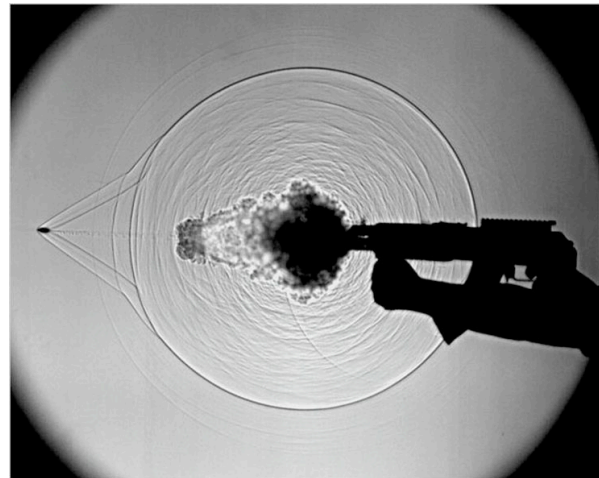


Lecture 15 281116

- Il pdf delle lezioni puo' essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1617/



MagnetoHydroDynamics (MHD)

We have seen the equations of hydrodynamics and some astrophysical applications of them, but we have neglected any electromagnetic phenomenon relative to the fluids

However, in astrophysics environments fluid temperatures are usually very high and the most of the atoms are completely ionized

This is particularly true for H and He, which make the most part cosmic matter, since their ionization potentials are quite low

Plasmas

Plasmas are gases in which the constituent particles are electrically charged

The first consequence is that the electric fields are not important, because the abundances of free electric charges ensure that any E field would be short-circuited by their motion ie the charges move in such a way to cancel the (external) field

But if the fluid is in a B field, its motion wrt B builds up E fields too by induction and this generates currents

In turn, currents are influenced by the B fields and generate new B fields, that influence again the fluid motion

Plasmas

The plasmas are made of particles of opposite sign

In a volume containing many charged particles, we expect that volume to be close to charge-neutral, since any charge imbalance would produce strong electrostatic forces to restore charge neutrality

Charge imbalances may exist only over a short distance (called Debye length) or for a short period of time (the inverse of the so called plasma frequency)

What makes the macroscopic behavior of plasmas so different from that of neutral gases is the fact that an electromotive force applied to a plasma can drive large currents → volumes of plasmas can sustain large currents in spite of being nearly charge neutral

Since many charged particles in a plasma can interact simultaneously through long range electromag interactions, there can be many phenomena in plasma which are caused by collective interactions

Plasmas

In neutral gases, the interactions between particles are mediated by collisions

In a completely ionized plasma, instead, the interactions are mediated by long range electromagnetic forces

In a partially ionized gas, both processes play a role

if we put a test charge q in the plasma, the presence of many charges of opposite sign around q screens the electromagnetic fields felt by "far" charges

The characteristic length over which the screening occurs is a measure of the assumption of charge neutrality of the overall gas:
the Debye length

Plasmas: Debye length

To understand how good is the assumption of charge neutrality is, let us consider the charge separation produced by introducing a charge q inside a plasma

If n_e and n_i are the number densities of e^- and ions, the charge density at a point x is

$$\nabla^2 \Phi = -4\pi(n_i - n_e)e$$

If the plasma is in thermodynamical equilibrium and n is the density of e^- and ions far away from the charge q , then we expect, according to Maxwell-Boltzmann distribution

$$n_i = n \times \exp\left(-\frac{e\Phi}{kT}\right) \quad n_e = n \times \exp\left(\frac{e\Phi}{kT}\right)$$

Substituting in the potential eqn $\nabla^2 \Phi = -4\pi n e \left(\exp\left(-\frac{e\Phi}{kT}\right) - \exp\left(\frac{e\Phi}{kT}\right) \right)$

Usually $e\Phi \ll kT \rightarrow$ we may expand in Taylor series the exp, neglecting higher terms

$$\nabla^2 \Phi = -\frac{\Phi}{\lambda_D^2} \quad \text{with} \quad \lambda_D = \left(\frac{kT}{8\pi n e^2}\right)^{1/2} \quad \text{Called Debye length}$$

Plasmas: Debye length

$$\nabla^2 \Phi = -\frac{\Phi}{\lambda_D^2} \quad \lambda_D = \left(\frac{kT}{8\pi n e^2}\right)^{1/2}$$

The solution is easy to find assuming a spherical symmetry $\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r})$

We get $\Phi(r) = \frac{q}{r} e^{-r/\lambda_D}$

It thus appears that the effect of the charge is screened beyond a distance λ_D

So a plasma can be considered charge neutral when distances larger than the Debye length are considered

Although the E field of a charge in principle extends to ∞ , the influence on a charged particle in a plasma is effectively felt to a distance λ_D , ie within a volume λ_D^3 , called the Debye volume

Hence the nbr of particles interacting with q is $n\lambda_D^3$: this is a measure of the number of particles which can interact simultaneously

Plasma parameter

The plasma parameter is defined as $g = 1/n\lambda_D^3 = \frac{(8\pi)^{3/2}e^3n^{1/2}}{(kT)^{3/2}}$

When g is smaller, there is more collective interaction in the plasma
(note g is smaller for smaller n)

Therefore the nbr of particles interacting collectively is more for a low density plasma, the Debye length being less effective so that Debye volume is much larger

The average distance between the particles of a plasma is of the order of $n^{-1/3} \rightarrow$ the average potential energy between a pair of nearby particles is of the order of $e^2n^{1/3}$

Hence the ratio of potential and average kinetic energy ($\sim kT$) is

$$\frac{\langle PE \rangle}{\langle KE \rangle} \sim \frac{e^2n^{1/3}}{kT} \propto n^{1/3}$$

Another interpretation of g is, therefore, that it is a measure of the potential energy of interactions compared to kinetic energy: when g is small (as for low n), the interaction amongst particles is weak, but a large nbr of particles interact simultaneously. On the other hand, a larger g implies few particles interacting collectively, but interacting strongly

The limit of small g is referred to as the ***plasma limit***

$$g = \frac{(8\pi)^{3/2} e^3 n^{1/2}}{(kT)^{3/2}}$$

$$\lambda_D = \left(\frac{kT}{8\pi n e^2} \right)^{1/2}$$

Types of plasmas

The characteristics of a plasma are determined by n and T

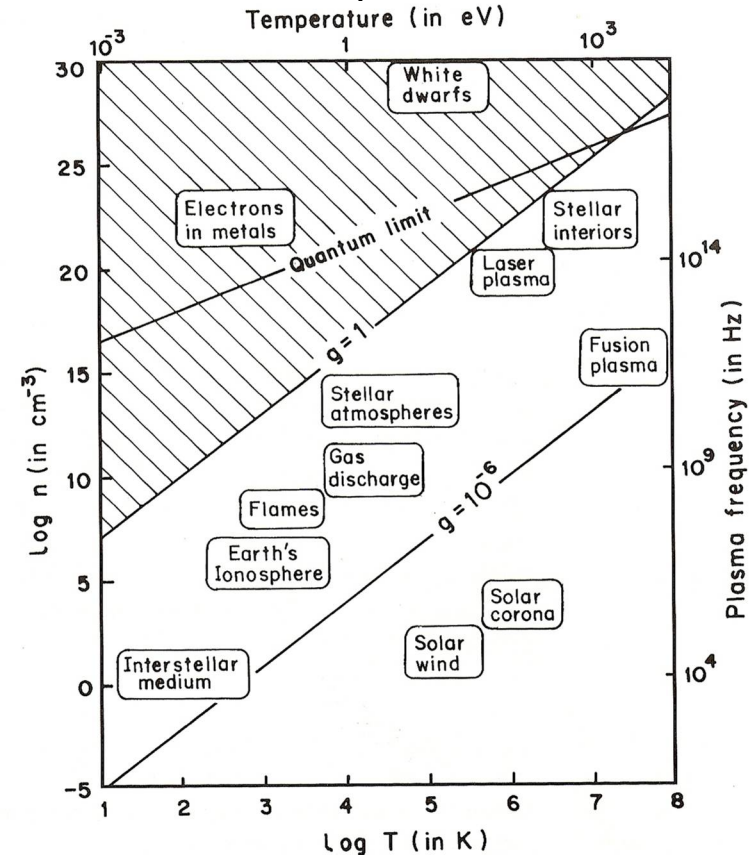
The condition $g < 1$ is a requirement for a gas to be defined a plasma

In the region above $g=1$, the gas do not behave like a plasma

Above quantum limit, we have to use proper quantum mechanics, as for electrons in a metal and white dwarfs interiors (as for neutron stars)

In between QL and $g=1$ we find solids, liquids and crystals which can not be considered plasmas

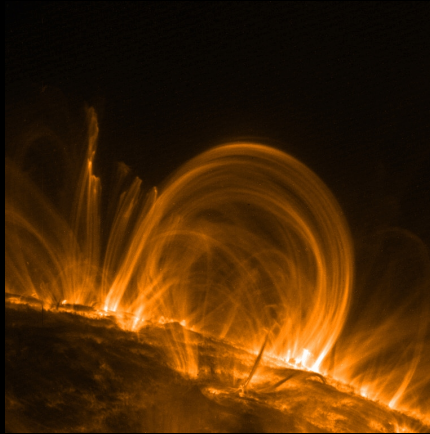
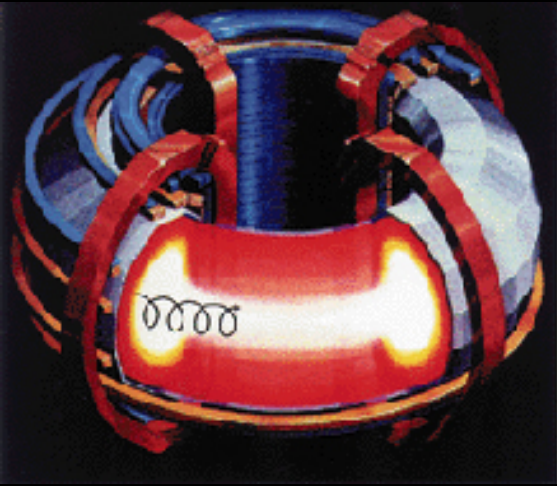
The ratio $\frac{\langle PE \rangle}{\langle KE \rangle} \sim \frac{e^2 n^{1/3}}{kT} \propto n^{1/3}$



Implies that for small g plasmas kinetic energy exceeds potential energy, so that it can be treated as a perfect gas

This holds even for material at the centres of stars!

The Wide Range of Plasmas



Plasmas

Plasma

Density low to neglect particles collisions...

...but high enough to allow elm interactions.

The overall charge is nearly zero → quasi-neutrality

Free charged particles → currents → self magnetic fields → auto-interactions
→ Magnetohydrodynamics

Plasma is the more common state - the 4th state - of matter
in universe:

99% of “normal” matter is plasma

MHD

The MHD studies the interactions of ensembles of charged particles, plasmas, and electromagnetic fields

Limiting cases:

- 1) external fields assigned \rightarrow determine the particle motion from Lorentz force, $m\mathbf{a} = (q/c)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ or $\rho d\mathbf{v}/dt = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}$, $\rho_c = qn$, $\rho = mn$
- 2) charges and current distributions assigned \rightarrow determine the fields from Maxwell equations

The MHD is different for two reasons:

- (a) it deals with many particles systems (plasmas) which exhibit collective behavior (bulk motion, oscillations, instabilities,...)
- (b) fields \mathbf{E} and \mathbf{B} are not prescribed but determined by the positions and motions of the particles

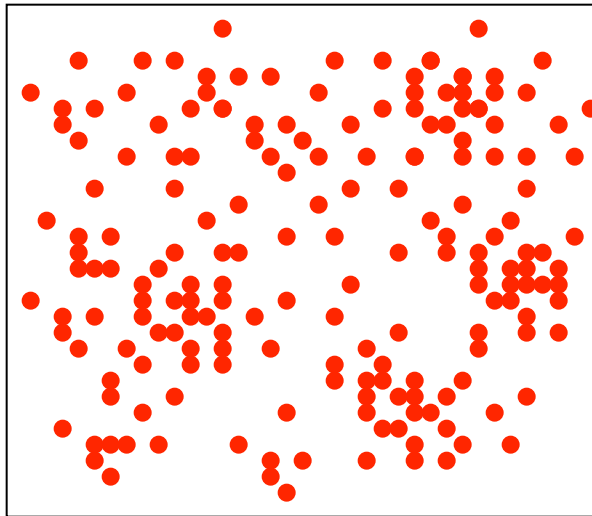
\rightarrow field and motion equations must be solved simultaneously and self-consistently: we are looking for a set of particle trajectories and fields patterns such that the particles generate the field patterns as they move along their orbits and the fields patterns force the particles to move in exactly these orbits

In a plasma the generated self-fields act as coupling device between the individual particles

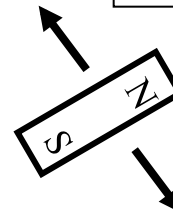
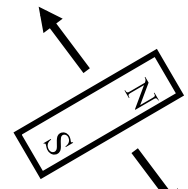
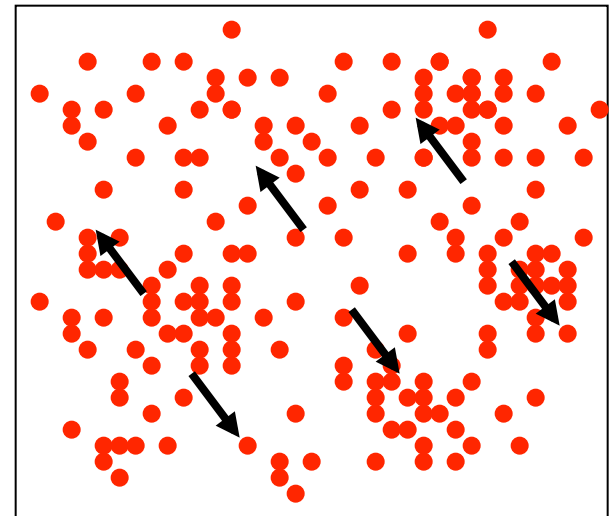
Plasmas Respond to B-Fields

• Magneto-HydroDynamics (MHD) =
hydrodynamics flow+electro-magnetic
phenomena

Regular Gas



Plasma



MHD

The problem is mechanical, thermodynamical and electromagnetic

We need to find the matter distribution ρ , charge distr. ρ_c , current density j , speed v and the fields E and B , with given boundary conditions

Assumptions:

- (a) the medium cannot be magnetized nor polarized $\varepsilon = \mu = 0$
- (b) small velocities compared to c
- (c) highly conductive medium, ie $E/B \gg 1$
- (d) displacement current small compared to induction current: $\partial E / \partial t \ll \mu_0 j$

Motion equations

To get the motion equations for the fluid we have to modify the hydrodynamic equations

The mass conservation law remains unchanged $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

In the momentum equation a new term appears due to the Lorentz force acting on the moving charges

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}$$

The energy equation must be modified too, since the presence of currents implies electrical resistance and this, in turn, implies dissipation and therefore heating

From elementary physics we know that the rate of heating (ie power dissipated) is given by Joule's law $P = dE/dt = j^2/\sigma$, being σ the fluid conductivity, therefore

While in absence of dissipation, we got for a fixed mass element $T \frac{Ds}{Dt} = 0$

In presence of resistive dissipation it becomes $\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$

$$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$$

Motion equations

Why should we have currents in the fluid?

Because elm fields could be present: from elementary physics, we get Ohm's law

In a stationary medium, an electric field generates a current density $j = \sigma E'$

In our case, the fluid is in motion wrt the laboratory frame (or, for astronomers, of the observer!) and the electric and magnetic fields are measured in this frame

From this it follows that the electric and magnetic fields felt by the fluid in its reference (rest) frame are different from those measured in laboratory

Remember that the Lorentz transformations for the fields are

$$\vec{E}' = \gamma[\vec{E} + (\vec{v} \times \vec{B})/c] \quad \text{being } E \text{ and } B \text{ the fields in the lab}$$

We assumed that $v \ll c$, therefore we can approximate as $\vec{E}' \approx [\vec{E} + (\vec{v} \times \vec{B})/c]$

This is the field felt by the fluid in its rest frame

Motion equations

$$\vec{E}' \approx [\vec{E} + (\vec{v} \times \vec{B})/c]$$

Therefore the Ohm's law becomes

$$\vec{j}' \approx \sigma[\vec{E} + (\vec{v} \times \vec{B})/c]$$

Generally speaking, the conductivity is a tensor and it's easy to argue that, in presence of strong B fields, σ (as diffusion coefficients, like heat transport or viscosity) is different if we consider the directions parallel or normal to the field, but it can be shown that the conduction coefficients along the two directions differ by $3\pi/32=0.295$, so that we will omit the distinction and treat σ as a scalar

Of course, j is the spatial component of a 4-vector $(c\rho, \vec{j})$ and is expected to transform when we go from rest frame to the lab frame

But the Lorentz transf mixes charge density and current, and we have seen that we can safely assume $\rho=0$ because of charge-neutrality, so in the limit $v \ll c \rightarrow$

$$\vec{j}' \approx \gamma \vec{j} \approx \vec{j}$$

In these hypothesis, then, current density does not change when passing from rest frame to the lab frame and the Ohm's law is

$$\vec{j}' \approx \sigma[\vec{E} + (\vec{v} \times \vec{B})/c]$$

Motion equations

Together with the fluid equations, we have to take into account also the Maxwell equations to describe the elm fields

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \approx 0 \quad \text{Due to charge neutrality } \rho \sim 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \text{It follows that the only way to have E fields in MHD is by induction!}$$

And that the induced field strength is, to order of magnitude, $E \sim vB/c$, being V a typical speed

$$\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \approx (4\pi/c)\vec{j} \quad \text{Since the displacement current is negligible}$$

Since the E field in MHD is due exclusively to induction, from rotor equation we find that $E \sim LB/cT$, where L and T are typical lengths and time scale over which the elm varies and L/T is clearly a typical speed

From this it follows that the displacement current is, to order of magnitude, $\sim LB/c^2T^2$

This must be compared with rotor of B , which, to order of magnitude is B/L

The ratio of the two terms is $L^2/(cT)^2 \sim (v/c)^2$, so that for $v \ll c$ can be neglected

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{Remains unchanged}$$

MHD eqns summary

In the simplest case of a single charge specie the equations are

Maxwell field equations	$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0$ $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Ohm's law	$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}/c)$
Equations of motion (Euler eqn for ideal fluids)	$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}/c$		
Mass conservation	$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$		
Energy equation	$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma}$		
Equation of state	$\frac{d(\frac{p}{\rho^{\gamma_a}})}{dt} = 0$		

MHD induction equation

Let assume that $\sigma = \text{const.}$ in time and space.

Combining Faraday and Ohm law we eliminate E and J from equations

$$\vec{E} = \vec{j}/\sigma - \vec{v} \times \vec{B}/c \quad \Rightarrow \quad \vec{\nabla} \times \vec{j} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c} \frac{\partial \vec{B}}{\partial t})$$

$$\vec{\nabla} \times (\vec{j}/\sigma - \vec{v} \times \vec{B}/c) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

From Ampere law $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j} \quad \Rightarrow \quad \frac{c}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{j}$

$$\frac{c}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \sigma(\vec{v} \times \vec{B}/c - \frac{1}{c} \frac{\partial \vec{B}}{\partial t})$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\nabla^2 \vec{B} + \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) = -\nabla^2 \vec{B} \quad \Rightarrow \quad -\frac{c^2}{4\pi} \nabla^2 \vec{B} = \sigma(\vec{v} \times \vec{B} - \frac{\partial \vec{B}}{\partial t})$$

$$\boxed{\frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial \vec{B}}{\partial t}}$$

The coefficient of laplacian $\lambda = \frac{4\pi\sigma}{c^2}$ Is called magnetic diffusivity

Or $\eta = c^2/4\pi\sigma$ is the magnetic diffusion coefficient

This equation, called induction equation, is very useful because only the B field appears in it

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

N.B.:

(i) \rightarrow B once v is known

(ii) In MHD, v and B are **primary variables**:
induction eqn + eqn of motion \rightarrow basic physics

(iii) $\mathbf{j} = c \nabla \times \mathbf{B} / 4\pi$ and $\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \mathbf{j} / \sigma$
are **secondary variables**

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{v} \times \mathbf{B})}_{\mathbf{A}} + \underbrace{\eta \nabla^2 \mathbf{B}}_{\mathbf{B}}$$

(iv) \mathbf{B} changes due to transport + diffusion

(v) $\frac{A}{B} = \frac{L_0 v_0}{\eta} = R_m$ -- * magnetic Reynold number*

eg, $\eta = 1 \text{ m}^2/\text{s}$, $L_0 = 10^5 \text{ m}$, $v_0 = 10^3 \text{ m/s}$ --> $R_m = 10^8$

(vi) $\mathbf{A} \gg \mathbf{B}$ in most of Universe -->

\mathbf{B} frozen to plasma -- keeps its energy

Except **SINGULARITIES** -- \mathbf{j} & $\nabla \mathbf{B}$ large

Form at **NULL POINTS**, $B = 0$

(a) If $R_m \ll 1$

■ The induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$$

■ B is governed by a diffusion equation

--> field variations on a scale L_0

diffuse away on time

$$t_d = \frac{L_0^2}{\eta}$$

with speed $v_d = L_0 / t_d = \frac{\eta}{L_0}$

■ E.g.: sunspot ($\eta = 1 \text{ m}^2/\text{s}$, $L_0 = 10^6 \text{ m}$), $t_d = 10^{12} \text{ sec}$;

for whole Sun ($L_0 = 7 \times 10^8 \text{ m}$), $t_d = 5 \times 10^{17} \text{ sec}$

MHD: ideal limit

$$\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial\vec{B}}{\partial t}$$

To understand what happens when we are not in ideal limit, let consider a fluid in which the second term in the LHS is negligible

$$\frac{c^2}{4\pi\sigma}\nabla^2\vec{B} = \frac{\partial\vec{B}}{\partial t}$$

This is perfectly analogous to the heat diffusion equation, which by definition represents the dissipation: even if there is some heat concentrated in a small region, after a while it diffuses into all the space

→ the B field diffuses in space on a time scale

$$T_d = \frac{4\pi\sigma L^2}{c^2}$$

MHD: ideal limit

$$T_d = \frac{4\pi\sigma L^2}{c^2}$$

To understand whether the ideal MHD approximation fits the astrophysical situations, let calculate T_d for some concrete situation

Assume a conductivity for a hydrogen gas completely ionized of $\sigma = 6.98 \times 10^7 \frac{T^{3/2}}{\ln\Lambda} \text{ s}^{-1}$

Where T is the temperature in K and $\ln\Lambda \sim 30$ is an approximate factor called Coulomb logarithm (it arises from semi-classical calculation)

For a stellar interior for which $T \sim 10^7$ K and $L \sim 10^{11}$ cm, we find $T_d \sim 3 \times 10^{11}$ years, greater than universe lifetime, T_U

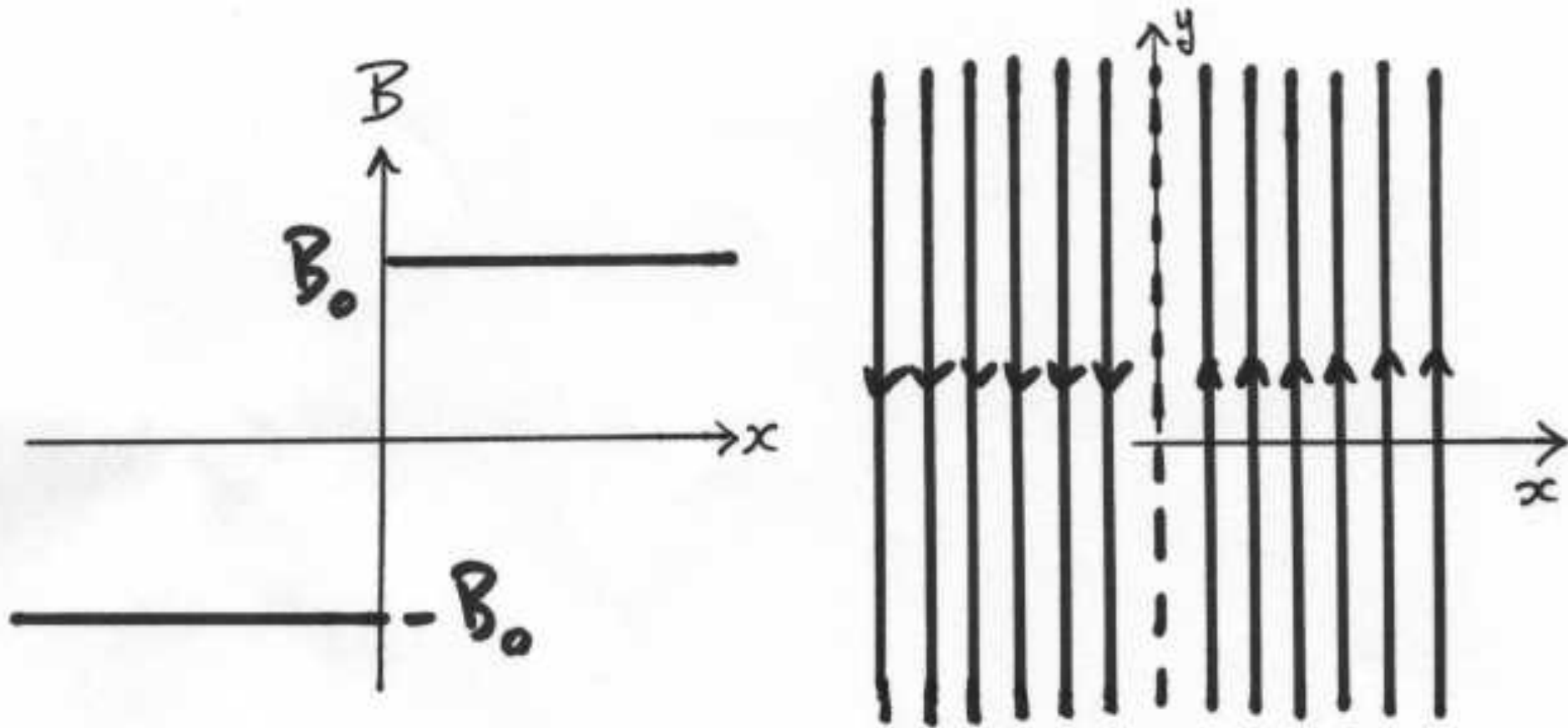
For a galaxy, L is very huge and $T_d > T_U$

But even for the smallest objects this time is long: in the inner regions of accretion disks around a black hole, with dimensions of ~ 100 Schwarzschild radii ($\sim 10^{15}$ cm), we find $T_d \sim 10^{20}$ years

In other words, the astrophysical fluids are (almost) all such that the dissipation time of the B field is greater than Universe lifetime: the B field is never dissipated

E. Fiandrini Cosmic Rays 1617 in typical astrophysical conditions

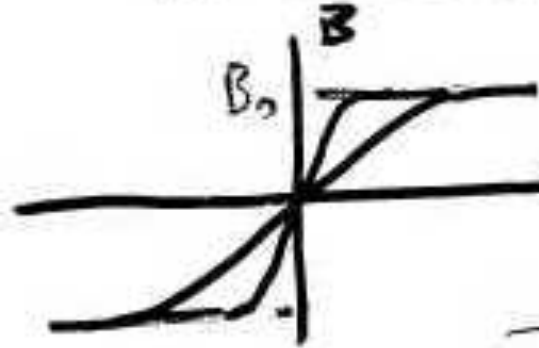
Example: given $B(x,0)$ as in figure, determine $B(x,t)$ in the hypothesis that diffusion dominates



To Solve

$$\frac{\partial B}{\partial t} = \epsilon \frac{\partial^2 B}{\partial x^2} \quad - (1)$$

for $B(x, t)$



$$\text{where } B(x, 0) = \begin{cases} B_0, & x > 0 \\ -B_0, & x < 0 \end{cases} \quad - (2)$$

Since initial conditions have no natural scale, seek a

Self-similar solution of the form

$$\underline{B(x, t) = f(V)}, \text{ where } \underline{V = \frac{x}{t^{1/2}}}$$

$$\text{Then } \frac{\partial B}{\partial t} = \frac{df}{dV} \left(-\frac{x}{2t^{3/2}} \right) = -\frac{V}{2t} \frac{df}{dV}$$

$$\frac{\partial B}{\partial x} = \frac{df}{dV} \frac{1}{t^{1/2}}, \quad \frac{\partial^2 B}{\partial x^2} = \frac{d^2 f}{dV^2} \frac{1}{t}$$

Thus (i) becomes
$$-\frac{V}{2} \frac{dF}{dV} = \frac{d^2 F}{dV^2}$$

Put $\frac{dF}{dV} = F$ and $V = U\sqrt{4\eta t} \Rightarrow \frac{dF}{dU} = -2UF$

Integrate $F = \text{const } e^{-U^2}$

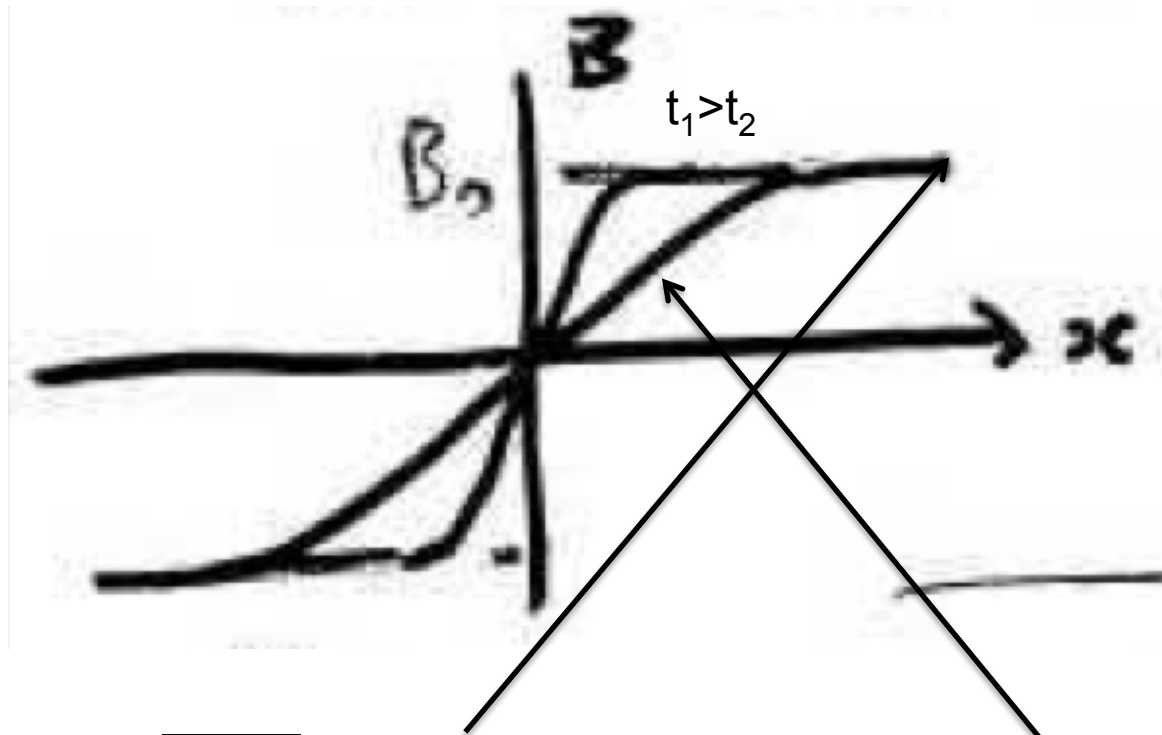
Integrate $f(U) = c \int_0^U e^{-u^2} du$

where $t \rightarrow 0, U \rightarrow \infty \Rightarrow c = \frac{2B_0}{\sqrt{\pi}}$
from (2)

$$\text{i.e. } B = \frac{2B_0}{\sqrt{\pi}} \int_0^{x/\sqrt{4\eta t}} e^{-u^2} du$$

For $x \gg \sqrt{4\eta t}$, $B \approx B_0$, *indep x*

For $x \ll \sqrt{4\eta t}$, $B \approx B_0 x / \sqrt{\pi\eta t}$, *so slope \downarrow with t*



For $x \gg \sqrt{4\eta t}$, $B \approx B_0$, *indep x*

For $x \ll \sqrt{4\eta t}$, $B \approx B_0 x / \sqrt{\pi\eta t}$, *so slope \downarrow with t*

MHD: ideal limit

The equations simplify in the so called "ideal MHD" when $R \gg 1$ or $\sigma \rightarrow \infty$

In such a case the energy equation turns back to the form of pure entropy conservation

$$\rho T \frac{Ds}{Dt} = \frac{j^2}{\sigma} \quad \longrightarrow \quad \rho T \frac{Ds}{Dt} = 0 \quad \longrightarrow \quad \frac{Ds}{Dt} = 0$$

Euler eqn remains the same $\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{j} \times \vec{B}/c$

With the aid of $(c/4\pi)\vec{\nabla} \times \vec{B} = \vec{j}$ $\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$

Mass conservation law is the same $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

The field equation simplifies

$$\frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} + \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial \vec{B}}{\partial t} \quad \longrightarrow \quad \vec{\nabla} \times \vec{v} \times \vec{B} = \frac{\partial \vec{B}}{\partial t}$$

These are the MHD equations in the ideal limit

(b) If $R_m \gg 1$

The induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

and Ohm's law -->

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}$$

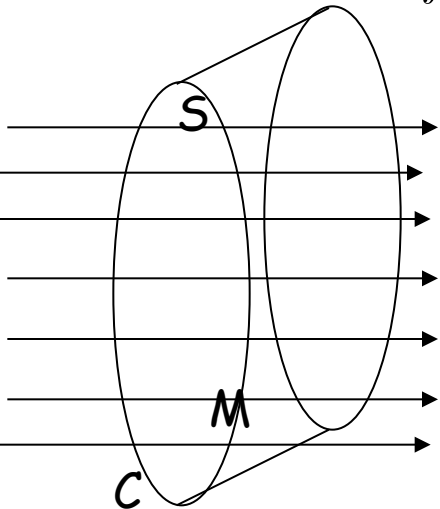
Magnetic field is “* frozen to the plasma *”

MHD: Frozen-in flux

Let assume that the B field is $B(r, t_0)$ at time t_0

The magnetic flux through a surface S enclosed by a curve C is

$$\Phi = \int_S \vec{B} \cdot d\vec{S}$$



if the curve moves, Φ changes because:
(a) the B field changes in time and (b) the field lines move into or out of S

As C moves, it creates a cylinder (a "flux tube") with a mantle surface M

Every field line leaving or entering C is associated with a flux of the SAME line through M

$$d\Phi = dt \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_M \vec{B} \cdot d\vec{S}_M$$

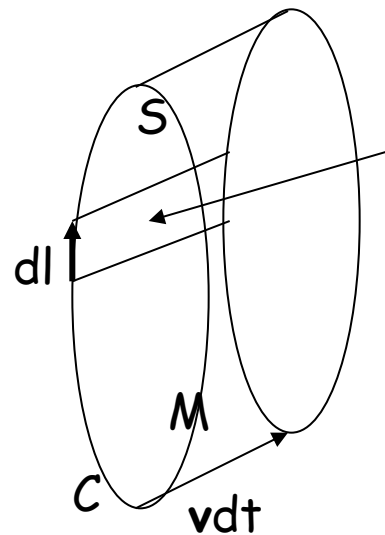
MHD

$$d\Phi = dt \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_M \vec{B} \cdot d\vec{S}_M$$

The surface element on M can be written as $d\vec{S}_M = \vec{v} \times d\vec{l} dt$

$$d\Phi = dt \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_C \vec{B} \cdot \vec{v} \times d\vec{l} dt \quad d\Phi/dt = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_C \vec{B} \cdot \vec{v} \times d\vec{l}$$

$$\vec{B} \cdot \vec{v} \times d\vec{l} = -\vec{v} \times \vec{B} \cdot d\vec{l} \quad d\Phi/dt = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} - \int_C \vec{v} \times \vec{B} \cdot d\vec{l}$$



$$\begin{aligned} d\vec{S}_M &= \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} - \int_S \vec{\nabla} \times \vec{v} \times \vec{B} \cdot d\vec{S} && \text{With the Stokes' theorem for the last term} \\ &= \int_S \left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{v} \times \vec{B} \right] \cdot d\vec{S} = - \int_S \vec{\nabla} \times \vec{j}/\sigma \cdot d\vec{S} = - \int_C \vec{j}/\sigma \cdot d\vec{l} \end{aligned}$$

In the convective limit, $R_M \gg 1$ or $S \ll 1$, ie $\sigma \rightarrow \infty$

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{v} \times \vec{B} = 0 \quad \Rightarrow \quad d\Phi/dt = 0$$

The magnetic flux cannot change \rightarrow therefore field lines must be swept away with the plasma motion, ie B is tied to the particles in the element: **FROZEN FIELD** \rightarrow flux tube does not break when plasma shuffles around!

Frozen-in Condition

In astrophysical systems with high R_m , we can imagine the magnetic flux to be frozen in the plasma and to move with the plasma flows

Suppose we have straight magnetic field lines going through a plasma column

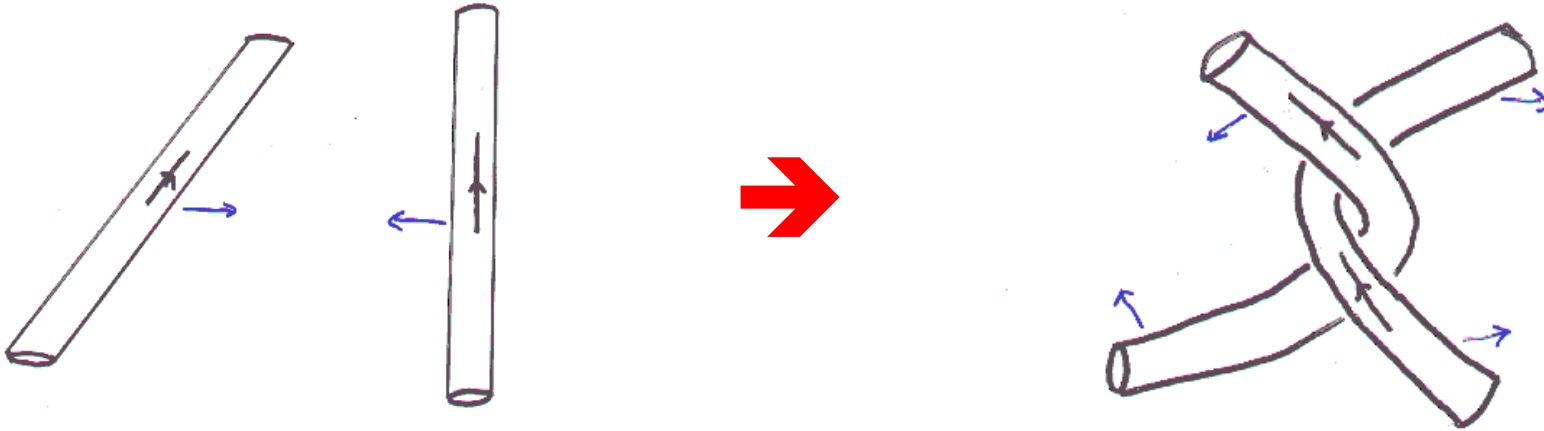
If the plasma column is bent, the magnetic field lines are bent too

On the other hand, if one end of the plasma column is twisted (because, for instance, is in rotation), then the magnetic field lines are twisted too

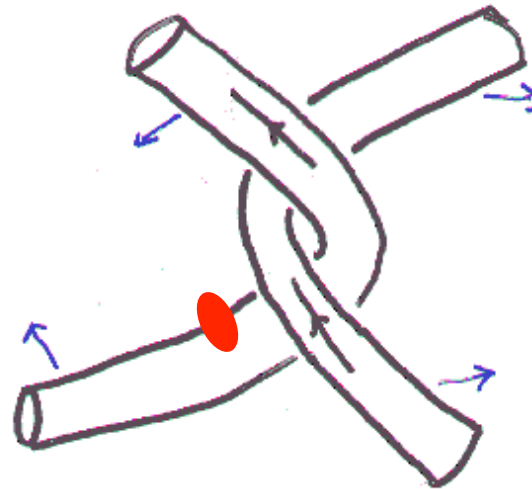
As a result of frozen-in theorem, the B field in a astrophysical system can be almost regarded as a plastic material which can be bent, twisted or distorted by making the plasma move appropriately

This view of a magnetic field is radically different from that we encounter in laboratory situations, where it appears as something rather passive which we can switch on or off by sending a current through a coil. In the astrophysical setting, the magnetic field appears to acquire a life of its own

Magnetic Field Lines Can't Break

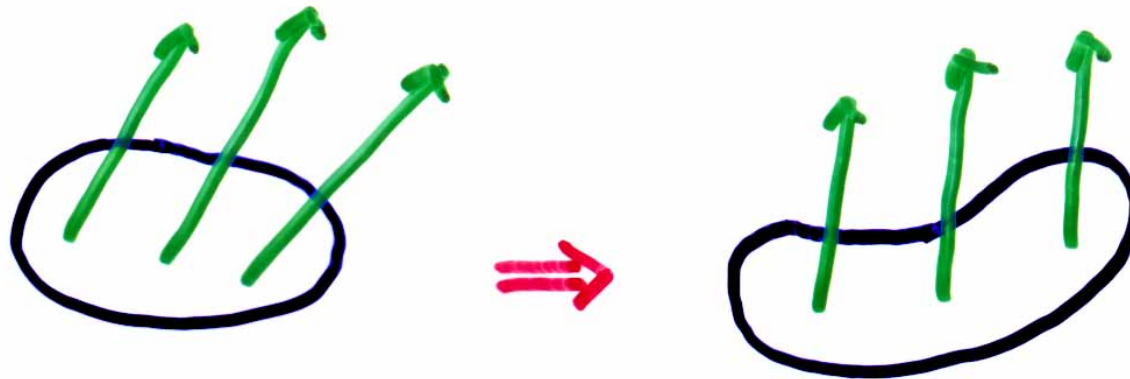


Everything
Breaks
Eventually



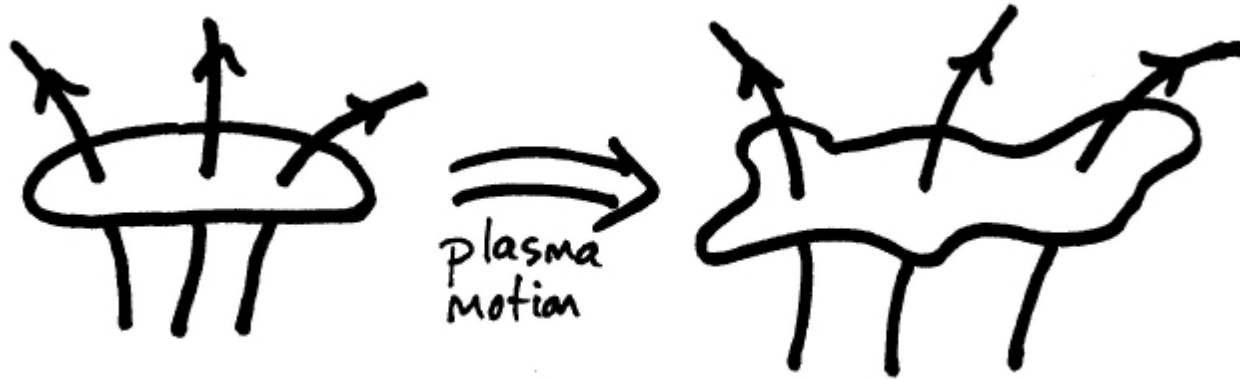
Frozen-in Condition

- In a simple form of plasma, the plasma moves so that the magnetic flux through any surface is preserved.

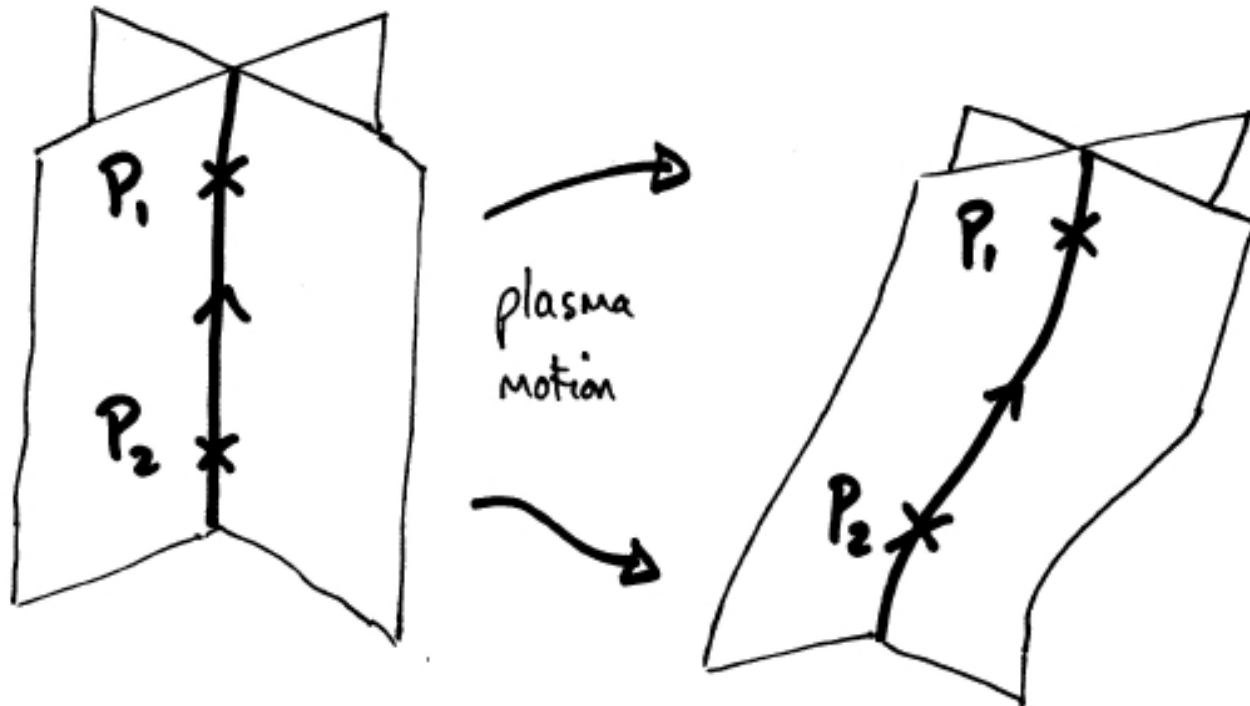


The frozen-in field condition is exactly the opposite of dissipation (ie magnetic diffusion), which tends to decrease the flux through the spire

Magnetic Flux Conservation:



Magnetic Field Line Conservation:



Frozen-in field: an example

When an astronomical object shrinks due to gravitational attraction, its magnetic field is expected to become stronger

If r is the radius of the equatorial cross-section of the body through which the magnetic field of the order of B is passing, then the flux is of the order of Br^2

If the field is perfectly frozen, then this flux should remain an invariant during the contraction of the object

Some neutron stars are believed to have B fields of the order of 10^{12} G

Let us see if we can explain this magnetic field by assuming that the neutron star formed due to the collapse of an ordinary star of which the B field got compressed

A star like the Sun has a radius $r=10^{11}$ cm and $B \sim 10$ G at poles

Since the radius of a typical NS is 10^6 cm, the equatorial area would decrease by a factor 10^{10} if an ordinary star were to collapse to become the NS

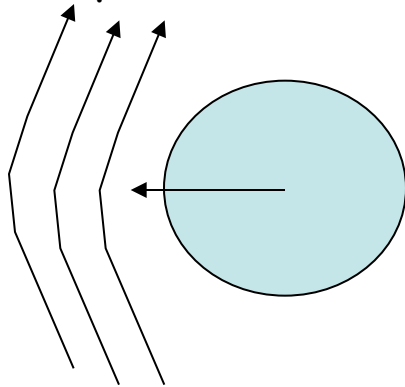
If the magnetic flux remained frozen during the collapse, the the initial field of 10 G would finally become 10^{11} G

MHD

$$d\Phi/dt = \int_S \left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{v} \times \vec{B} \right] \cdot d\vec{S} - \int_S \vec{\nabla} \times \vec{j}/\sigma \cdot d\vec{S} = - \int_C \vec{j}/\sigma \cdot d\vec{l}$$

A reversal of the concept, the frozen-out field, exists

If a field free plasma bubble moves toward a region filled with a magnetic field, then it pushes the field away until the pressures equals. The field cannot enter into the bubble because the magnetic flux inside the bubble would change. This happens with the solar wind frozen-out of the magnetosphere (here the field inside in the convective limit... $(R_M)_{\text{mgt sph}} \gg 1$)



Magnetic force:

$$\begin{aligned}\mathbf{j} \times \mathbf{B} &= (\nabla \times \mathbf{B}) \times \frac{\mathbf{B}}{\mu} \\ &= (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{\mu} - \nabla \left(\frac{B^2}{2\mu} \right)\end{aligned}$$

Magnetic field lines have a

Tension B^2 / μ ----> force when lines curved

Pressure $B^2 / (2\mu)$ ----> force from high to low B^2

Magnetic force(s)

The internal forces acting on a charged fluid are the mechanical pressure, already seen and the Lorentz force, which is new because due to the particle charge and the B field

$$\vec{f} = (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Using again the identity $(\vec{\nabla} \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2$

$$\vec{f} = \frac{1}{4\pi}[(\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2]$$

The first term can be rewritten

We note that it represents B times the derivative along the field line

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B \frac{\partial \vec{B}}{\partial s}$$

Where s is a curvilinear coordinate along the field line

Magnetic force(s) $\vec{f} = \frac{1}{4\pi} [(\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2]$

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B \frac{\partial \vec{B}}{\partial s}$$

Where s is a curvilinear coordinate along the field line

In general, B varies both in intensity and direction $\vec{B} = B\hat{b}$

With \hat{b} is the local tangent versor to B

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B \frac{\partial(B\hat{b})}{\partial s} = B^2 \frac{\partial \hat{b}}{\partial s} + B\hat{b} \frac{\partial B}{\partial s} \quad \text{but} \quad \frac{\partial B^2}{\partial s} = 2B \frac{\partial B}{\partial s}$$

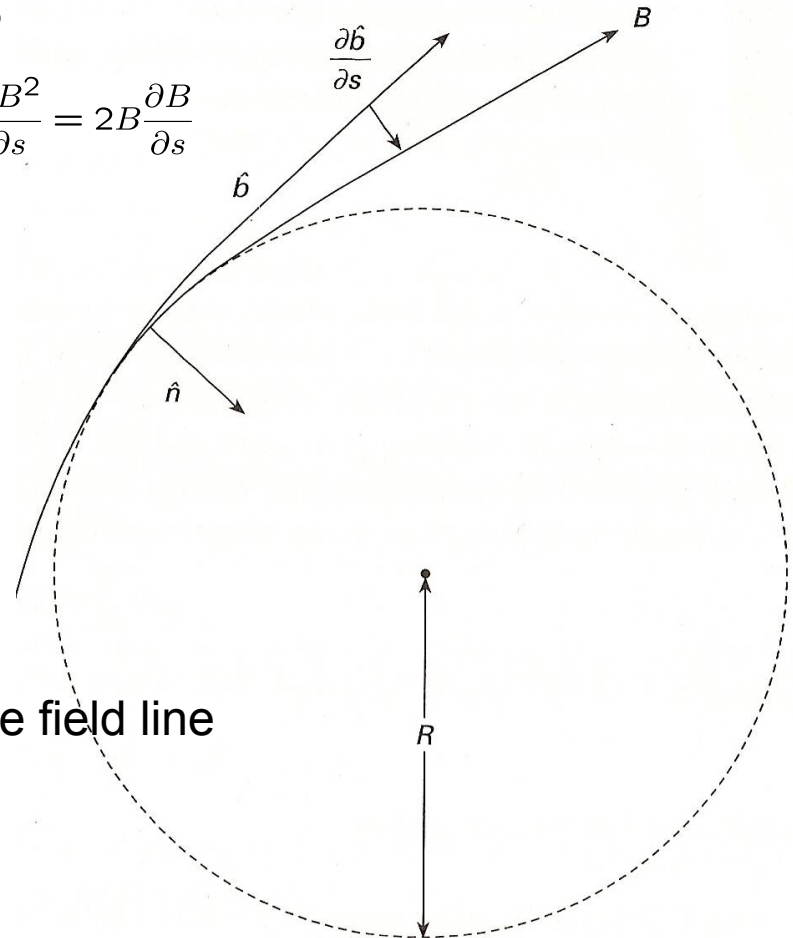
→
$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B^2 \frac{\partial \hat{b}}{\partial s} + \hat{b} \frac{\partial B^2}{2\partial s}$$

The first term is due to the change of the field direction. The versor derivative is, by definition, perp to the versor (ie field line) and measures the curvature of the field $\frac{\partial \hat{b}}{\partial s} = \frac{\hat{n}}{R}$

The 2nd term may rewritten as $\hat{b} \frac{\partial B^2}{2\partial s} = \hat{b} \vec{\nabla}_{\parallel} \left(\frac{B^2}{2} \right)$

Where grad_{\parallel} is the gradient component along the field line

→
$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B^2 \frac{\hat{n}}{R} + \hat{b} (\vec{\nabla})_{\parallel} \frac{B^2}{2}$$



Magnetic force(s)

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B^2 \frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel} \frac{B^2}{2}$$

Lorentz force is therefore $\vec{f} = \frac{1}{4\pi} [B^2 \frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel} \frac{B^2}{2} - \frac{1}{2} \vec{\nabla} B^2]$

Writing grad as $\vec{\nabla} = \hat{n} \nabla_{\perp} + \hat{b} \nabla_{\parallel}$ We get

$$\vec{f} = \frac{1}{4\pi} [B^2 \frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel} \frac{B^2}{2} - (\hat{n} \nabla_{\perp} + \hat{b} \nabla_{\parallel}) \frac{B^2}{2}] = \frac{1}{4\pi} [B^2 \frac{\hat{n}}{R} - \hat{n} \nabla_{\perp} \frac{B^2}{2}]$$

Parallel component of the gradient cancels with the corresponding pressure term, leaving only the perp component: both terms give forces normal to the field line (as it is reasonable to expect since the Lorentz force is $\sim \mathbf{v} \times \mathbf{B}$)

Magnetic force(s)

$$\vec{f} = \frac{1}{4\pi} \left[B^2 \frac{\hat{n}}{R} + \hat{b}(\vec{\nabla})_{\parallel} \frac{B^2}{2} - (\hat{n}\nabla_{\perp} + \hat{b}\nabla_{\parallel}) \frac{B^2}{2} \right] = \frac{1}{4\pi} \left[B^2 \frac{\hat{n}}{R} - \hat{n}\nabla_{\perp} \frac{B^2}{2} \right]$$

The first term represents a force directed along the instantaneous center of curvature: it tends therefore to shorten the field line

This term is very similar to the elastic force exerted on a guitar string when is tweaked: it exerts a force that tends to bring the string back to its minimum length

That's why it's called **magnetic tension**

The 2nd term is the gradient of a scalar quantity $\rightarrow B^2/2$ can be interpreted as a **magnetic pressure**, opposite to the gradient: the more the field is increasing along perp dir, the more is the force that tends to decrease the field...in terms of field lines this means when you try to compress field lines, there is a force which opposes to the compression

Curvature can be eliminated since

$$\vec{\nabla}_{\perp} B = -\frac{B}{R} \hat{m}$$

In P, $\vec{B}(0,0,B)$ e $\boxed{\frac{\partial B_x}{\partial z} = 0}$

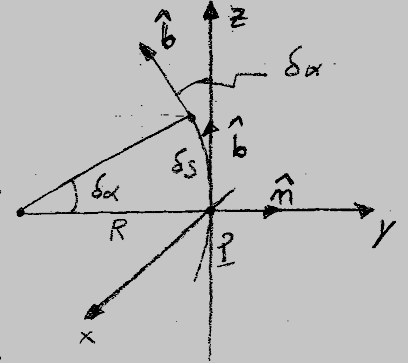
$$\vec{\nabla} \times \vec{B} = 0$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = 0$$

$$\boxed{-\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} = 0}$$

$$\boxed{\frac{\partial B_z}{\partial x} = 0}$$

Due to osculator circle in P



$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

$$\rightarrow \vec{\nabla} B = \frac{\partial}{\partial x} [(B_x^2 + B_y^2 + B_z^2)^{1/2}] = \frac{2B_x \frac{\partial B_x}{\partial x} + 2B_y \frac{\partial B_y}{\partial x} + 2B_z \frac{\partial B_z}{\partial x}}{2B} = \left(\frac{B_z}{B}\right) \frac{\partial B_z}{\partial x} \stackrel{1}{=} 0$$

$$\frac{\partial}{\partial y} [(\dots)^{1/2}] = \frac{B_x \frac{\partial B_x}{\partial y} + B_y \frac{\partial B_y}{\partial y} + B_z \frac{\partial B_z}{\partial y}}{B} = \frac{B_z}{B} \frac{\partial B_z}{\partial y} = \frac{\partial B}{\partial y}$$

$$\frac{\partial}{\partial z} [(\dots)^{1/2}] = \frac{\partial B}{\partial s} = (\hat{b} \cdot \vec{\nabla}) B \equiv \nabla_{\parallel} B$$

Gradient lies in the yz-plane

$$\text{But } \vec{\nabla} = \vec{\nabla}_{\parallel} + \vec{\nabla}_{\perp} \Rightarrow \nabla_{\perp} B = \frac{\partial B}{\partial y} \Rightarrow \vec{\nabla} B = \nabla_{\parallel} B \hat{b} + \nabla_{\perp} B \hat{m}$$

The trasverse comp (B_y) changes when moving along field line

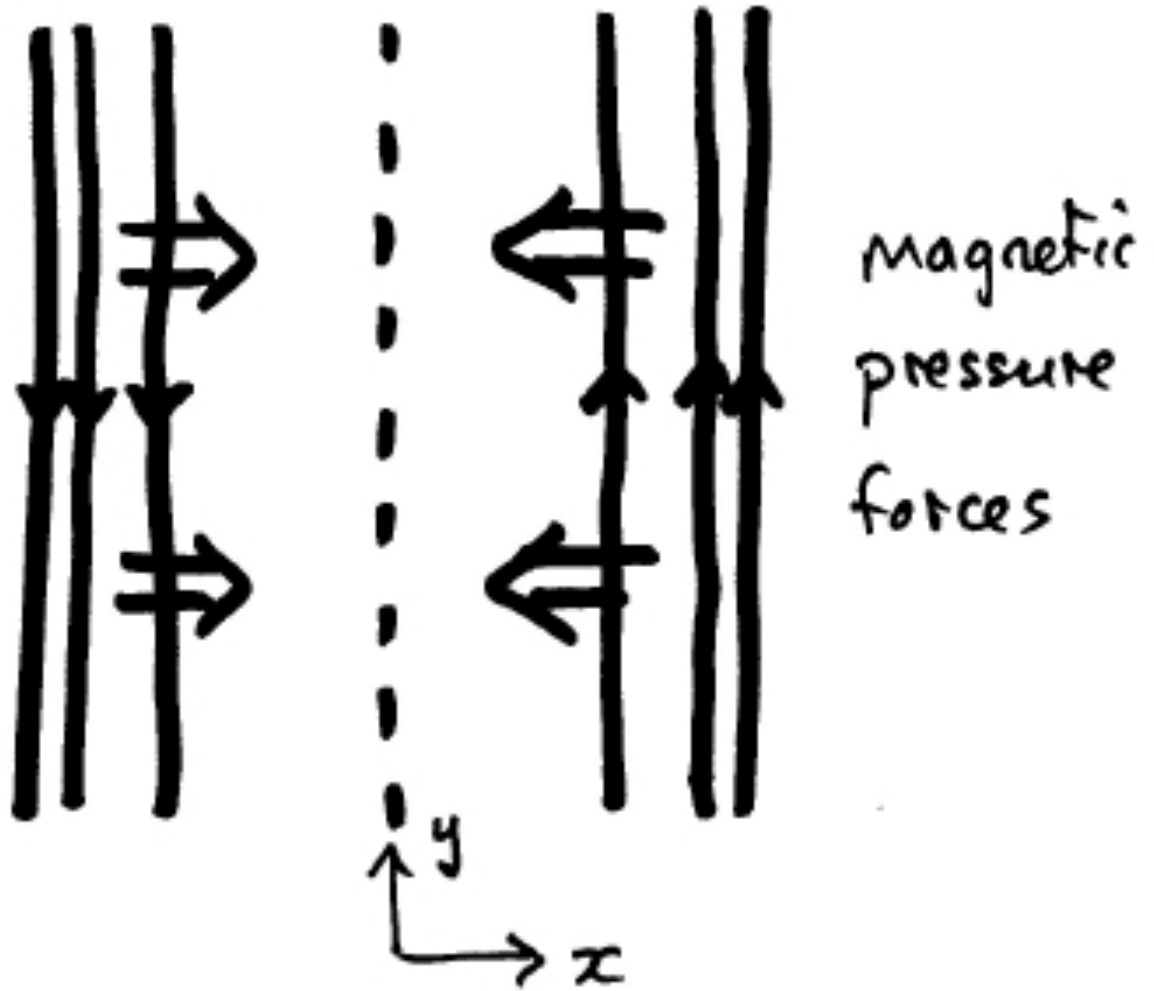
$$dB_y \approx -B \sin \delta \alpha \approx -B \delta \alpha \approx -B \frac{\delta s}{R} \Rightarrow \frac{\partial B_y}{\partial s} = \frac{\partial B_y}{\partial z} \approx -\frac{B}{R}$$

$$\text{But } \frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y} \text{ from } \vec{\nabla} \times \vec{B} = 0 \Rightarrow \frac{\partial B_z}{\partial y} \approx -\frac{B}{R}$$

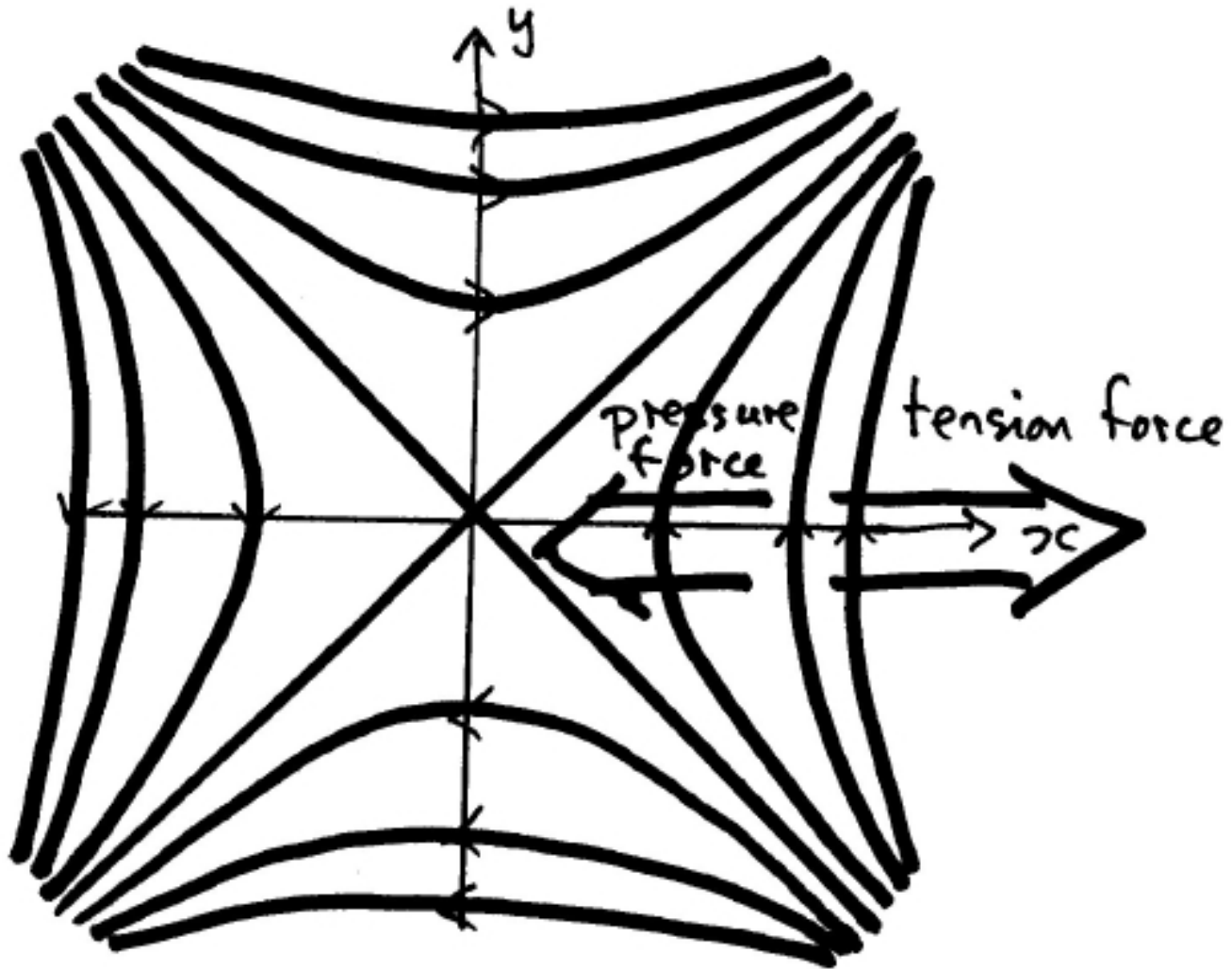
E. Fiandrini Cosmic Rays 1617 $\nabla_{\perp} B \approx -\frac{B}{R}$ or in vector form $\frac{\vec{\nabla}_{\perp} B}{B} = -\frac{\hat{n}}{R}$ (with $\vec{\nabla}_{\perp} = \hat{m} \nabla_{\perp}$)

Ex.

$$\mathbf{B} = x \hat{\mathbf{y}}$$



Ex. $\mathbf{B} = y \hat{\mathbf{x}} + x \hat{\mathbf{y}}$



Conservative form

We have already seen that the equation of hydrodynamics can be written in a manifestly conservative form

$$\frac{\partial(\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{V} + P \mathbf{I}) = -\rho \nabla \Phi = 0 \quad \text{or in components} \quad \frac{\partial(\rho v_i)}{\partial t} + \nabla_i(\rho v_i v_j + p \delta_{ij}) = 0$$

Einstein convention adopted

The tensor $\mathbf{R}_{ik} = \rho \mathbf{V}_i \mathbf{V}_k + p \delta_{ik}$ is the Reynolds stress tensor for an ideal fluid and represents the mechanical momentum flux $\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial R_{ik}}{\partial x_k}$

This is possible for MHD too

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} p + (1/4\pi)(\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Make use of the identity $(\vec{\nabla} \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \vec{\nabla}) \vec{B} - \frac{1}{2} \vec{\nabla} B^2$

The double vector prod can be written as the divergence of a tensor

$$[(\vec{\nabla} \times \vec{B}) \times \vec{B}]_i = B_j \nabla_j B_i - \frac{1}{2} \nabla_i B^2 \quad \text{Now} \quad \nabla_j (B_j B_i) = B_i \nabla_j B_j + B_j \nabla_j B_i \quad \text{but} \quad \nabla_j B_j = \vec{\nabla} \cdot \vec{B} = 0$$

$$\implies \nabla_j (B_j B_i) = B_j \nabla_j B_i \implies [(\vec{\nabla} \times \vec{B}) \times \vec{B}]_i = \nabla_j (B_j B_i) - \frac{1}{2} \nabla_i B^2 = \nabla_j (B_j B_i) - \nabla_j \frac{1}{2} B^2 \delta_{ij}$$

$$= \nabla_j [(B_j B_i) - \frac{1}{2} B^2 \delta_{ij}]$$

$$\implies \rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} p + (1/4\pi) \nabla_j [(B_j B_i) - \frac{1}{2} B^2 \delta_{ij}]$$

Conservative form

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i(\rho v_i v_j + p \delta_{ij}) = 0 \quad \mathbf{R}_{ik} = \rho \mathbf{V}_i \mathbf{V}_k + p \delta_{ik} \quad \frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial R_{ik}}{\partial x_k}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \frac{1}{4\pi} \nabla_j [(B_j B_i) - \frac{1}{2} B^2 \delta_{ij}]$$

The Euler eqn can be put therefore in the same form as for hydrodynamics case defining

$$M_{ij} = -\frac{1}{4\pi} [(B_j B_i) - \frac{1}{2} B^2 \delta_{ij}]$$

And $T_{ij} = R_{ij} + M_{ij} = \rho v_i v_j + p \delta_{ij} - \frac{1}{4\pi} [(B_j B_i) - \frac{1}{2} B^2 \delta_{ij}] \quad \Rightarrow \quad \frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial T_{ik}}{\partial x_k}$

M_{ij} describes flux per time unit of the i-th component of the B field momentum through a surface oriented along the j-th direction and it's called Maxwell stress tensor

The relative importance in concrete astrophysical situations, as pulsar winds and black hole jets, of the mechanical and magnetic momentum flux is, at the moment, uncertain and of great interest

Conservative form

It is well known that elm field transports energy too

The energy density of the field is $\epsilon_B = B^2/8\pi$

The energy flux is described by Poynting vector $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$

We have to eliminate the E field

Consider the Ohm's law $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}/c)$

In the ideal limit $\sigma \rightarrow \infty$, to avoid infinite currents we must have

$$\vec{E} + \vec{v} \times \vec{B}/c = 0$$

$$\Rightarrow \vec{E} = -\vec{v} \times \vec{B}/c \quad \Rightarrow \quad \vec{S} = -\frac{1}{4\pi}(\vec{v} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi} \vec{B} \times (\vec{v} \times \vec{B})$$

Conservative form

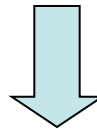
$$\epsilon_B = B^2/8\pi \qquad \vec{S} = -\frac{1}{4\pi}(\vec{v} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}\vec{B} \times (\vec{v} \times \vec{B})$$

We have to add these terms into the energy equation of the hydro case

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho V^2 + \rho e + \rho \Phi \right) + \nabla \cdot \left[\rho \mathbf{V} \left(\frac{1}{2} V^2 + h + \Phi \right) \right] = \mathcal{H}_{\text{eff}}$$

$$h = \epsilon + P/\rho$$

$$\mathcal{H}_{\text{eff}} \equiv \mathcal{H} + \rho \frac{\partial \Phi}{\partial t}$$



$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho e + \rho \Phi + \frac{1}{8\pi} B^2 \right) + \vec{\nabla} \cdot \left[\vec{v} \left(\frac{1}{2} \rho v^2 + \rho h + \rho \Phi \right) + \frac{1}{4\pi} \vec{B} \times (\vec{v} \times \vec{B}) \right] = H_{\text{eff}}$$

With no irreversible losses and no time dependent gravitational potential $H_{\text{eff}}=0$

Summary in the ideal limit

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0 \quad \text{Mass conservation law}$$

M density M flux

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_i[\rho v_i v_j + p \delta_{ij} - \frac{1}{4\pi}(B_j B_i - \frac{1}{2}B^2 \delta_{ij})] = 0 \quad \text{Momentum conservation law}$$

p density p flux

Energy conservation law

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2 + \rho e + \rho \Phi + \frac{1}{8\pi}B^2) + \vec{\nabla} \cdot [\vec{v}(\frac{1}{2}\rho v^2 + \rho e + P + \rho \Phi) + \frac{1}{4\pi}\vec{B} \times (\vec{v} \times \vec{B})] = 0$$

E density

E flux

Small perturbations

We will treat the small perturbations of a magnetohydrodynamical equilibrium

Let choose an equilibrium configuration of an omogeneous, infinite medium at rest for sake of simplicity (but the method can be applied to any equilibrium configuration, even non omogeneous and time dependent)

Let therefore consider a configuration in which $p=p_o$, $\rho=\rho_o$, $\mathbf{v}=0$ and $\mathbf{B}=B_o\mathbf{z}$

Consider now a situation in which the quantities are perturbed from equilibrium, that is $p=p_o+\delta p$, $\rho=\rho_o+\delta\rho$, $\mathbf{v}=\delta\mathbf{v}$ and $\mathbf{B}=B_o\mathbf{z}+\delta\mathbf{B}$

Our goal is find perturbations corresponding to acoustic waves, therefore let assume that perturbations are iso-entropic, so that we have a relation between pressure and density:

$$\delta p = c_s^2 \delta \rho$$

With $c_s = \sqrt{\gamma p_o / \rho_o}$ iso-entropic sound speed

Small perturbations

The motion equations, under the assumption that perturbations are small, can be linearized

$$\begin{cases} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 & (i) \\ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{1}{4\pi\rho} (\vec{\nabla} \times \vec{B}) \times \vec{B} & (ii) \\ \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) & (iii) \end{cases}$$

$$\begin{aligned} \rho &= \rho_0 + \delta\rho & p &= p_0 + \delta p & \vec{v} &= \delta\vec{v} & \vec{B} &= B_0 \hat{z} + \delta\vec{B} \\ \delta\rho &\ll \rho_0 & \delta p &\ll p_0 & \delta\vec{v} &\ll 1 & \delta\vec{B} &\ll B_0 \hat{z} \\ \delta p &= c_s^2 \delta\rho & c_s^2 &= \frac{\delta p}{\delta\rho} \end{aligned}$$

$$(i) \quad \frac{\partial}{\partial t} (\rho_0 + \delta\rho) + \vec{\nabla} \cdot [(\rho_0 + \delta\rho) \delta\vec{v}] = \frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla} \cdot \delta\vec{v} + \vec{\nabla} \cdot (\delta\rho \delta\vec{v}) \approx \frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla} \cdot \delta\vec{v} \Rightarrow \frac{\partial}{\partial t} \left(\frac{\delta\rho}{\rho_0} \right) \approx -\vec{\nabla} \cdot \delta\vec{v}$$

$$(ii) \quad \frac{\partial \delta\vec{v}}{\partial t} + (\delta\vec{v} \cdot \vec{\nabla}) \delta\vec{v} = -\frac{\vec{\nabla}(\rho_0 + \delta\rho)}{(\rho_0 + \delta\rho)} + \frac{1}{4\pi(\rho_0 + \delta\rho)} \left[\vec{\nabla} \times (B_0 \hat{z} + \delta\vec{B}) \right] \times (B_0 \hat{z} + \delta\vec{B})$$

$$\begin{aligned} \frac{\partial \delta\vec{v}}{\partial t} &\approx -\frac{\vec{\nabla} \delta\rho}{\rho_0 + \delta\rho} + \frac{1}{4\pi(\rho_0 + \delta\rho)} \left[\underbrace{(\vec{\nabla} \times B_0 \hat{z}) \times B_0 \hat{z}}_{=0} + \underbrace{(\vec{\nabla} \times B_0 \hat{z}) \times \delta\vec{B}}_{=0} + (\vec{\nabla} \times \delta\vec{B}) \times B_0 \hat{z} + \underbrace{(\vec{\nabla} \times \delta\vec{B}) \times \delta\vec{B}}_{\ll 1} \right] \approx \\ &\approx -\frac{\vec{\nabla} \delta\rho}{\rho_0} + \frac{1}{4\pi\rho_0} (\vec{\nabla} \times \delta\vec{B}) \times B_0 \hat{z} = -c_s^2 \vec{\nabla} \left(\frac{\delta\rho}{\rho_0} \right) + \frac{1}{4\pi\rho_0} (\vec{\nabla} \times \delta\vec{B}) \times B_0 \hat{z} \end{aligned}$$

$$(iii) \quad \frac{\partial \delta\vec{B}}{\partial t} = \vec{\nabla} \times [\delta\vec{v} \times (B_0 \hat{z} + \delta\vec{B})] = \vec{\nabla} \times (\delta\vec{v} \times B_0 \hat{z}) + \vec{\nabla} \times (\delta\vec{v} \times \delta\vec{B}) \approx \vec{\nabla} \times (\delta\vec{v} \times B_0 \hat{z})$$

(iv) $Ds/Dt=0$ is automatically satisfied since entropy is assumed constant

The magnetic field equation can be furtherly simplified noting that B_0 has null all the spatial derivatives

$$\begin{aligned}\vec{\nabla} \times (\delta \vec{v} \times \vec{B}) &= (\vec{B} \cdot \nabla) \delta \vec{v} - (\delta \vec{v} \cdot \nabla) \vec{B} - (\vec{B} \nabla) \cdot \delta \vec{v} + \delta \vec{v} (\nabla \cdot \vec{B}) \\ &= (\vec{B} \cdot \nabla) \delta \vec{v} - (\vec{B} \nabla) \cdot \delta \vec{v} \quad \text{The terms } \text{grad}(B)=0 \text{ and } (\vec{v} \cdot \text{grad})B=0 \\ &= B_0 \frac{\partial}{\partial z} \delta \vec{v} - B_0 \hat{z} \nabla \cdot \delta \vec{v}\end{aligned}$$

$$\frac{\partial}{\partial t} \delta \vec{B} = B_0 \frac{\partial}{\partial z} \delta \vec{v} - B_0 \hat{z} \nabla \cdot \delta \vec{v}$$

Small perturbations

$$\frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho_o} \right) + \nabla \cdot \delta \vec{v} = 0$$

This is a system of partial derivative equations

$$\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla \left(\frac{\delta \rho}{\rho_o} \right) + \frac{1}{4\pi \rho_o} (\nabla \times \delta \vec{B}) \times B_o \hat{z}$$

Since the coefficients are constant we can develop the unknowns in Fourier series and find a solution for each Fourier amplitude

$$\frac{\partial \delta \vec{B}}{\partial t} - \vec{\nabla} \times (\delta \vec{v} \times B_o \hat{z}) = 0 \text{ or } \frac{\partial}{\partial t} \delta \vec{B} = B_o \frac{\partial}{\partial z} \delta \vec{v} - B_o \hat{z} \nabla \cdot \delta \vec{v}$$

Derive wrt t the momentum equation

$$\frac{\partial^2 \vec{v}}{\partial t^2} = -c_s^2 \nabla \left(\frac{\partial}{\partial t} \left[\frac{\delta \rho}{\rho_o} \right] \right) + \frac{1}{4\pi \rho_o} (\nabla \times \frac{\partial}{\partial t} \delta \vec{B}) \times B_o \hat{z} \quad \text{Use 1 and 3 to eliminate time deriv in RHS}$$

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_s^2 \nabla (\nabla \cdot \delta \vec{v}) + \frac{1}{4\pi \rho_o} (\nabla \times (\vec{\nabla} \times (\delta \vec{v} \times B_o \hat{z}))) \times B_o \hat{z}$$

Introducing the Alven speed

$$\vec{v}_A \equiv \frac{\vec{B}_o}{\sqrt{4\pi \rho_o}}$$

We get

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_s^2 \nabla (\nabla \cdot \delta \vec{v}) + (\nabla \times (\vec{\nabla} \times (\delta \vec{v} \times \vec{v}_A))) \times \vec{v}_A$$

This is the wave equation for MHD

Small perturbations

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_s^2 \nabla (\nabla \cdot \delta \vec{v}) + (\nabla \times (\vec{\nabla} \times (\delta \vec{v} \times \vec{v}_A))) \times \vec{v}_A$$

$$\vec{v}_A \equiv \frac{\vec{B}_o}{\sqrt{4\pi\rho_o}}$$

In absence of B field, it reduces to the acoustic wave equation:

In fact taking 1-dim case, ie $\delta \mathbf{v}$ directed along the x-axis $(\delta v, 0, 0)$, we get

$$\nabla \cdot \delta \vec{v} = \frac{\partial v}{\partial x} \quad \nabla = \frac{\partial}{\partial x} \quad \longrightarrow \quad \frac{\partial^2 v}{\partial t^2} = c_s^2 \frac{\partial^2 v}{\partial x^2}$$

When B field is present, the situation is much more complicated:

- i) There are two speeds involved in the wave propagation, c_s and v_A
- ii) There are waves types which are not acoustic, since a speed completely dependent on the unperturbed B field appears

(i) Sound Waves ($\mathbf{B}_0 = \mathbf{0}$)

Uniform medium of pressure p_0 , density ρ_0

Disturbance $\mathbf{v} = \mathbf{v}_1$, $p = p_0 + p_1$, $\rho = \rho_0 + \rho_1$

Linearise eqns motion, continuity, energy

$$(p / \rho^\gamma = c)$$

Fourier analyse $\mathbf{v}_1, p_1, \rho_1 \approx \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$

→

$$\omega^2 = k^2 c_s^2$$

Dispersion Relation

→

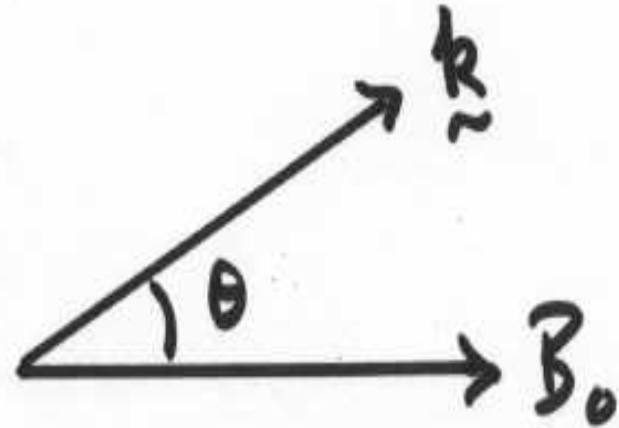
Waves propagate with speed

$$\omega / k = c_s$$

(ii) Magnetic Waves ($p_0 = 0$)

Repeat, but uniform (B_0)

- include $\mathbf{j} \times \mathbf{B}$ force
- assume wave propagates at angle to B_0



Either $\omega^2 = k^2 v_A^2 \cos^2 \theta$ *Alfvén Waves*

Incompressible - due to magnetic tension

Or $\omega^2 = k^2 v_A^2$ Compressional Alfvén Waves

Compressible - due to magnetic pressure
- propagate at speed v_A

(iii) MHD Waves (p_0 and B_0 nonzero)

- Alfvén Wave is unaffected
- Compressional Alfvén Wave and Sound Wave are coupled:

Slow Magnetoacoustic Wave (Slow-Mode)
+ **Fast Magnetoacoustic Wave** (Fast-Mode)

Propagate slower/faster than Alfvén Wave

Small perturbations

Therefore let look for solutions of type

$$\frac{\delta\rho}{\rho_o} = r e^{i\Phi} \quad \delta\vec{v} = \vec{V} e^{i\Phi} \quad \frac{\delta\vec{B}}{B_o} = \vec{b} e^{i\Phi} \quad \Phi = \vec{k} \cdot \vec{r} - \omega t$$

With k and ω completely arbitrary at this moment

Inserting these solutions into the equations we get

$$\omega r = \vec{k} \cdot \vec{V}$$

$$\omega \vec{V} = c_s^2 r \vec{k} - \frac{(\vec{k} \times \vec{b}) \times \hat{z} B_o^2}{4\pi\rho_o}$$

$$\omega \vec{b} = -k_z \vec{V} + \hat{z} \vec{k} \cdot \vec{V}$$

In this way the coupled equation system becomes a simple omogeneous linear system in the unknowns r, \vec{V} and \vec{b}

The system admits only the trivial solution $r=\vec{V}=\vec{b}=0$, unless the determinant of the system is zero \rightarrow the condition for

non zero solutions to exist is that $\det=0$

Small perturbations

It is convenient to decompose the wave vector \mathbf{k} in its components \parallel and perp to \mathbf{B}_0 (directed along z dir) $\vec{k} = k_{\parallel} \hat{z} + k_{\perp} \hat{y}$

After a little bit of algebra, we find the determinant of the system

$$\det \begin{pmatrix} \omega^2 - v_A^2 k_{\parallel}^2 & 0 & 0 \\ 0 & \omega^2 - v_A^2 k_{\parallel}^2 - (c_s^2 + v_A^2) k_{\perp}^2 & -c_s^2 k_{\parallel} k_{\perp} \\ 0 & -c_s^2 k_{\parallel} k_{\perp} & \omega^2 - c_s^2 k_{\perp}^2 \end{pmatrix} = 0$$

$$(\omega^2 - v_A^2 k_{\parallel}^2)(\omega^4 - \omega^2(c_s^2 + v_A^2)k^2 + c_s^2 v_A^2 k_{\parallel}^2 k^2) = 0$$

This establishes a relation between the wave frequency ω and the wave vector k , which is called dispersion relation

$v_f = \omega/k$ is not the real wave speed (but only the phase speed), which is the group speed $v_g = \partial\omega/\partial k$

There is a dispersion (ie v depends on ω) every time phase and group velocity differ

Small perturbations

Occasionally it may happens that, for a given k , ω has an imaginary part (ie is complex)

Remembering that all the physical quantities depend on time as $e^{i\omega t}$, this means that the phenomenon either is strongly dumped or is strongly amplified

When even only one solution is amplified, one speaks of **instabilities** of the 0th order solutions: perturbations, even if initially small, tend to increase without bound with time

This is not the case for previous relation dispersion

Small perturbations

$$(\omega^2 - v_A^2 k_{\parallel}^2)(\omega^4 - \omega^2(c_s^2 + v_A^2)k^2 + c_s^2 v_A^2 k_{\parallel}^2 k^2) = 0$$

This equation has three distinct solutions in ω^2 (ie six independent solutions in ω) but the pure k^2 dependence in the dispersion relation tells us that conjugated solutions $\pm \omega$ represent the same wave propagating in the $+B$ and $-B$ directions

The first is $\omega^2 = v_A^2 k_{\parallel}^2$ and it is called Alfvén's wave

Its peculiarity is that it is transmitted exclusively along the B field direction with the characteristic speed v_A , as shown by the presence of the $\parallel k$ component only in the spatial part

The 2nd and 3rd are $\omega^2 = \frac{k^2}{2}(c_s^2 + v_A^2 \pm \sqrt{(c_s^2 - v_A^2)^2 + 4c_s^2 v_A^2 k_{\perp}^2})$

Called magneto-sonic waves, fast or slow, depending on the sign in front of square root

The fact that the dispersion relation is even in k , that depends exclusively on k^2 tells us that the two propagation directions are completely equivalent

Small perturbations

$$\omega^2 = v_A^2 k_{\parallel}^2 \quad \text{Alfven's wave}$$

$$\omega^2 = \frac{k^2}{2} (c_s^2 + v_A^2 \pm \sqrt{(c_s^2 - v_A^2)^2 + 4c_s^2 v_A^2 k_{\perp}^2}) \quad \text{magneto-sonic waves, fast or slow}$$

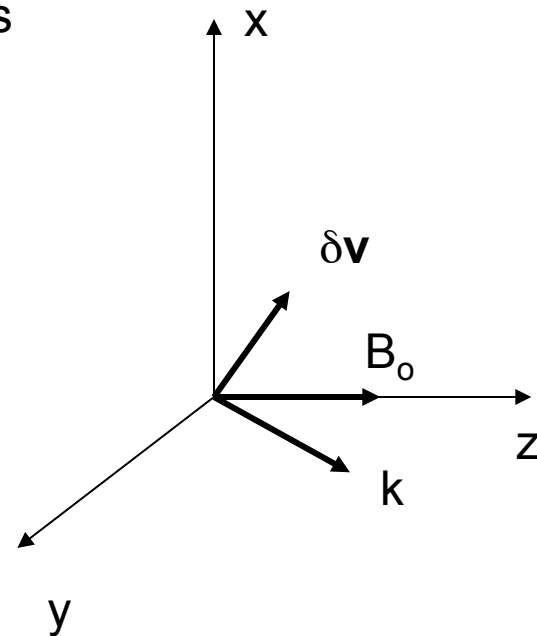
Since there are three independent solution there must be three distinct modes of oscillation

In principle, these modes could be identified finding the solutions, linearly independent and orthogonal of the system, but there is a simpler, more physical way to follow

Small perturbations

Let choose a suitable reference frame: z axis oriented along the B field, k in the yz plane (this simplifies the representation with no loss of generality)

The speed perturbation has three components



With the reference choice k_n lies in the xy plane and $k_{||}$ is along z -axis

Small perturbations

Let consider the perturbed momentum equation $\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla \left(\frac{\delta \rho}{\rho_o} \right) + \frac{1}{4\pi \rho_o} (\nabla \times \delta \vec{B}) \times B_o \hat{z}$

Get the z component, that is along the unperturbed field direction and put the solutions in

$$\frac{\delta \rho}{\rho_o} = r e^{i\Phi} \quad \delta \vec{v} = \vec{V} e^{i\Phi} \quad \frac{\delta \vec{B}}{B_o} = \vec{b} e^{i\Phi} \quad \Phi = \vec{k} \cdot \vec{r} - \omega t$$

We find that $\omega \delta v_z = k_z \frac{\delta p}{\rho_o}$ (the double vector prod is by construction perp to z)

So, every perturbation of the speed along the field direction is exclusively due to pressure gradient and is not connected to any perturbation of the B field and can be considered as an acoustic wave

Let consider now the y comp of the equation $\omega \delta B_z = k_\perp \delta p + \frac{B_o}{4\pi} (k_\perp \delta B_z - k_\parallel \delta B_y)$

The div equation $\text{div} \vec{B} = 0$ for the field becomes $k_\parallel \delta B_z = -k_\perp \delta B_y$

Small perturbations

$$\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla \left(\frac{\delta \rho}{\rho_o} \right) + \frac{1}{4\pi \rho_o} (\nabla \times \delta \vec{B}) \times B_o \hat{z}$$

Z comp

$$\omega \delta v_z = k_z \frac{\delta p}{\rho_o}$$

Y comp

$$\omega \delta B_z = k_{\perp} \delta p + \frac{B_o}{4\pi} (k_{\perp} \delta B_z - k_{\parallel} \delta B_y)$$

$$k_{\parallel} \delta B_z = -k_{\perp} \delta B_y \quad \text{Insert this condition in y comp eqn}$$

$$\omega \rho_o k_{\perp} \delta v_y = k_{\perp}^2 \delta p + k^2 \frac{B_o \delta B_z}{4\pi} = k_{\perp}^2 \delta p + k^2 \frac{\delta B_z^2}{8\pi} \quad \text{Use now the z comp eqn and finally we get}$$

$$\vec{k} \cdot \delta \vec{v} = \frac{k^2}{\omega \rho_o} \delta (p + B^2/8\pi)$$

From this we see this wave has as restoring force the total pressure of the fluid, that is mechanical + magnetic pressure

For this reason they are called **magneto-sonic** waves

The reason why there are slow and fast modes depends on the fact that p and $B^2/8\pi$ may have signs equal or opposite: when with same sign, the restoring force is greater and the waves are faster, when with opposite signs the two restoring forces δp and δB^2 tend to cancel and the wave will propagate more slowly

Small perturbations

$$\frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla \left(\frac{\delta \rho}{\rho_o} \right) + \frac{1}{4\pi \rho_o} (\nabla \times \delta \vec{B}) \times B_o \hat{z}$$

$$\begin{aligned} \frac{\delta \rho}{\rho_o} &= r e^{i\Phi} \\ \frac{\delta \vec{B}}{B_o} &= \vec{b} e^{i\Phi} \\ \delta \vec{v} &= \vec{V} e^{i\Phi} \\ \Phi &= \vec{k} \cdot \vec{r} - \omega t \end{aligned}$$

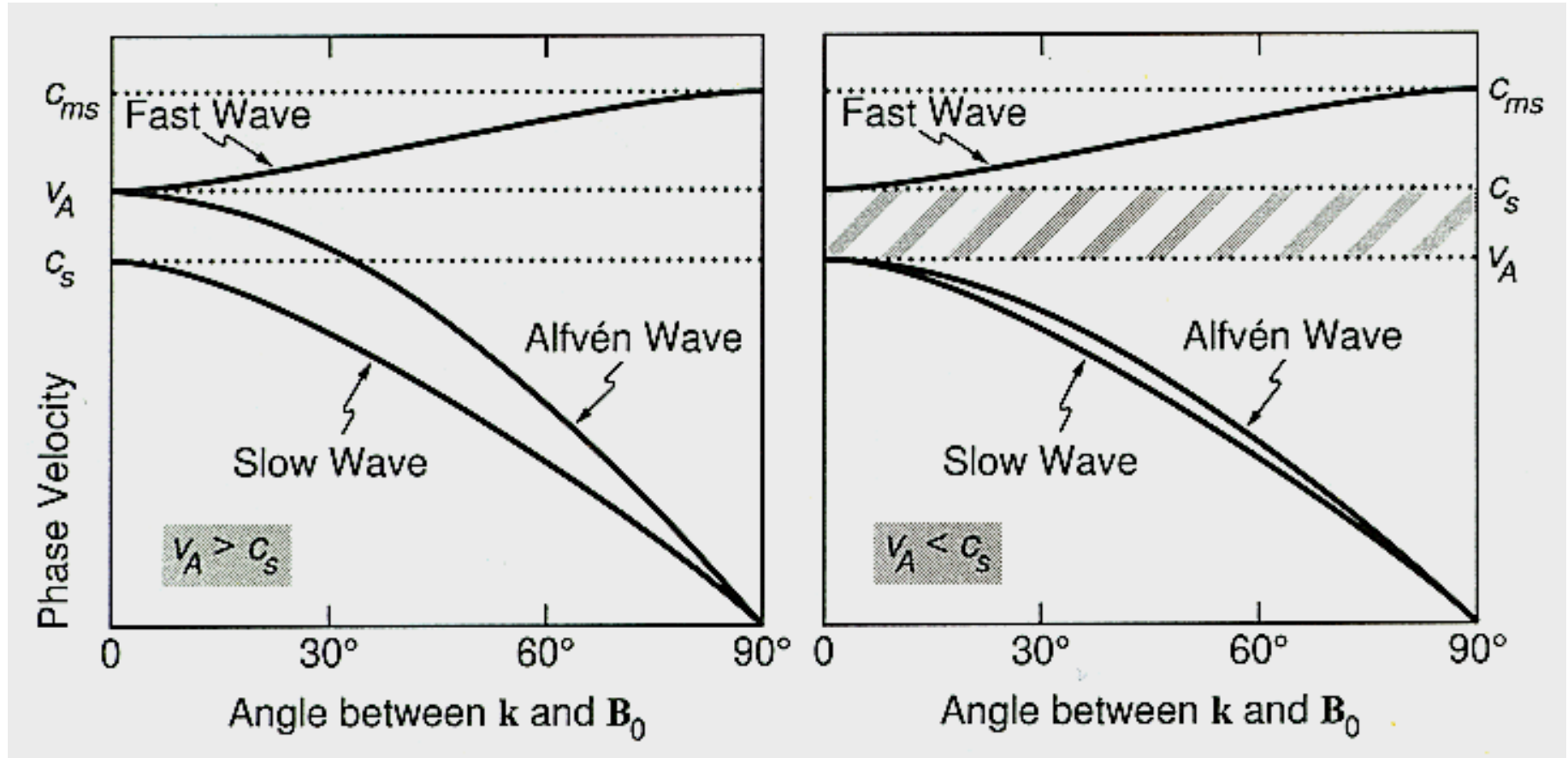
To examine the 3rd wave type, consider the 3rd component of the motion equation

This is directed along x direction, perp B_o and k

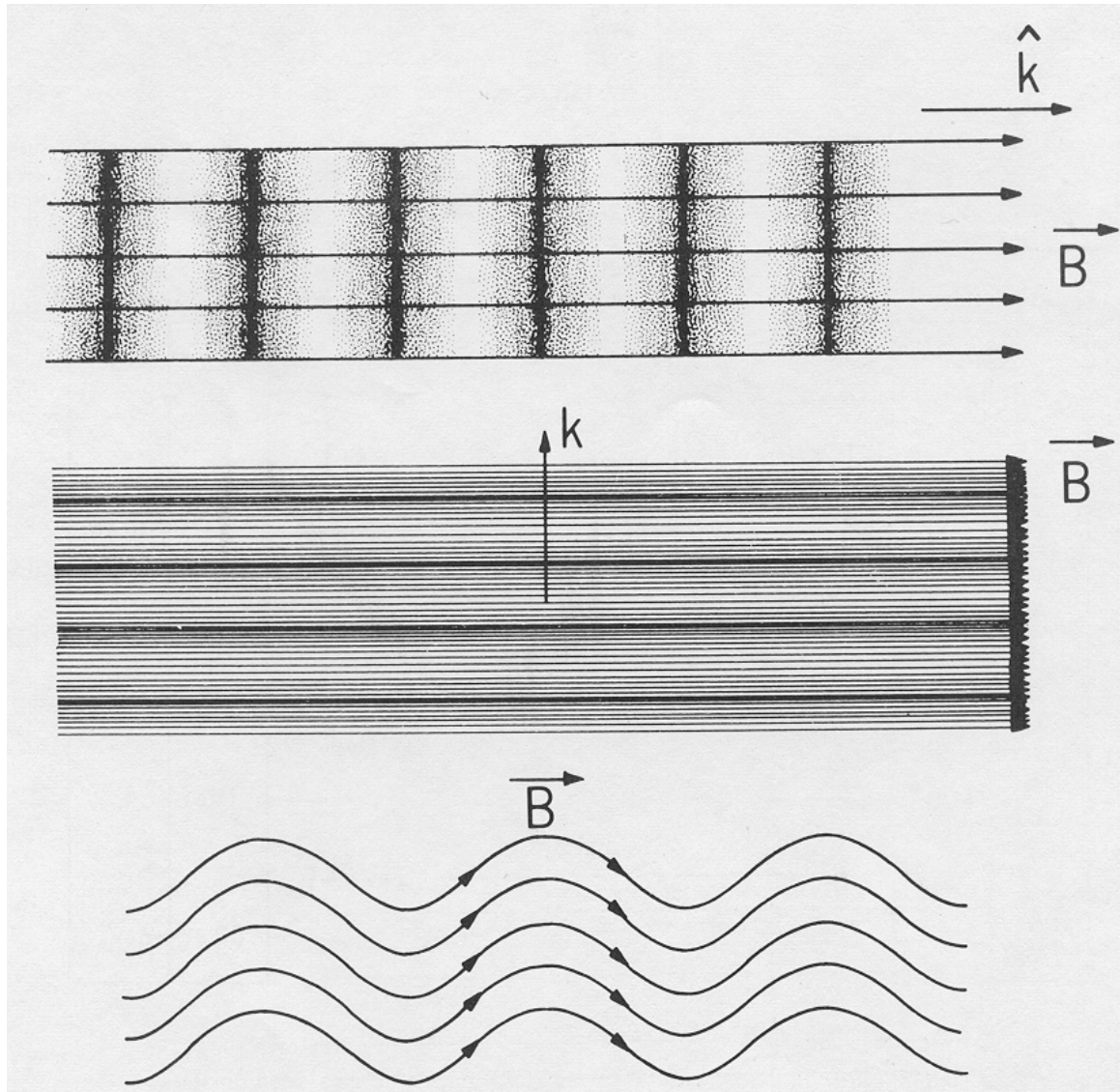
$$\omega \rho_o \delta v_x = -\delta B_x \frac{k_{\parallel} B_o}{4\pi}$$

From this we see that the pressure is not involved at all in the Alfvén wave, neither the mechanical or the magnetic pressure: the restoring force is entirely due to the magnetic tension and the wave is transverse

Dependence of phase velocity on propagation angle



Magnetohydrodynamic waves



- Magnetosonic waves

compressible

- parallel slow and fast
- perpendicular fast

$$c_{ms} = (c_s^2 + v_A^2)^{-1/2}$$

- Alfvén wave

incompressible

parallel and oblique

$$v_A = B/(4\pi\rho)^{1/2}$$

Shock waves

The description of discontinuities in MHD is more complex of pure hydrodynamics, due to the presence of magnetic fields.

The MHD shock waves form

However, in astrophysics exists the following circumstance that simplifies considerably the problem

Except that around the pulsars, astrophysical fluids are never dominated by the magnetic field, in the sense that the plasma- β par is such that

$$\frac{B^2}{8\pi p} = \frac{\gamma v_A^2}{2c_s^2} < 0.1 - 0.3 \quad (\text{take the defs of } v_A \text{ and } c_s)$$

This implies that the dynamical importance of B field can be approximated as small

Thanks to this we can also neglect the fact that the shock waves must supereffvenic, $V_s > v_A$

In effect, since $v_A = (B^2/4\pi\rho)^{1/2}$ and from previous relation we have $B^2/8\pi < \rho c_s^2$, the condition $V_s > v_A$ is equivalent to $V_s > c_s$, which is absolutely necessary for shock waves

Equation of Motion

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

(1)

(2)

(3)

(4)

(i) $\frac{(2)}{(3)} = \beta = \frac{p}{B^2 / (2\mu)}$ * **Plasma beta** *

When $\beta \ll 1$, $\mathbf{j} \times \mathbf{B}$ dominates

(ii) $(1) \approx (3) \rightarrow v \approx v_A = \frac{B}{\sqrt{\mu\rho}}$ * **Alfvén speed** *

Shock waves

The jump conditions at the discontinuity surface are the same as for hydrodynamics:

Conservation of:

- i) mass flux
- ii) momentum flux (parallel and normal flux)
- iii) energy flux

Putting $[X] = X_2 - X_1$ We get

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0$$

$$[\rho v_n (v^2/2 + w) + \frac{1}{4\pi} (v_n B^2 - B_n (\vec{v} \cdot \vec{B}))] = 0$$

There additional conditions due to electromagnetic fields:

From Maxwell eqns, at the discontinuity, we have B_n and E_t are continuous \rightarrow

$$[B_n] = 0 \quad [E_t] = 0 \quad \text{Since we are in the ideal limit } E = -v \times B/c \quad \text{we get}$$

$$[B_n v_t - B_t v_n] = 0$$

Shock waves

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0 \quad [B_n] = 0$$

$$[\rho v_n(v^2/2 + w) + \frac{1}{4\pi}(v_n B^2 - B_n(\vec{v} \cdot \vec{B}))] = 0 \quad [B_n v_t - B_t v_n] = 0$$

The analysis is simple in two limiting cases

The first is one in which the B field before the shock is normal to the shock surface, $B_{t1}=0$

This case is called "parallel" shock since the B field is parallel to shock normal direction

In such a case it is easy to show that $B_{t2}=0$ behind the shock

Since both B_n and B_t are continuous with $B_{n1}=B_{n2}$ and $B_{t1}=B_{t2}=0$, it follows that the parallel shocks reduce to the pure hydrodynamic case, as if the B field is not present

Shock waves

$$[\rho v_n] = 0$$

$$[p + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)] = 0 \quad [\rho v_n v_t + \frac{1}{4\pi} B_t B_n] = 0$$

$$[B_n] = 0$$

$$[\rho v_n (v^2/2 + w) + \frac{1}{4\pi} (v_n B^2 - B_n (\vec{v} \cdot \vec{B}))] = 0$$

$$[B_n v_t - B_t v_n] = 0$$

The analysis is simple in two limiting cases

The 2nd case is when the B field is parallel to the shock surface, ie perpendicular to the normal, $B_{n1} = 0$

In such a case, from 3rd equation we see that v_t is continuous \rightarrow therefore we can choose a reference frame in which $v_t = 0$ and the shock is a normal shock

From $[B_n v_t - B_t v_n] = 0$ And $[\rho v_n] = 0$ We get

$$B_{t1}/B_{t2} = \rho_1/\rho_2 \quad \text{While the others reduce to}$$

$$[\rho v] = 0 \quad [p + \rho v^2 + \frac{B_t^2}{8\pi}] = 0 \quad [v^2/2 + w + \frac{B_t^2}{4\pi\rho}] = 0$$

Which are the hydrodynamics eqns with the additional magnetic terms is momentum and energy flux

If the plasma is made of electrons and protons (or ions), there is the question about the particle temperature after the shock, that is if we have $T_e = T_i$

We have seen that what transforms the the ordered kinetic energy in internal kinetic energy are not the collisions, but are time-varying induced electromagnetic fields

The electrons are subjected to same forces as ions, but, due to their much lower mass, the accelerations are much higher

In there conditions, it is perfectly possible that electron irradiate so that they do not retain, ie dissipate, the internal kinetic energy transferred by the shock wave \rightarrow it is possible that the electrons come out form shock front with $T_e < T_i$ or even $T_e \ll T_i$

This means that the shock dissipates a fraction of its energy, but since before the shock, where the kinetic energy is mainly ordered, the electrons have only a fraction $m_e/m_p \ll 1$ of the total energy, even if almost all this energy would be dissipated, we have not important dynamic consequences on the jump conditions

We can say that, without radiative losses within the shock thickness and with no heat transfer from protons to electrons, the electron temperature behind the shock is a factor m_e/m_p lower than for protons, which is the temperature we get from RH conditions

Consider a subsonic disturbance moving through a conventional neutral fluid. As is well-known, *sound waves* propagating ahead of the disturbance give advance warning of its arrival, and, thereby, allow the response of the fluid to be both smooth and adiabatic. Now, consider a supersonic disturbance. In this case, sound waves are unable to propagate ahead of the disturbance, and so there is no advance warning of its arrival, and, consequently, the fluid response is sharp and non-adiabatic. This type of response is generally known as a *shock*. Let us investigate shocks in MHD fluids.

Since information in such fluids is carried via three different waves--namely, *fast* or compressional-Alfvén waves, *intermediate* or shear-Alfvén waves, and *slow* or magnetosonic waves we might expect MHD fluids to support three different types of shock, corresponding to disturbances traveling faster than each of the aforementioned waves. This is indeed the case.

(iv) Shock Waves

- Nonlinear sound wave can steepen to a shock wave
 - propagates at speed $> c_s$

- In MHD 3 modes:

(1) Slow-mode shock

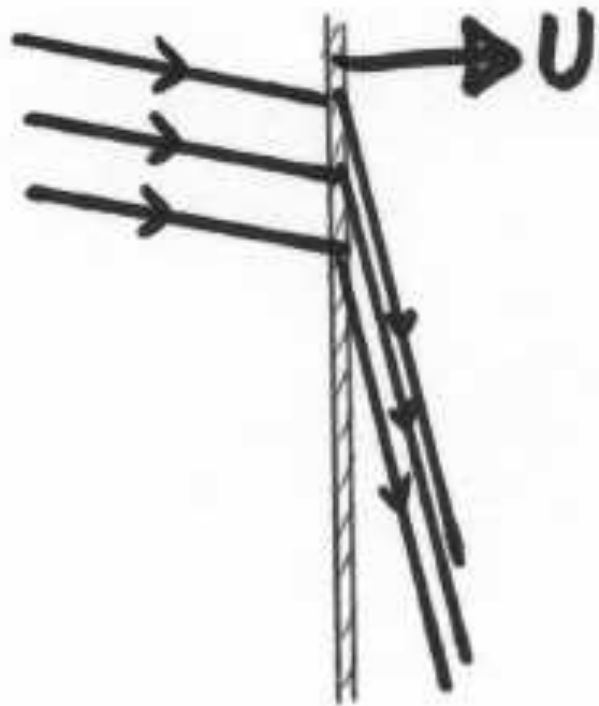
- propagates faster than slow-mode speed
- turns B towards normal

(2) Fast-mode shock

- propagates faster than fast-mode speed
- turns B away from normal

(3) Finite-amplitude Alfvén Wave

- no change in p - reverses tangential magnetic field



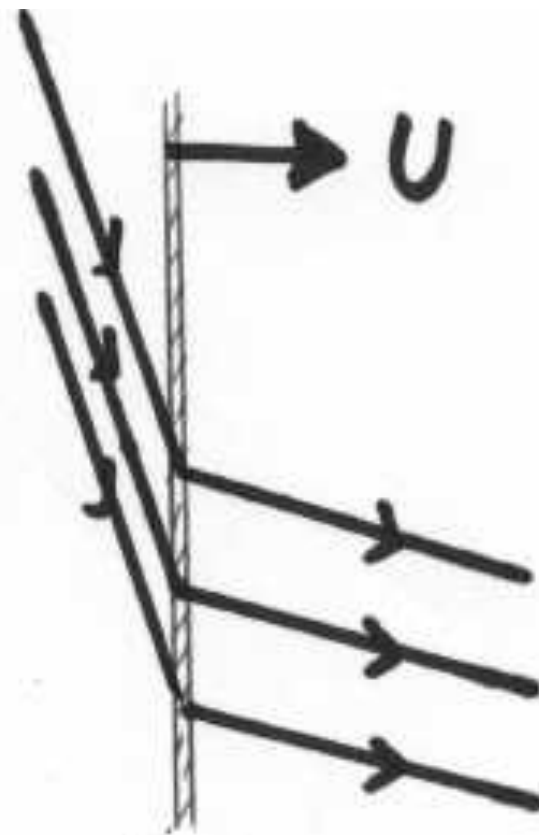
(1)

Slow-mode



(3)

Alfvén



(2)

Fast-mode

- Conservation of momentum $\left[\rho v_n \vec{v}_t - \frac{B_n}{\mu_0} \vec{B}_t \right] = 0$. The subscript t refers to components that are transverse to the shock (i.e. parallel to the shock surface).

- Conservation of energy $\left[\rho v_n \left(\frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} \right) + v_n \frac{B^2}{\mu_0} - \vec{v} \cdot \vec{B} \frac{B_n}{\mu_0} \right] = 0$

There we have used $p\rho^{-\gamma} = \text{const.}$

The first two terms are the flux of kinetic energy (flow energy and internal energy) while the last two terms come from the electromagnetic energy flux $\vec{E} \times \vec{B} / \mu_0$

- Gauss Law $\nabla \cdot \vec{B} = 0$ gives $[B_n] = 0$

- Faraday's Law $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$ gives $[v_n \vec{B}_t - B_n \vec{v}_t] = 0$

- The jump conditions are a set of 6 equations. If we want to find the downstream quantities given the upstream quantities then there are 6 unknowns ($\rho, v_n, v_t, p, B_n, B_t$).
- The solutions to these equations are not necessarily shocks. These are conservation laws and a multitude of other discontinuities can also be described by these equations.

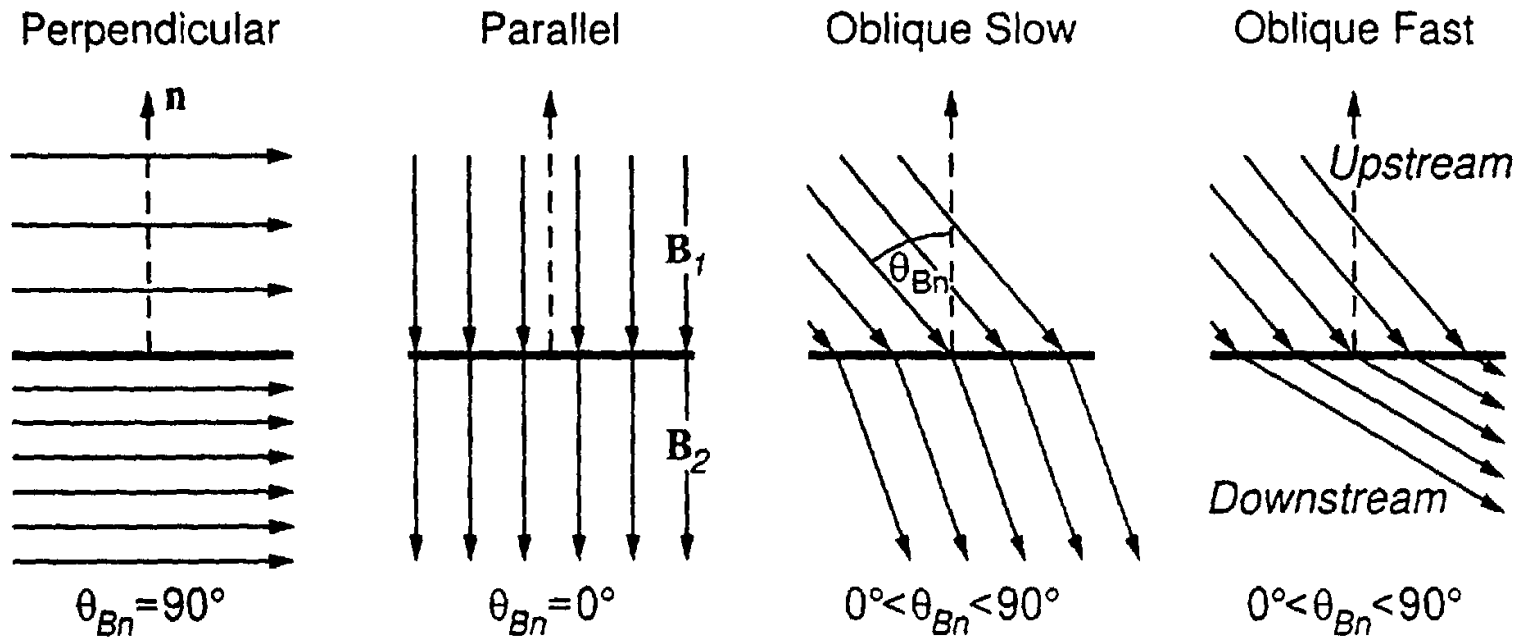
Types of Discontinuities in Ideal MHD

Contact Discontinuity	$v_n = 0, B_n \neq 0$	Density jumps arbitrary, all others continuous. No plasma flow. Both sides flow together at v_t .
Tangential Discontinuity	$v_n = 0, B_n = 0$	Complete separation. Plasma pressure and field change arbitrarily, but pressure balance
Rotational Discontinuity	$v_n \neq 0, B_n \neq 0$ $v_n = B_n / (\mu_0 \rho)^{1/2}$	Large amplitude intermediate wave, field and flow change direction but not magnitude.

Types of Shocks in Ideal MHD

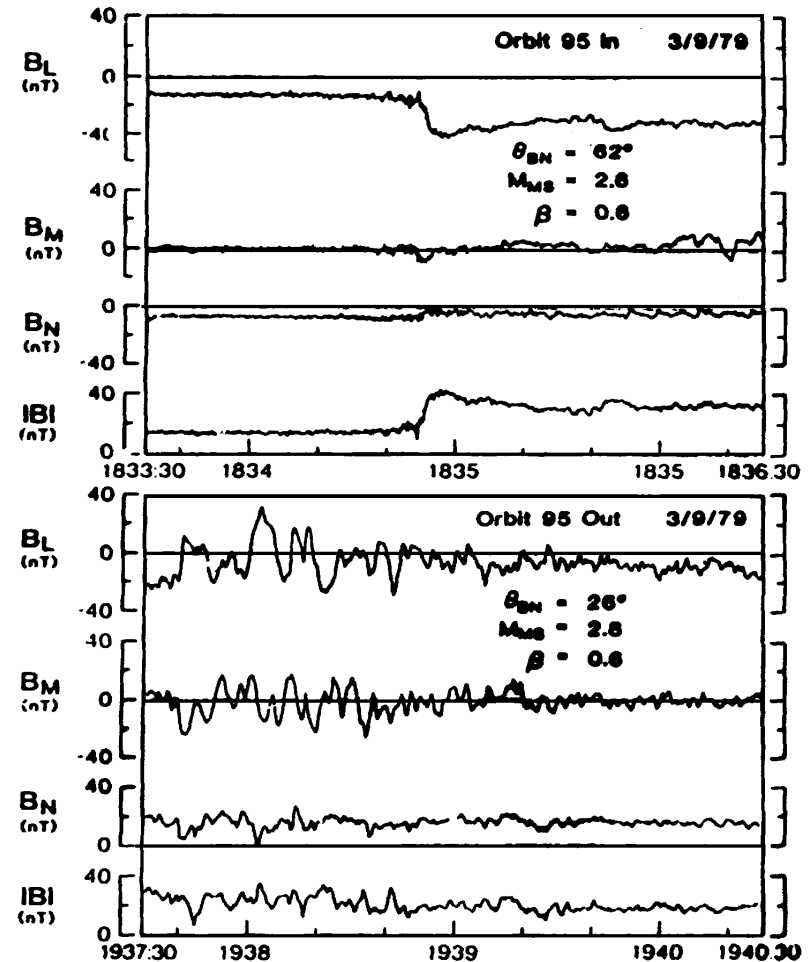
Shock Waves	$v_n \neq 0$	Flow crosses surface of discontinuity accompanied by compression.
Parallel Shock \vec{B} (along shock normal)	$B_t = 0$	B unchanged by shock.
Perpendicular Shock	$B_n = 0$	P and B increase at shock
Oblique Shocks	$B_t \neq 0, B_n \neq 0$	
Fast Shock		P, and B increase, B bends away from normal
Slow Shock		P increases, B decreases, B bends toward normal.
Intermediate Shock		B rotates 180° in shock plane. [p]=0 non-compressive, propagates at u_A , density jump in anisotropic case.

- Configuration of magnetic field lines for fast and slow shocks. The lines are closer together for a fast shock, indicating that the field strength increases.



- Quasi-perpendicular and quasi-parallel shocks.

- Call the angle between \vec{B} and the normal θ_{Bn} .
- Quasi-perpendicular shocks have $\theta_{Bn} > 45^\circ$ and quasi-parallel have $\theta_{Bn} < 45^\circ$.
- Perpendicular shocks are sharper and more laminar.
- Parallel shocks are highly turbulent.
- The reason for this is that perpendicular shocks constrain the waves to the shock plane while parallel shocks allow waves to leak out along the magnetic field
- In these examples of the Earth's bow shock – N is in the normal direction, L is northward and M is azimuthal.



- Examples of the change in plasma parameters across the bow shock

- The solar wind is super-magnetosonic so the purpose of the shock is to slow the solar wind down so the flow can go around the obstacle.

- The density and temperature increase.

- The magnetic field (not shown) also increases.

- The maximum compression at a strong shock is 4 but 2 is more typical.

