, \mathbf{r}, t)\,d\gamma d\Omega_\alpha$ as the density of electrons at $\mathbf{r}$ at time t with pitch angle within $d\Omega_\alpha$ and energy within $d\gamma$.

Now, consider the electrons within the volume element $r^2 dr d\Omega$ shown in Fig. 15. Letting α' be the angle between $\mathbf{B}$ and the observer, we see that $\alpha' = \alpha + O(\gamma^{-1})$. This figure therefore shows that if the electron spends a time t within $r^2 dr d\Omega$, it is observed to radiate only for a time,

$$t_{\text{obs}} = t(1-\beta\cos\alpha\cos\alpha') \approx t\sin^2\alpha. \qquad (4.53)$$

An electron within that volume element is therefore observed to radiate only for a fraction $\sin^2\alpha$ of the time it spends within the volume. Taking into account retardation effects, one can relate N and N_{obs}. The

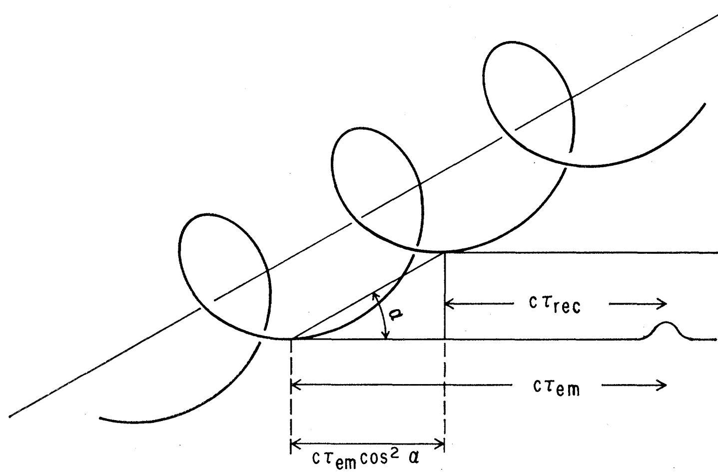

FIG. 14. Distance between received pulses, $c\tau_{rec}$.

result obtained by (GSS68) is

$$N_{obs}(\gamma, \alpha, \mathbf{r}, t) = \sin^2\alpha\, N(\gamma, \alpha, \mathbf{r}, t-r/c). \quad (4.54)$$

Using this, along with the result (4.48) in (4.52), we obtain for the received spectrum

$$dW/d\nu dt = \iint P_{emitted}(\nu) N(\gamma, \alpha, \mathbf{r}, t-r/c)\, d\gamma d\Omega_\alpha. \quad (4.55)$$

But this is just the synchrotron *emission* spectrum per unit volume at the time $t-r/c$. Therefore, when the distribution function N is not time dependent, the emitted and received synchrotron spectra from a fixed volume in space are equal. If synchrotron emission from a given moving volume of electrons is being considered, then the integral over all space of equation (4.55) will give the correctly Doppler-shifted result for the total received flux. Indeed, if the distribution function for a single electron (a delta function) is used in equation (4.55), the result (4.48) is recovered.

We now calculate the total synchrotron emission spectrum per unit volume from a power-law electron energy distribution. We further assume that the distribution function can be put in the form

$$N(\gamma, \alpha, \mathbf{r}, t) = k\gamma^{-p} N(\alpha)/4\pi, \quad (4.56)$$

with γ contained within some range $\gamma_1 < \gamma < \gamma_2$. Then, if k contains no time dependence, substituting this form of N into (4.55) also gives the total received spectrum per unit volume. The factor of $1/4\pi$ appears above so that the spectrum reduces to $k\gamma^{-p}$ when there is no pitch angle dependence.

Substituting the above spectrum into (4.55) for the total synchrotron emission yields

$$\frac{dW}{d\nu dt} = \frac{\sqrt{3}ke^3B}{4\pi mc^2} \int d\Omega_\alpha N(\alpha) \sin\alpha \times \int_{\gamma_1}^{\gamma_2} d\gamma \gamma^{-p} \frac{\nu}{\nu_c} \int_{\nu/\nu_c}^{\infty} d\xi K_{5/3}(\xi). \quad (4.57)$$

In order that the result be expressible in terms of known functions, we make the assumption that the end points of the electron energy spectrum do not contribute. This will be true when γ_1 and γ_2 are such that $\nu_c(1) \ll \nu$ and $\nu_c(2) \gg \nu$. Then, the limits on the γ integration may be replaced by zero and infinity. This integral is then performed in (W59) to yield

$$\frac{dW}{d\nu dt} = \frac{\sqrt{3}ke^3B}{4\pi mc^2} \left(\frac{2\pi mc\nu}{3eB}\right)^{-(p-1)/2} (p+1)^{-1} \Gamma[\tfrac{1}{12}(3p-1)] \times \Gamma[\tfrac{1}{12}(3p+19)] \int d\Omega_\alpha (\sin\alpha)^{(p+1)/2} N(\alpha). \quad (4.58)$$

In the case of local isotropy, $N(\alpha) = 1$, and the integral over α can be performed. The final result is

$$\frac{dW}{d\nu dt} = \frac{4\pi ke^3 B^{(p+1)/2}}{mc^2} \left(\frac{3e}{4\pi mc}\right)^{(p-1)/2} a(p) \nu^{-(p-1)/2}, \quad (4.59)$$

where

$$a(p) = \frac{2^{(p-1)/2}\sqrt{3}\,\Gamma[(3p-1)/12]\Gamma[(3p+19)/12]\Gamma[(p+5)/4]}{8\pi^{1/2}(p+1)\Gamma[(p+7)/4]}. \quad (4.60)$$

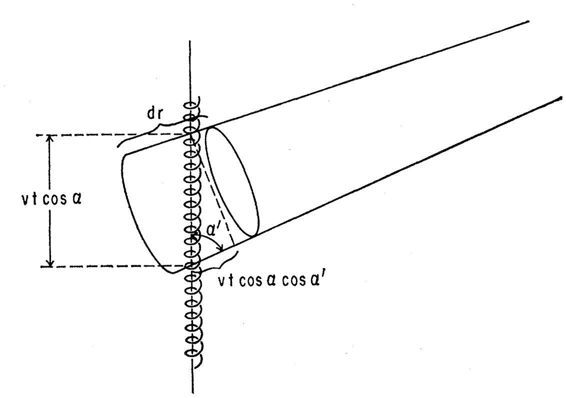

FIG. 15. An electron of pitch angle α spiraling through the fixed volume element $r^2 dr d\Omega$.

A table of values for $a(p)$ is given in Table III, but for $1.5<p<5$, $a(p)$ is approximately equal to 0.1. Equation (4.59) then represents the total synchrotron emission of an isotropic distribution of electrons.

Finally, it is worthwhile noting from (4.58) that starting from a power-law electron spectrum with index p, the emitted (and received) synchrotron distribution becomes a power law with index $(p-1)/2$. This is exactly the same situation as for Thomson scattering found in Sec. 2 of this review.

TABLE III. $a(p)$ in Eq. (4.60) [from (GS65)].

p	$a(p)$
1	0.283
1.5	0.147
2	0.103
2.5	0.0852
3	0.0742
4	0.0725
5	0.0922

5. EFFECTS ON THE SPECTRUM OF HIGH-ENERGY ELECTRONS

So far in this review, we have calculated the photon spectrum due to Compton scattering, bremsstrahlung, and synchrotron radiation from a single very high-energy electron. We have also used a power-law electron energy spectrum in order to calculate the *total* radiation spectrum for the above processes. In general, this radiation spectrum depends very strongly upon the electron energy spectrum.

In this section it will be seen that the electron energy spectrum is in turn determined by the processes causing the electrons to lose energy. When electrons lose only a very small portion of their energy in one collision, their energy spectrum is governed by a differential equation. On the other hand, when electrons are likely to lose a significant portion of their energy in one collision, it becomes necessary to use an integro-differential equation to calculate their spectrum. The latter occurs in the case of bremsstrahlung or Compton scattering in the Klein–Nishina limit when an electron's energy may change with time as in Fig. 5. Previous articles (FM66, K62) have often used just a differential equation to obtain an approximate solution in this case, but the results of this section indicate that, for example, for bremsstrahlung losses, the exact result may differ by about thirty percent.

After considering the general equation for the electron distribution function, we shall present solutions for certain special cases of astrophysical interest. Because this section is rather mathematical, no attempt is made to present all possible solutions to these equations; instead the reader is referred to other sources (FM66, GS64a, K62, M61). Perhaps this section could be used as a starting point for future, more detailed treatments of these problems.

5.1 Continuity and Integro-differential Equations for the Electron Distribution

a. Continuity Equation

In certain energy-loss mechanisms for high energy electrons, such as Thomson scattering or synchrotron radiation, the energy lost by one electron or the emitted photon energy in one collision is much less than the energy of the electron. Denoting $\dot{E}$ = the total rate of energy loss for an electron as a function of E, we have the condition

$$-\dot{E}/E\ll Nc\sigma, \tag{5.1}$$

where σ is the total cross section for the processes considered and N is the number density of objects scattering off the electrons. We assume $\beta\rightarrow 1$, or the relative velocity is equal to c. The above inequality states that the fractional loss of energy in a given time interval must be much less than the number of collisions in that time interval.

When the above relation holds, it is possible to write a differential equation for the electron energy spectrum. If $N_e(E,t)dE$ is the number of electrons with energy within dE at time t, then $N_e(E,t)\dot{E}(E)$ is the flux of electrons entering the internal dE, and $N_e(E+dE,t)\dot{E}(E+dE)$ is the flux leaving the interval. Equating the increase of electrons within dE to the total flux entering the interval, we obtain

$$\partial N_e/\partial t+(\partial/\partial E)(\dot{E}N_e)=\sum_i Q_i(E,t), \tag{5.2}$$

where the $Q_i(E,t)$ represent sources and sinks of high-energy electrons corresponding to possible production, annihilation, or gradual leakage from the region of space considered. The above equation is the continuity equation in energy space for electrons. Often, when there is a loss of electrons to the system by either annihilation or leakage, it is possible to set one of the $Q_i(E,t)=-N_e/T$, where T is the characteristic time of the loss of electrons.[26] The continuity equation then becomes

$$\partial N_e/\partial t+(\partial/\partial E)(\dot{E}N_e)+N_e/T=\sum_i Q_i(E,t). \tag{5.3}$$

Ginzburg and Syrovatskii (GS64A) obtain the general solution of (5.3):

$$N_e(E,t)=\int dE_0\int_{-\infty}^{t}dt_0G(E,t;E_0,t_0)\sum_i Q_i(E_0,t_0), \tag{5.4}$$

where the Green's function is given by

$$G(E,t;E_0\,t_0)=|\,\dot{E}\,|^{-1}e^{-\tau/T}\delta(t-t_0-\tau), \tag{5.5}$$

[26] For example, if electrons are completely removed from the system by collisions, $T\sim(Nc\sigma)^{-1}$, or if leakage occurs, T is the characteristic time for an electron to leave the system.

with

$$\tau=\int_{E_0}^{E}\frac{dE}{\dot{E}(E)}. \quad (5.6)$$

For most energy-loss processes for electrons, there is a spread in the photon distribution emitted from a single electron. Therefore, an electron loses energy in random increments determined by the photon spectrum. For this reason it may be necessary to include on the left-hand side of (5.3) an energy-space diffusion term of the form, $-\frac{1}{2}\partial^2(DN_e)/\partial E^2)$. Here D represents the mean-squared energy change per unit time. For the electron interactions considered in this review, the fluctuations are small, and the diffusion term becomes unimportant. This term is usually kept, however, if statistical acceleration is being considered. This is described in further detail in (GS64A) and (M61).

b. Integro-differential Equation

Processes such as bremsstrahlung and Compton scattering in the Klein–Nishina limit cause an electron to give up a large fraction of its energy to one photon, and therefore condition (5.1) will not hold. It then becomes necessary to consider the random-walk characteristics of an electron's energy and to use an integral to represent the flux of electrons into the interval dE.

Let $P(E, E-\epsilon)d\epsilon dt$ be the probability that an electron with energy E will undergo a collision causing it to lose an amount of energy between ϵ and $\epsilon+d\epsilon$ in time dt. Again, equating the flux of electrons entering dE to the increase in density, we obtain the integro-differential equation

$$\frac{\partial N_e(E,t)}{\partial t}+\frac{\partial}{\partial E}[\dot{E}N_e(E,t)]+N_e(E,t)\times\int_{mc^2}^{E}dE'P(E,E')-\int_{E}^{\infty}dE'N_e(E',t)P(E',E)=\sum_i Q_i(E,t). \quad (5.7)$$

Here $\dot{E}$ represents the total energy loss due to those processes for which (5.1) is valid, while $P(E, E')$ represents the total probability of emitting a photon within $d\epsilon$ by those processes for which (5.1) does not hold.

5.2 Solution for Synchrotron and Thomson-Scattering Losses

When an electron loses energy by Thomson scattering and synchrotron radiation only, it is not likely to lose a large fraction of its energy to one photon, and condition (5.1) holds. This follows from (2.18) since

$$-\dot{E}/E=\tfrac{4}{3}\sigma_T cN[\gamma\langle\epsilon\rangle/mc^2]\ll\sigma_T cN. \quad (5.8)$$

Since (2.18) is valid for both Thomson scattering and synchrotron radiation, it follows that the continuity equation (5.2) may be used to describe the electron distribution function. Using (2.18) for the total energy loss

$$-\dot{E}=\tfrac{4}{3}\sigma_T c(E^2/m^2c^4)\langle\mathcal{E}\rangle\equiv aE^2, \quad (5.9)$$

we obtain the continuity equation

$$\partial N_e/\partial t-(\partial/\partial E)(aE^2N_e)=\sum_i Q_i(E,t). \quad (5.10)$$

In many astrophysical applications, the source function for high-energy electrons takes the form (GB67, GS64)

$$Q(E,t)=KE^{-\Gamma}. \quad (5.11)$$

This is essentially the source function for secondary cosmic-ray electrons (cf. GB67) or for particles accelerated by statistical mechanisms (cf. GS64A). Then, assuming steady-state conditions, as we will for the remainder of this section,[27] (5.10) becomes

$$-(\partial/\partial E)[aE^2N_e(E)]=KE^{-\Gamma}, \quad (5.12)$$

with solution

$$N_e(E)=[K/a(\Gamma-1)]E^{-(\Gamma+1)}. \quad (5.13)$$

Thus, both synchrotron radiation and Thomson scattering steepen the electron distribution spectral index by one. This is just the power-law electron spectrum used earlier to obtain the total flux of photons due to these processes.

When the leakage loss, $-N_e/T$, is present along with synchrotron and Thomson scattering, it is possible to use the Green's function (5.5) to obtain the solution[28]

$$N_e(E)=\frac{K\exp(-1/aTE)}{aE^2}(aT)^{\Gamma-1}\int_0^{1/aTE}d\xi\xi^{\Gamma-2}e^{\xi}. \quad (5.14)$$

If the losses due to leakage are much less than those due to radiation, then $T\gg1/aE$, and the exponentials may be expanded to obtain to first order

$$N_e(E)=[K/a(\Gamma-1)]E^{-(\Gamma+1)}[1-(aTE\Gamma)^{-1}]. \quad (5.15)$$

5.3 Solution for Klein–Nishina (Compton) Losses

Compton scattering in the Klein–Nishina limit may become an important energy-loss mechanism for electrons sufficiently energetic so that $\langle\epsilon\rangle E\gg m^2c^4$, where $\langle\epsilon\rangle$ is the characteristic photon energy of the field in which the electron is immersed. In Sec. 2.8 it was shown that in this case an electron is likely to give up a significant portion of its energy to one photon. It therefore becomes necessary to use the integro-differential equation (5.7) to describe the electron distribution function.

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[27] For a discussion of the solutions of the continuity equation as a function of time for various source functions and for various energy-loss mechanisms, see (K62).

[28] We assume $\Gamma>1$. For $\Gamma\leq1$, one must use different limits of integration in (5.14).

Then, assuming a power-law source function for the electrons and assuming that no other energy-loss mechanisms are important, (5.7) becomes

$$N_e(E)\int_{mc^2}^{E}P(E,E')\,dE' - \int_E^\infty N_e(E')P(E',E)\,dE' = KE^{-\Gamma}. \quad (5.19)$$

Here, the probability $P(E,E')$ is related to the photon spectrum by the relation[29]

$$P(E,E')=P(E,E-\epsilon_1)=\int_L^\infty\left(\frac{dN_{\gamma,\epsilon}}{dtd\epsilon d\epsilon_1}\right)d\epsilon, \quad (5.17)$$

where the lower limit L corresponds to the lowest value of ϵ from which a photon of energy $\epsilon_1=E-E'$ may be obtained. From (2.50),

$$\min(\epsilon)=L=m^2c^4/4E'. \quad (5.18)$$

Thus, the total probability $P(E,E')$ is obtained by integrating over the initial photon spectrum, $n(\epsilon)$. To simplify the solution of (5.16), we first assume a monoenergetic initial photon distribution of the form $n(\epsilon)=n\delta(\epsilon-\epsilon_0)$. In this case, using (2.48) the probability becomes

$$P(E,E')=\frac{\pi m^2c^5r_0^2n}{\epsilon_0}\times\left(\frac{E^2+E'^2}{E^3E'}+\frac{m^2c^4(E-E')}{4\epsilon_0E^4E'^2}(4EE'-E^2-E'^2)\right). \quad (5.19)$$

In performing the first integral in (5.16) it is now necessary to replace the lower limit by the minimum value of E' that can occur when an electron of energy E Compton scatters with a photon of energy ϵ_0. This is obtained from (5.18):

$$E'\geq m^2c^4/4\epsilon_0 \quad (5.20)$$

Then, to lowest order in $m^2c^4/4E\epsilon_0$, the integral equation becomes

$$\frac{\pi m^2c^5r_0^2n}{\epsilon_0E}N_e(E)\left(\ln\frac{4\epsilon_0E}{m^2c^4}-\tfrac{1}{2}\right) = KE^{-\Gamma}+\int_E^\infty dE'N_e(E')P(E',E). \quad (5.21)$$

Although an exact solution to this equation would be very difficult to obtain, an approximate solution can be obtained by assuming $N_e(E)$ to be nearly a power law of the form

$$N_e(E)=A(E)E^{-p}. \quad (5.22)$$

Here, the function $A(E)$ is assumed to vary much more slowly as a function of E than E^{-p}. Then, if

$$\frac{dA/dE}{A}\ll p/E, \quad (5.23)$$

or if the relative rate of change of $A(E)$ is much less than the relative rate of change of E^{-p}, $A(E')$ may be removed from the integral in (5.21), and the integration can be performed over a pure power-law spectrum. This gives, to lowest order,

$$\int_E^\infty P(E',E)N_e(E')\,dE'=\frac{2\pi m^2c^5r_0^2n}{\epsilon_0}AE^{-(p+1)}\frac{(p+1)}{p(p+2)}. \quad (5.24)$$

Substituting this into (5.21) and equating equal powers of E, we find $p=\Gamma-1$ and obtain the spectrum

$$N_e(E)=\frac{K\epsilon_0E^{-\Gamma+1}}{\pi m^2c^5r_0^2n}\left(\ln\frac{4\epsilon_0E}{m^2c^4}-\frac{1}{2}-\frac{2\Gamma}{\Gamma^2-1}\right)^{-1}. \quad (5.25)$$

This is the steady-state distribution for electrons undergoing Compton scattering in the Klein–Nishina limit with a monoenergetic photon field. Note that since $A(E)$ is essentially logarithmic and since the argument of the logarithm is much greater than one,

$$\frac{dA/dE}{A}\sim E^{-1}\left[\ln\frac{4\epsilon E}{m^2c^4}\right]^{-2}\ll\frac{p}{E}, \quad (5.26)$$

and therefore condition (5.23) is satisfied.

When electrons undergo Compton scattering with a black-body photon field,

$$n(\epsilon)=[\pi^2(\hbar c)^3]^{-1}[\epsilon^2/(e^{\epsilon/kT}-1)], \quad (5.27)$$

and it becomes necessary to perform two integrals for each term in the integral equation. An integration of the photon spectrum must be performed to obtain $P(E,E')$ as in (5.17) and integrals over $P(E,E')$ must be performed in Eq. (5.16). The integration in the first term of (5.16) becomes

$$\int_{mc^2}^{E}P(E,E')\,dE'=\int_{mc^2}^{E}dE'\int_{m^2c^4/4E'}^{\infty}d\epsilon\left(\frac{dN_{\gamma,\epsilon}}{d\epsilon dtd(E-E')}\right). \quad (5.28)$$

Interchanging integrals, we obtain

$$\int_{mc^2}^{E}P(E,E')\,dE'=\int_{m^2c^4/4E}^{mc^2/4}d\epsilon\int_{m^2c^4/4\epsilon}^{E}dE'\left(\frac{dN_{\gamma,\epsilon}}{d\epsilon dtd(E-E')}\right)+\int_{mc^2/4}^{\infty}d\epsilon\int_{mc^2}^{E}dE'\left(\frac{dN_{\gamma,\epsilon}}{d\epsilon dtd(E-E')}\right) \quad (5.29)$$

If the mean photon energy in the distribution (5.27) is not too high or if $\langle\epsilon\rangle\sim\langle kT\rangle\ll mc^2/4$, the second integral in (5.29) can be set approximately equal to zero. Furthermore, the upper limit of the ϵ integration of the first term can be put equal to infinity. Then, since

[29] Note that $dN_{\gamma,\epsilon}/dtd\epsilon_1$ is the spectrum from *one* electron with Lorentz factor γ, or, equivalently, with energy E.

$\langle\epsilon\rangle E\gg m^2c^4/4$, (5.29) becomes

$$\int_{mc^2}^{E} P(E,E')\,dE' = \int_0^\infty d\epsilon \int_{m^2c^4/4\epsilon}^{E} dE' \left(\frac{dN_{\gamma,\epsilon}}{d\epsilon dt d(E-E')}\right). \quad (5.30)$$

With these simplifications, the procedure for obtaining the electron distribution in the case of a thermal photon spectrum, (5.27), is exactly the same as in the monoenergetic case. The electron spectrum is then given by

$$N_e(E) = \frac{6\hbar^2 KE^{-\Gamma+1}}{\pi m^2c^2r_0^2(kT)^2} \times \left[\ln\frac{4EkT}{m^2c^4} + \frac{1}{2} - \frac{2\Gamma}{\Gamma^2-1} - C_E - C_l\right]^{-1}, \quad (5.31)$$

where $C_E=0.5772$ and $C_l=0.5700$. Besides the Klein–Nishina condition $kTE\gg m^2c^4$, we have imposed the additional condition $kT\leq mc^2/4$ to obtain the result (5.31).

Thus, we have seen in this section that in the extreme Klein–Nishina limit the steady-state electron spectrum can be represented as a power law times a function whose dependence on E is slow enough for it to be taken out of the integral in (5.16). However, in Sec. (2.9) we assumed a pure power-law electron spectrum when we integrated $dN_{\gamma,\epsilon}/dtd\epsilon$ over the electron distribution to obtain the total photon spectrum. But since the factor in brackets in (5.31) has such a slow relative rate of change, it could have been taken out of the integral in Sec. (2.9) anyhow. In that case, it would be necessary to substitute $E\approx\epsilon_1$ in the argument of the logarithm in both (5.31) and (5.25) in order that the total Klein–Nishina photon flux not be a function of E.

It should be noted that at *very* high electron energies, losses in pair-producing electron–photon collisions will dominate Klein–Nishina losses. We have not treated this process here.

5.4 Solution for Bremsstrahlung Losses

When an electron traversing a dilute gas loses energy predominantly by means of bremsstrahlung, it once again becomes necessary to employ the integro-differential equation (5.7) to determine the electron spectrum. Even though the bremsstrahlung infrared divergence $d\sigma\sim k^{-1}dk$ implies that a large number of low-energy photons are radiated, it is seen from Fig. 10 that the energy emitted by an electron per unit photon energy, the energy emission spectrum, does not drop off very fast until $k\sim E$. Therefore, an electron is likely to lose a significant portion of its energy to one photon, and its energy will change in discrete jumps with time.

In this section we consider the case where bremsstrahlung is the only significant energy-loss mechanism. With a power-law source spectrum for electrons and with leakage losses present, the steady-state integral equation becomes

$$N_e(E)\int_1^E dE'P(E,E') - \int_E^\infty dE'N_e(E')P(E',E) + \frac{N_e(E)}{T} = KE^{-\Gamma}. \quad (5.32)$$

As in Sec. 3, energy is expressed in terms of mc^2. The probabilities

$$P(E,E') = c\sum_s n_s(d\sigma_s/dk), \quad (5.33)$$

where $k=E-E'$. We consider here only the case of electrons traversing a gas of pure hydrogen. The generalization of the results derived here to higher Z elements readily follows from the considerations of Sec. 3.5.

In the case of extremely high-energy electrons ($E>30$) traversing neutral hydrogen, it is appropriate to use the strong shielding ($\Delta\ll 1$) expression for the cross section. Then, since $\varphi_1\approx\varphi_2\equiv\varphi$, the probability matrix (5.33) becomes

$$P(E,E') = \alpha r_0^2 cn_H\varphi(E-E')^{-1} \times[1-\tfrac{2}{3}(E'/E)+(E'^2/E^2)]. \quad (5.34)$$

Because of the infrared divergence or $(E-E')^{-1}$ factor in $P(E,E')$, it is clear that both integrals in (5.32) will exhibit a logarithmic divergence at $E'=E$. However, since $(E-E')P(E,E')$ is well behaved near $E'=E$, the divergence problem can be avoided by integrating the first integral from 1 to $E-\eta$, and integrating the second from $E+\eta$ to infinity. Then, in the limit of $\eta\rightarrow 0$, the diverging contributions of both integrals exactly cancel.

To solve (5.32), we assume the solution to be a pure power law:

$$N_e(E) = AE^{-p}. \quad (5.35)$$

Then, with corrections of order E^{-1}, the first integral in (5.32) becomes

$$N_e(E)\int_1^{E-\eta} dE'P(E,E') = -AE^{-p}\alpha r_0^2 cn_H\varphi\left(\frac{4}{3}\ln\frac{\eta}{E}+\frac{5}{6}\right). \quad (5.36)$$

Meanwhile, the second integral gives

$$\int_{E+\eta}^\infty dE'P(E',E)N_e(E') = -AE^{-p}\alpha r_0^2 cn_H\varphi \times\left[\frac{4}{3}\ln\frac{\eta}{E}+\frac{4}{3}C_E+\psi(p)-\frac{2}{3}\psi(p+1)+\psi(p+2)\right], \quad (5.37)$$

where Euler's constant$=C_E=0.577$, and the psi function $\psi(p)$ is the logarithmic derivative of the gamma function:

$$\psi(p)=(d/dp)\ln\Gamma(p). \qquad (5.38)$$

Thus, the diverging terms in (5.36) and (5.37) do indeed cancel each other. Finally, by substituting the above terms into the integral (5.32) and equating powers of E, we find that $p=\Gamma$ and obtain the steady-state electron spectrum

$$N_e(E)=\frac{KE^{-\Gamma}}{\alpha r_0^2cn_H\varphi[-\frac{5}{6}+\frac{4}{3}C_E+\psi(\Gamma)-\frac{2}{3}\psi(\Gamma+1)+\psi(\Gamma+2)]+T^{-1}}. \qquad (5.39)$$

The spectrum (5.39) is valid for electrons losing energy by bremsstrahlung collisions with hydrogen atoms in the strong-shielding limit $E>30$. This solution is nearly exact, the first-order correction being of order E^{-1}. The effects of gradual leakage, the $T^{-1}N_e(E)$ term in (5.32), have been included in the electron spectrum. The distribution (5.39) would not be valid if T were a function of E but for very high-energy electrons this is not usually the case.[30]

We now consider the spectrum of very high-energy electrons in a region of completely ionized hydrogen. In Sec. 3.4 we showed that the electron–electron bremsstrahlung spectrum for $k\gg1$ is essentially the same as the pure Coulomb bremsstrahlung spectrum. For smaller k, exchange effects must be considered. However, since exchange effects are important only in the low-energy end of the spectrum, their effect upon the electron distribution corresponds to a *continuous* energy loss process. Furthermore, since $E\gg1$, this energy loss will be much less important than high energy bremsstrahlung in determining the electron spectrum. We therefore assume $d\sigma_{e\text{-}e}=d\sigma_{e\text{-}p}$, and obtain the transition probabilities

$$P(E,E')=8\alpha n_pr_0^2ck^{-1}[1+(E'^2/E^2)-\tfrac{2}{3}(E'/E)]\times\{\ln(2EE'/k)-\tfrac{1}{2}\}, \qquad (5.40)$$

where $k=E-E'$. Since the integral equation cannot be readily solved with this form of $P(E,E')$, it is useful to reduce (5.40) to a simpler function. Most of the k dependence of $P(E,E-k)$ is in the k^{-1} factor, and therefore if we substitute the characteristic photon energy $k\approx E'\approx E/2$ into the logarithm in (5.40), we obtain

$$P(E,E')\approx8\alpha n_pr_0^2c(E-E')^{-1}\ln 2E, \qquad (5.41)$$

where the $\frac{1}{2}$ has been ignored as small compared to $\ln 2E$. Also, the bracket in (5.40) has been set equal to unity; when the total energy loss $[\int kP(E,E')dk]$ is computed, the terms with $(E'/E)^2$ and $-2E'/3E$ effectively cancel, so this should be a good approximation. This approximation is equivalent to assuming the energy spectra in Fig. 10 to be horizontal straight lines. The *mean absolute* error due to this assumption is about 20% for $E=10^3$. However, one can see from Fig. 10 that most of this error occurs in the low-energy end of the spectrum where $\ln(2EE'/k)$ becomes large. This error can then be corrected by adding a *continuous* energy-loss term to the integral equation. Since the difference between the total energy loss predicted by (5.40) and that found from (5.41) is nearly zero anyhow, we will ignore the continuous energy-loss correction term.[31]

To solve the integral equation (5.32), we define

$$\phi(E)=N_e(E)\ln 2E. \qquad (5.42)$$

An integration of (5.32) by parts yields

$$8\alpha n_pr_0^2c\ln E\,\phi(E)+8\alpha n_pr_0^2c\times\int_0^\infty dx\ln x\,\phi'(x+E)+\frac{\phi(E)}{T\ln 2E}=KE^{-\Gamma}. \qquad (5.43)$$

In obtaining (5.43) the divergence problem is handled in the same way as in the strong-shielding case. Using the same approach as in Sec. 5.3 for Klein–Nishina losses, we assume

$$\phi(E)=A(E)E^{-p}, \qquad (5.44)$$

where $A(E)$ is a very slowly varying function of E. Again, substituting this into (5.43), we find $p=\Gamma$ and

$$A(E)=K\{8\alpha n_pr_0^2c[C_E+\psi(\Gamma)]+(T\ln 2E)^{-1}\}^{-1}. \qquad (5.45)$$

Therefore, $A(E)$ is indeed much more slowly varying than E^{-p}, and the steady-state electron spectrum

$$N_e(E)=KE^{-\Gamma}/\{8\alpha n_pr_0^2c\ln 2E\,[C_E+\psi(\Gamma)]+T^{-1}\}, \qquad (5.46)$$

is a self-consistent solution of (5.43). As in the Klein–Nishina case, the bremsstrahlung solution (5.46) is not the pure power law used in Sec. 3 to calculate the total photon spectrum from a distribution of electrons. However, the denominator in (5.46) can be taken out of the integrals in Sec. 3.7 by setting $E\approx2k$ in the argument of the logarithm.

Finally, when high-energy electrons traverse a

[30] If the electrons leak from a region of space of dimension R, the characteristic leakage time, $T\approx Rc^{-1}\beta^{-1}$. But since $E>30$ in our discussion, β can be set equal to 1 for all the electrons.

[31] This correction term can, however, be estimated. Setting $P(E,E-k)=k^{-1}f(E)$, $f(E)$ is obtained by demanding that $\langle k^2\rangle$, the mean-squared photon energy, be the same as that found from (5.40). Then, the continuous energy-loss correction is given by the difference of (3.53) and the energy loss found from $P(E,E-k)=k^{-1}f(E)$. Using $\langle k^2\rangle$ to normalize $P(E,E')$ weights the high-energy (discrete loss) end of the spectrum in $P(E,E')$.

partially ionized gas, the electron spectrum is obtained by setting $P(E, E')$ equal to the sum of the contributions due to the two components. Then, the electron distribution becomes

$$N_e(E) = (KE^{-\Gamma}/\alpha r_0^2 c)$$
$$\times\{n_H\varphi[-\tfrac{5}{6}+\tfrac{4}{3}C_E+\psi(\Gamma)-\tfrac{2}{3}\psi(\Gamma+1)+\psi(\Gamma+2)]$$
$$+8n_p \ln 2E\,[C_E+\psi(\Gamma)]+(\alpha r_0^2 cT)^{-1}\}^{-1}. \quad (5.47)$$

Thus, in general, both bremsstrahlung and leakage losses do not change the power of the electron injection spectrum.

We have assumed in this section that electrons lose energy only through bremsstrahlung. If, for example, synchrotron losses are also present, the electron spectrum is more difficult to obtain. However, $(dE/dt)_{\text{brem}} \propto E$, while $(dE/dt)_{\text{synch}} \propto E^2$. Therefore synchrotron losses dominate for $E \gg E_0$, where E_0 is the energy at which the two energy losses are equal, and bremsstrahlung is most important for $1 \ll E \ll E_0$. The electron spectrum is then given by the synchrotron spectrum (a $\Gamma+1$ power law) for $E \gg E_0$, and by the bremsstrahlung loss electron spectrum for $E \ll E_0$. The spectrum near E_0 can be estimated by treating the less important energy loss as a perturbation.

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APPENDIX: SOFT PHOTON EMISSION—VIRTUAL PHOTON SPECTRUM

A very useful formula can be derived in radiation theory which gives the probability of the emission of a soft photon of energy within $\hbar d\omega$ accompanying a change in velocity of a charged particle. The nonrelativistic formula is derived in a number of textbooks (cf. J62). We give a simple outline of its derivation here for the case where we are not interested in the direction of emission of the photon; that is, we give the emission probability integrated over the emission solid angle. This simplifies the derivation considerably. Also we give the relativistic generalization of the formula for the special case where the change in velocity of the particle is approximately perpendicular to its initial direction, corresponding to small angle scatterings.

Consider the radiation flow through a perpendicular element of area $dA = R^2 d\Omega$, where $d\Omega$ is the element of solid angle defined by dA at a large (and *retarded*) distance R from a moving charge. Let the unit vector $\mathbf{n}$ denote the direction of $d\Omega$ from the radiating charge. In a finite time interval, the energy flux $J_W = dW/dA$ can be written as the time integral of the radial Poynting vector, or as an integral over a frequency spectrum:

$$J_W = \int S(t)\, dt = \int I(\omega)\, d\omega; \quad \text{(A1)}$$

here $I(\omega)$ is the energy flux per unit frequency or the Fourier amplitude of the energy flux. In terms of the electric and magnetic fields which are mutually perpendicular, perpendicular to $\mathbf{n}$, and of equal magnitude

$$S(t) = (c/4\pi)\,|\,E(t)\,|^2 = (c/4\pi)\,|\,B(t)\,|^2. \quad \text{(A2)}$$

The field intensities are given in terms of their Fourier components:

$$E(t) = \int E_\omega e^{-i\omega t}\, d\omega,$$
$$E_\omega = (2\pi)^{-1} \int E(t)\, e^{i\omega t}\, dt, \quad \text{(A3)}$$

(similarly with $B(t)$ and B_ω). Substituting (A3) into (A2) and (A1), and making use of

$$\int e^{-i\omega t} e^{i\omega' t} dt = 2\pi\delta(\omega - \omega'), \quad \text{(A4)}$$

we get $I(\omega) = \frac{1}{2}c\,|\,E_\omega\,|^2 = \frac{1}{2}c\,|\,B_\omega\,|^2$. Now both positive and negative frequencies are contained in $I(\omega)$. If we include these together by writing $I_\omega = I(\omega) + I(-\omega) = 2I(\omega)$, the total Fourier amplitude of the energy flow per unit area is

$$I_\omega = c\,|\,E_\omega\,|^2 = c\,|\,B_\omega\,|^2. \quad \text{(A5)}$$

Relating the energy dW_ω radiated in $d\omega$ to the probability of emitting a photon of energy within $\hbar d\omega$ by

$$dW_\omega = \hbar\omega dw_\omega \quad \text{(A6)}$$

and since

$$dW_\omega = dJ_W dA = I_\omega d\omega R^2 d\Omega, \quad \text{(A7)}$$

we get the basic result:

$$dw_\omega = d\omega (cR^2/\hbar\omega) \int [\,|\,B_\omega\,|^2 \quad \text{or} \quad |\,E_\omega\,|^2]\, d\Omega. \quad \text{(A8)}$$

This last expression, which is exact, can be put in a convenient form by relating B_ω to the velocities $\mathbf{v}_0$ and $\mathbf{v}_f$ of the charge(s) before and after scattering. Introducing the vector potential

$$B_\omega = (2\pi)^{-1} \int |\,\text{curl}\,\mathbf{A}(t)\,|\, e^{i\omega t}\, dt, \quad \text{(A9)}$$

making use of the relation

$$\text{curl}\,\mathbf{A}(t) = c^{-1}\dot{\mathbf{A}} \times \mathbf{n} \quad \text{(A10)}$$

valid for a propagating plane wave for which $\mathbf{A}$ is a function of $\mathbf{n}\cdot\mathbf{r} - ct$ ($\mathbf{n}$ is the direction of propagation; $\dot{\mathbf{A}} = \partial\mathbf{A}/\partial t$), and taking the low-frequency limit [$e^{i\omega t} \to 1$ in (A9)—this is the first approximation made], we get

$$B_\omega \underset{\omega\to 0}{\longrightarrow} (2\pi c)^{-1}\,|\,\Delta\mathbf{A} \times \mathbf{n}\,|. \quad \text{(A11)}$$

Here $\Delta\mathbf{A}$ is the change in the vector potential; in terms of the particle velocity

$$\mathbf{A}=\left[\frac{e}{R}\frac{\mathbf{v}}{1-\mathbf{n}\cdot\mathbf{v}/c}\right]_{\text{ret}}. \quad \text{(A12)}$$

For a nonrelativistic particle, (A8) then yields, on integration over $d\Omega$, the very simple expression ($\alpha=e^2/\hbar c\approx 1/137$)

$$dw_\omega(\text{nonrel})=(2\alpha/3\pi)(\Delta\mathbf{v}/c)^2(d\omega/\omega). \quad \text{(A13)}$$

This formula can also be derived by the methods of quantum electrodynamics (cf. JR55).

We are interested in a formula of the type (A13) applicable to the scattering of a *highly relativistic* particle. In particular, the case of small angle scattering is most important. For this case the final and initial velocities are related by

$$\mathbf{v}_f=\mathbf{v}_0+\Delta\mathbf{v}, \quad \text{(A14)}$$

with $\Delta\mathbf{v}\ll\mathbf{v}_0$, $\mathbf{v}_f$ and $\Delta\mathbf{v}$ approximately perpendicular to $\mathbf{v}_0$. One can then derive the relativistic generalization of (A13) by substituting (A11) and (A12) into (A8), integrating over $d\Omega$, and then expanding the integrand to lowest order in $\Delta\mathbf{v}$. This integration over the photon emission angle is a little tedious but yields the simple and more general result

$$dw_\omega(|\Delta\mathbf{v}|\ll c)=(2\alpha/3\pi)(\gamma\Delta\mathbf{v}/c)^2(d\omega/\omega), \quad \text{(A15)}$$

that is, just γ^2 times (A13). That this should be the result can be seen from a simple invariance consideration. Since dw_ω is a *probability*, it must be an invariant:

$$dw_\omega(\Delta\mathbf{v})=dw'_{\omega'}(\Delta\mathbf{v}'). \quad \text{(A16)}$$

We can consider one reference frame (K') where the particle velocities are small and (A13) applies. Now consider the process in the lab frame (K) which has its x axis aligned with the x axis of K', with the relative motion along these axes. The initial particle velocity is along the x axis and $\Delta\mathbf{v}$ is taken to be in the y direction. Then, from the Lorentz transformation of velocities,

$$\Delta v_y'=(\gamma)^{-1}\frac{\Delta v_y}{1-vv_x/c^2}. \quad \text{(A17)}$$

Now $v_x\approx v$, so that $\Delta v_y'\approx\gamma\Delta v_y$. Then, since $d\omega/\omega=dw'/\omega'$, we see how (A15) follows from (A16) and (A13).

The developments we have just outlined here can be applied to obtain quite readily an important result associated with the so-called *Weizsäcker–Williams* method. The idea behind this special method or approach, which is applied to the bremsstrahlung problem in Sec. 3.2, is based on the following result. The electric and magnetic fields produced at a fixed point by a highly relativistic charged particle of charge q passing at an impact parameter b are principally the *transverse* fields[32] $E(t)$ and $B(t)$ which are mutually perpendicular and both equal to, in the extreme relativistic limit,

$$E(t)=B(t)=q\gamma b/(b^2+\gamma^2c^2t^2)^{3/2}. \quad \text{(A18)}$$

This result is derived in a number of textbooks (cf. J62, PP62) as well as in the original papers (W34, W35) and follows from a straightforward Lorentz transformation of the static Coulomb field of the particle in its rest frame. Because of this result, the effects of the fields of the incident particle are the same as produced by a pulse of incident photons with the total time-dependent fields given by (A18). The spectrum and equivalent number of these photons can be easily found from our previous developments. If dN is the differential number of these photons incident on an element of area dA, then

$$\hbar\omega dN/dA\,d\omega=I_\omega=c\,|\,E_\omega\,|^2. \quad \text{(A19)}$$

The Fourier component E_ω is given by

$$E_\omega=(2\pi)^{-1}\int E(t)e^{i\omega t}\,dt$$

$$=\frac{q}{2\pi cb}\int_{-\infty}^{\infty}\frac{\exp(i\omega bx/\gamma c)}{(1+x^2)^{3/2}}\,dx. \quad \text{(A20)}$$

For $\omega\ll\gamma c/b$ the integral in (A20) approaches 2, so that for a given ω, say,

$$E_\omega\rightarrow q/\pi cb, \qquad b\ll b_{\max}\approx\gamma c/\omega. \quad \text{(A21)}$$

If the charge is incident at random impact parameters, we can take $dA=2\pi b db$. Then, for the number of virtual photons incident in the frequency interval $d\omega$ due to incident charges ($q=Ze$) with random impact parameters within db, we get

$$dN=(2\alpha Z^2/\pi)(db/b)(d\omega/\omega). \quad \text{(A22)}$$

We emphasize that this result holds with the restriction (A21) on b and ω. For *b larger than* $b_{\max}$, for example, the spectrum (A22) drops off due essentially to the oscillatory exponential factor in the integral in (A20).

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[32] These and other quantities introduced in the rest of this Appendix are referred to the electron's rest frame, and should, for consistency with the remainder of this article, be primed. For simplicity in the formalism the primes are omitted.

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