Lecture 12 161116

- Il pdf delle lezioni puo’ essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1617/

Energy Conservation

The total energy of a fluid element is given by the mechanical energy (kinetic + potential) and the internal energy (thermodynamical)

$$\epsilon_{tot} = \epsilon_{mech} + \epsilon_{int} = \epsilon_{kin} + \epsilon_{pot} + \epsilon_{int}$$

The mechanical part can be obtained by the motion equation

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right] = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi$$

Using the vectorial identity

$$(\vec{V} \cdot \nabla)\vec{V} = \nabla(\frac{V^2}{2}) - \vec{V} \times (\nabla \times \vec{V})$$

And multiplying scalarly the motion equation by **V**, we get

$$\rho\vec{V}\cdot(\frac{\partial\vec{V}}{\partial t}+\nabla(\frac{V^2}{2})-\vec{V}\times(\nabla\times\vec{V})) = -\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

The right hand side corresponds to the power of the force densities acting on the system

Energy Conservation

$$\rho\vec{V}\cdot\left(\frac{\partial\vec{V}}{\partial t}+\nabla\left(\frac{V^2}{2}\right)-\vec{V}\times(\nabla\times\vec{V})\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

The scalar prod with 2nd term in the lefthand side is = 0 because the vectors are perp →

$$\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}+\rho\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

$$\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}=\frac{\partial}{\partial t}(\rho V^2/2)-\frac{V^2}{2}\frac{\partial\rho}{\partial t}\quad=\frac{\partial}{\partial t}(\rho V^2/2)+\frac{V^2}{2}\nabla\cdot(\rho\vec{V})$$

Using the mass conservation law to eliminate ∂ρ/∂t

$$\frac{\partial}{\partial t}(\rho V^2/2)+\frac{V^2}{2}\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

$$=\nabla\cdot\left(\frac{\rho V^2\vec{V}}{2}\right)$$

$$\Rightarrow\quad\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot\left(\frac{\rho V^2\vec{V}}{2}\right)=-\vec{V}\cdot\nabla p-\rho\vec{V}\cdot\nabla\Phi$$

This equation shows how the kinetic energy of the fluid changes due to work done by pressure forces and by gravitational force: they act as sources of kinetic energy

The energy flux merely redistributes kinetic energy over space

Energy Conservation

$$\frac{\partial}{\partial t}(\rho V^2/2) + \nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p - \boxed{\rho\vec{V}\cdot\nabla\Phi}$$

Now it's the turn of potential energy

$$\rho\vec{V}\cdot\nabla\Phi = \nabla\cdot(\rho\Phi\vec{V}) - \Phi\nabla\cdot(\rho\vec{V}) \quad = \nabla\cdot(\rho\Phi\vec{V}) + \Phi\frac{\partial\rho}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p-\nabla\cdot(\rho\Phi\vec{V})-\boxed{\Phi\frac{\partial\rho}{\partial t}}$$

$$\Phi\frac{\partial\rho}{\partial t} = \frac{\partial(\rho\Phi)}{\partial t} - \rho\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2}) = -\vec{V}\cdot\nabla p-\nabla\cdot(\rho\Phi\vec{V})-\frac{\partial(\rho\Phi)}{\partial t}+\rho\frac{\partial\Phi}{\partial t}$$

Rearranging terms:

$$\frac{\partial}{\partial t}(\rho V^2/2)+\nabla\cdot(\frac{\rho V^2\vec{V}}{2})+\nabla\cdot(\rho\Phi\vec{V})+\frac{\partial(\rho\Phi)}{\partial t} = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

This equation states how the mechanical energy changes due to the work done by the pressure forces and due to non conservative gravitational field (ie explicitly time dependent)

Energy Conservation

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p+\rho\frac{\partial\Phi}{\partial t}$$

The thermodynamics is in the pressure term

Let rewrite the I principle of TD in terms of unit mass variables (specific variables)

$$dq = Tds = d\epsilon + pdv$$

Where dq is the specific heat exchanged, ds the specific entropy, dε is the specific internal energy and v the specific volume **v= V/m=1/ρ**

a) $Tds = d\epsilon + pd(1/\rho) \equiv d(\epsilon + p/\rho) - \frac{dp}{\rho}$ But ε +p/ρ = h = specific enthalpy ➔

b) $Tds = dh - \frac{dp}{\rho}$ ➔ $\rho dh - \rho T ds = dp$ ➔ The pressure gradient is $\rho\nabla h - \rho T\nabla s = \nabla p$

Energy Conservation

To complete the energy balance we have to add the internal energy to the equation

The explicit time variation of internal energy density is $\frac{\partial(\rho\epsilon)}{\partial t} = \rho\frac{\partial\epsilon}{\partial t} + \epsilon\frac{\partial\rho}{\partial t}$

Taking the time derivative of a), we get

$$T\frac{\partial s}{\partial t} = \frac{\partial\epsilon}{\partial t} + p\frac{\partial(1/\rho)}{\partial t} = \frac{\partial\epsilon}{\partial t} - \frac{p}{\rho^2}\frac{\partial\rho}{\partial t} \Rightarrow \rho T\frac{\partial s}{\partial t} = \rho\frac{\partial\epsilon}{\partial t} - \frac{p}{\rho}\frac{\partial\rho}{\partial t}$$

$$\rho\frac{\partial\epsilon}{\partial t} = \rho T\frac{\partial s}{\partial t} + \frac{p}{\rho}\frac{\partial\rho}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\nabla\cdot(\rho\vec{V}) \qquad \frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{V})$$

$$\frac{\partial(\rho\epsilon)}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\nabla\cdot(\rho\vec{V}) - \epsilon\nabla\cdot(\rho\vec{V}) = \rho T\frac{\partial s}{\partial t} - (\frac{p}{\rho} + \epsilon)\nabla\cdot(\rho\vec{V}) = \rho T\frac{\partial s}{\partial t} - h\nabla\cdot(\rho\vec{V})$$

We add ∂ε/∂t on both sides of mech E equation and substitute the grad(p)

$$\frac{\partial}{\partial t}(\frac{\rho V^2}{2} + \rho\Phi) + \nabla\cdot[(\frac{\rho V^2}{2} + \rho\Phi)\vec{V}] = -\vec{V}\cdot\nabla p + \rho\frac{\partial\Phi}{\partial t} \qquad \rho\nabla h - \rho T\nabla s = \nabla p$$

$$\frac{\partial(\rho\epsilon)}{\partial t} + \frac{\partial}{\partial t}(\frac{\rho V^2}{2} + \rho\Phi) + \nabla\cdot[(\frac{\rho V^2}{2} + \rho\Phi)\vec{V}] = \rho T\frac{\partial s}{\partial t} - h\nabla\cdot(\rho\vec{V}) - \vec{V}\cdot(\rho\nabla h - \rho T\nabla s) + \rho\frac{\partial\Phi}{\partial t}$$

Energy Conservation

To complete the energy balance we have to add the internal energy to the equation

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = \rho T\left(\frac{\partial s}{\partial t}+\vec{V}\cdot\nabla s\right)-\left(h\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla h\right)+\rho\frac{\partial\Phi}{\partial t}$$

$$\frac{\partial s}{\partial t}+\vec{V}\cdot\nabla s = \frac{Ds}{Dt} \qquad h\nabla\cdot(\rho\vec{V})+\rho\vec{V}\cdot\nabla h = \nabla\cdot(\rho h\vec{V})$$

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi)\vec{V}] = \rho T\frac{Ds}{Dt}-\nabla\cdot(\rho h\vec{V})+\rho\frac{\partial\Phi}{\partial t}$$

Move the grad on right hand side to left side and get

$$\frac{\partial}{\partial t}(\rho\epsilon+\frac{\rho V^2}{2}+\rho\Phi)+\nabla\cdot[(\frac{\rho V^2}{2}+\rho\Phi+\rho h)\vec{V}] = \rho T\frac{Ds}{Dt}+\rho\frac{\partial\Phi}{\partial t}$$

Energy density — Energy flux — "Net heating rate density"

The 1st term in RHS in the true heating (or cooling) due to "external" irreversible processes as radiation losses. The 2nd "gravitational heating" $\rho\partial\Phi/\partial t$ corresponds to the process known as violent relaxation in a time-varying gravitational potential, which plays an important role in the dynamics of galaxies, where it acts in a way analogous to a heating mechanism

Energy Conservation

Energy density is a scalar! → the energy flux is a vector

Kinetic energy density

Internal energy density

Gravitational potential energy density

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\rho \boldsymbol{V}\left(\frac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\rm eff}$$

$$\mathcal{H}_{\rm eff} \equiv \mathcal{H} + \rho\,\frac{\partial\Phi}{\partial t}$$

Irreversibly lost/gained energy per unit volume

Energy Conservation

$$e = \frac{P}{(\gamma - 1)\,\rho} \;\; \text{if} \;\; P\,\rho^{-\gamma} = \text{constant}$$

Internal energy per unit mass

$$h = e + \frac{P}{\rho} = \frac{\gamma P}{(\gamma - 1)\,\rho}$$

Specific enthalpy

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho\,V^2 + \rho e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{V}\left(\tfrac{1}{2}V^2 + h + \Phi\right)\right] = \mathcal{H}_{\rm eff}$$

$$\mathcal{H}_{\rm eff} \equiv \mathcal{H} + \rho\,\frac{\partial\Phi}{\partial t}$$

Irreversible gains/losses, e.g. radiation losses

"Dynamical Friction"

Conservazione dell'energia

Nei fluidi ideali, in cui sono assenti fenomeni di dissipazione dovuti ad attrito "interno" (cioe' viscosita') e nell'ipotesi di assenza di conduzione termica (che puo' trasferire calore da una regione all'altra), l'unico fenomeno di scambio di energia non meccanica (ie non esprimibile come pdV) puo' essere solo attraverso l'irraggiamento

Nel caso in cui anche i processi radiativi siano assenti o trascurabili, il processo e' adiabatico (se reversibile) → dQ = 0 → TdS = 0

In tal caso la conservazione dell'energia di un elemento di massa e' equivalente alla conservazione dell'entropia del sistema dello stesso elemento

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Conservazione dell'energia

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

La densita' dei fluidi astrofisici e' tipicamente molto bassa (~1 idrogeno/cm^3) per cui un fotone emesso dalle particelle del fluido non viene mai riassorbito (a meno che vi siano certe condizioni (cfr. Autoassorbimento e spessore ottico) e lascia il sistema, facendo perdere energia attraverso processi radiativi

In tal caso l'entropia non e' piu' conservata

Anche in questo caso pero' l'equazione di sopra ha un suo ambito di validita': infatti i processi radiativi hanno tempi scala caratteristici e se l'evoluzione del sistema avviene su scale di tempo << di quelli radiativi, il processo puo' essere considerato adiabatico

Conservazione dell'energia

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Vi sono situazioni in cui conduzione e viscosita' giocano un ruolo importante (p es all'interno delle stelle, in dischi di accrescimento e piu' in generale in oggetti compatti e/o densi in cui il libero cammino medio diventa "piccolo" e quindi lo spessore ottico diventa "grande" (cfr. Autoassorbimento)

Ma in genere il meccanismo per cui un sistema si discosta dalla isoentropia e' il fatto che il fluido viene riscaldato o raffreddato da una varieta' di processi radiativi

Se definiamo Γ e Λ i coefficienti di riscaldamento e raffreddamento per unita' di massa e di tempo

$$\frac{Dq}{Dt} = T\frac{Ds}{Dt} = T\left(\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s\right) = \Gamma - \Lambda$$

Steady Flows: no explicit time-dependence:

$$\frac{\partial}{\partial t} = 0$$

mass conservation: $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}) = 0 \;;$

momentum conservation: $\boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{V} \otimes \boldsymbol{V} + P \, \boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi \;;$

energy conservation: $\boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{V} \left(\tfrac{1}{2} V^2 + h + \Phi \right) \right] = 0 \;.$

Energy conservation

$$\text{mass conservation:} \quad \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}) = 0 \; ;$$

$$\text{momentum conservation:} \quad \boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{V} \otimes \boldsymbol{V} + P \, \boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi \; ;$$

$$\text{energy conservation:} \quad \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{V} \left(\tfrac{1}{2} V^2 + h + \Phi\right)\right] = 0 \; .$$

$$\boldsymbol{\nabla} \cdot (f \boldsymbol{A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{A} \cdot \boldsymbol{\nabla}) f$$

$$\boldsymbol{A} = \rho \boldsymbol{V} \quad , \quad f = \tfrac{1}{2} V^2 + h + \Phi$$

$$\nabla \cdot \mathbf{A} = 0$$

For mass conservation law

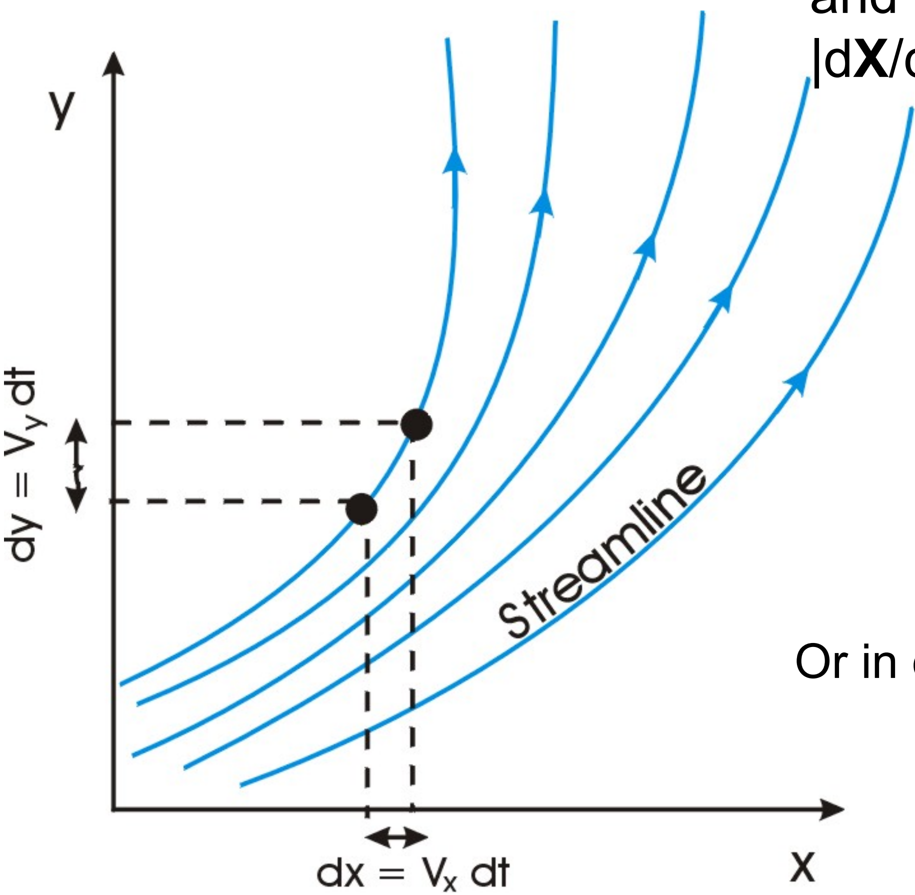
$$(\boldsymbol{V} \cdot \boldsymbol{\nabla}) \left(\tfrac{1}{2} V^2 + h + \Phi\right) = 0$$

The quantity f is (obviously) the specific energy (energy/mass)

Variation along flow lines in steady flows

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left(\tfrac{1}{2}V^2+h+\Phi\right)=0$$

Consider the flowlines defined as a trajectory **x**=**X**(l) such that the tangent vector (versor), d**X**/dl, is || the local **V**, d**X**/dl || **V**(**X**) and it can be choosen such that |d**X**/dl| = 1 → dl/dt=V



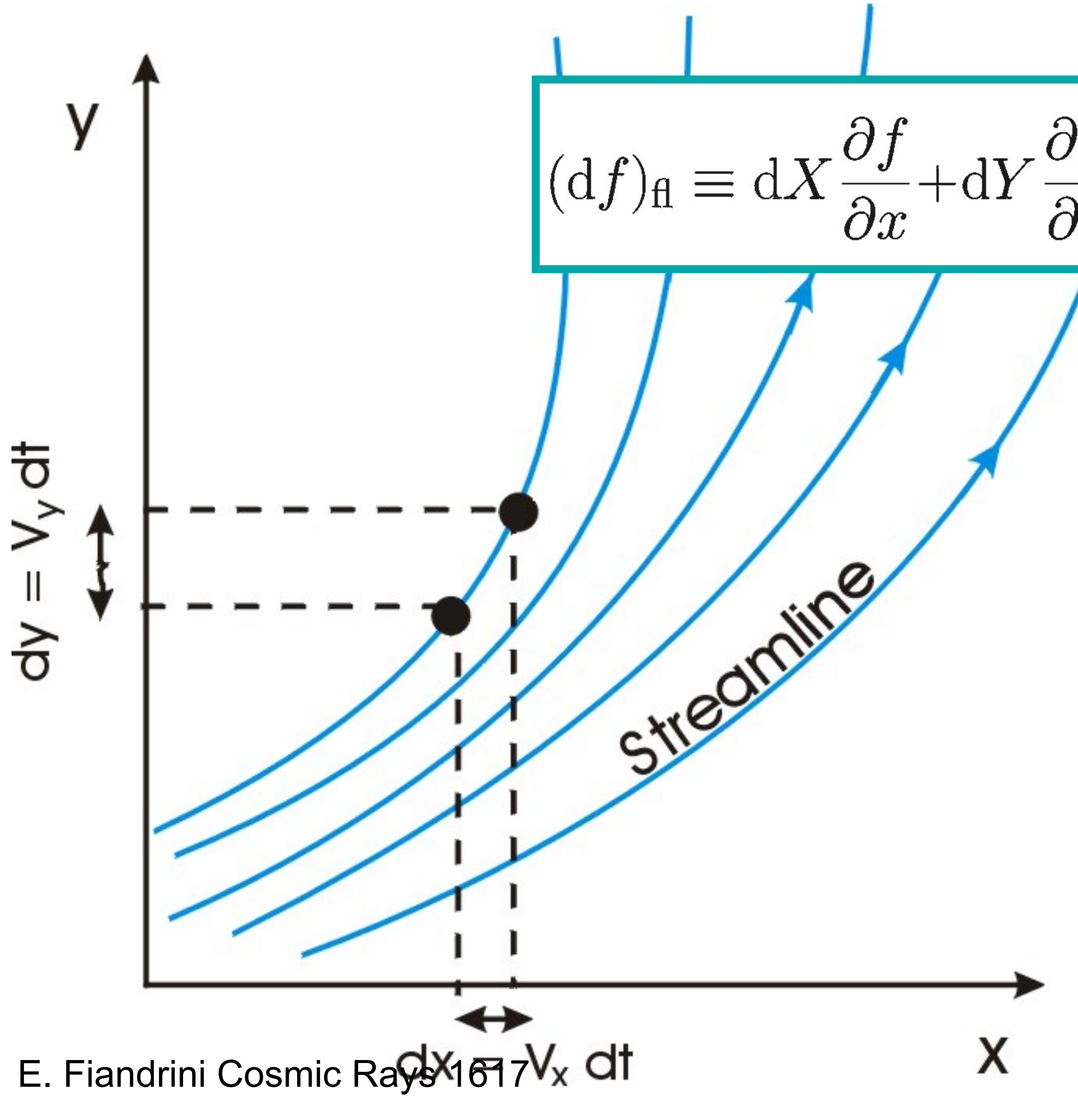
The coordinates of points on a given flow line satisfy the relation

$$\mathrm{d}\boldsymbol{X}=\boldsymbol{V}(\boldsymbol{x}=\boldsymbol{X})\,\mathrm{d}t$$

Or in components $\dfrac{\mathrm{d}X}{V_x(\boldsymbol{X})}=\dfrac{\mathrm{d}Y}{V_y(\boldsymbol{X})}=\dfrac{\mathrm{d}Z}{V_z(\boldsymbol{X})}=\mathrm{d}t$

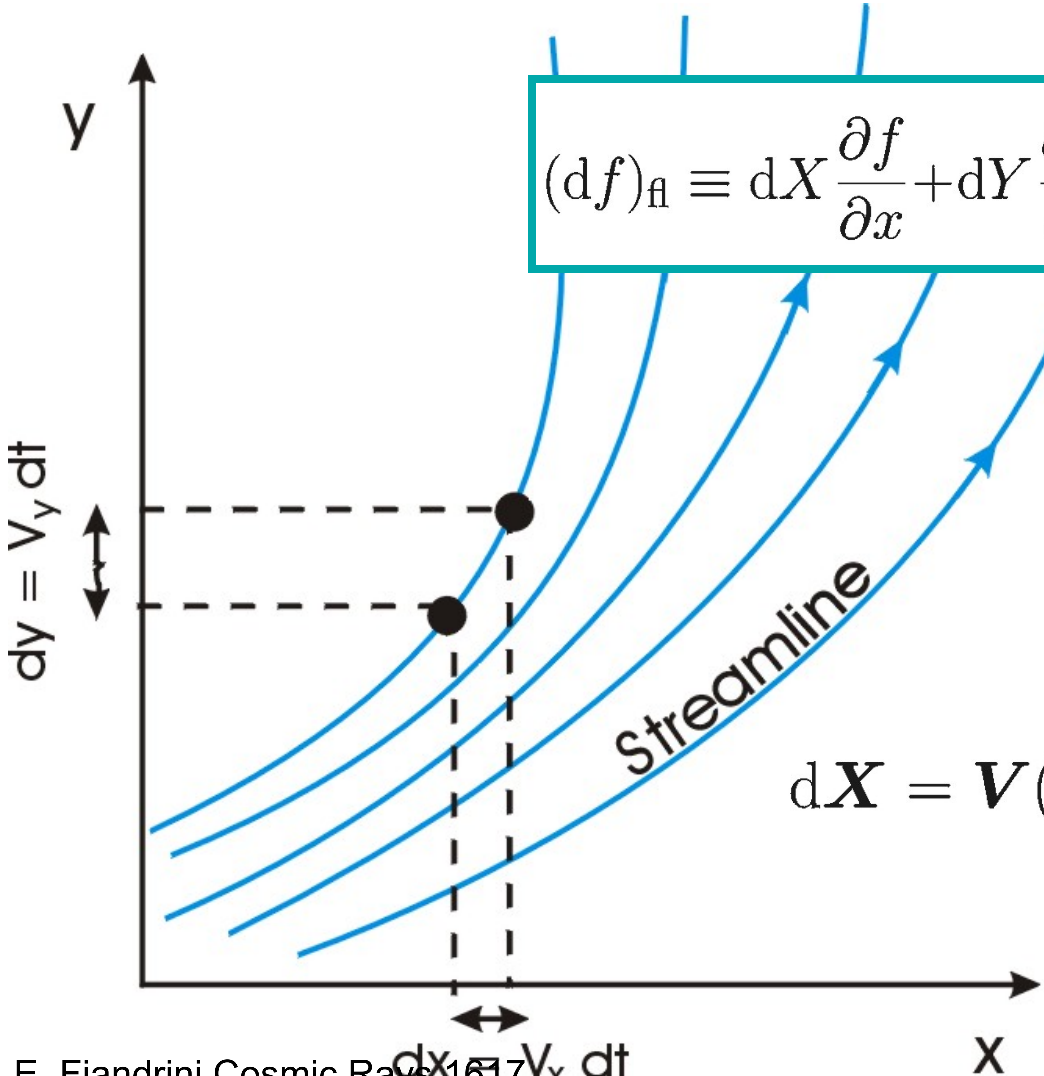
When is a function $f(x,y,z)$ constant along flow lines ?

$$(\mathrm{d}f)_{\mathrm{fl}} \equiv \mathrm{d}X\frac{\partial f}{\partial x}+\mathrm{d}Y\frac{\partial f}{\partial y}+\mathrm{d}Z\frac{\partial f}{\partial z} = (\mathrm{d}\boldsymbol{X}\cdot\boldsymbol{\nabla} f) = 0$$



When is a function *f(x,y,z)* constant along flow lines ?

$$(\mathrm{d}f)_{\mathrm{fl}} \equiv \mathrm{d}X\frac{\partial f}{\partial x}+\mathrm{d}Y\frac{\partial f}{\partial y}+\mathrm{d}Z\frac{\partial f}{\partial z} = (\mathrm{d}\boldsymbol{X}\cdot\boldsymbol{\nabla} f) = 0$$



$$\mathrm{d}\boldsymbol{X} = \boldsymbol{V}(\boldsymbol{x}=\boldsymbol{X})\,\mathrm{d}t$$

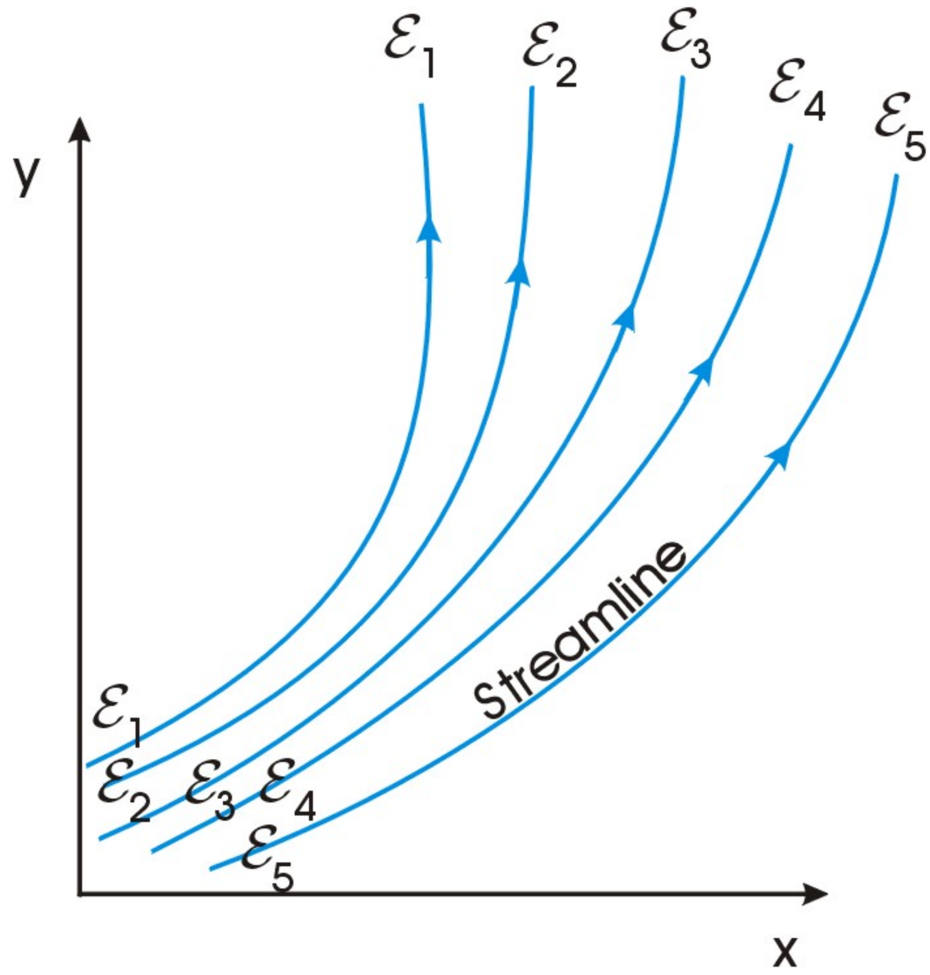
$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})f(\boldsymbol{x}) = 0$$

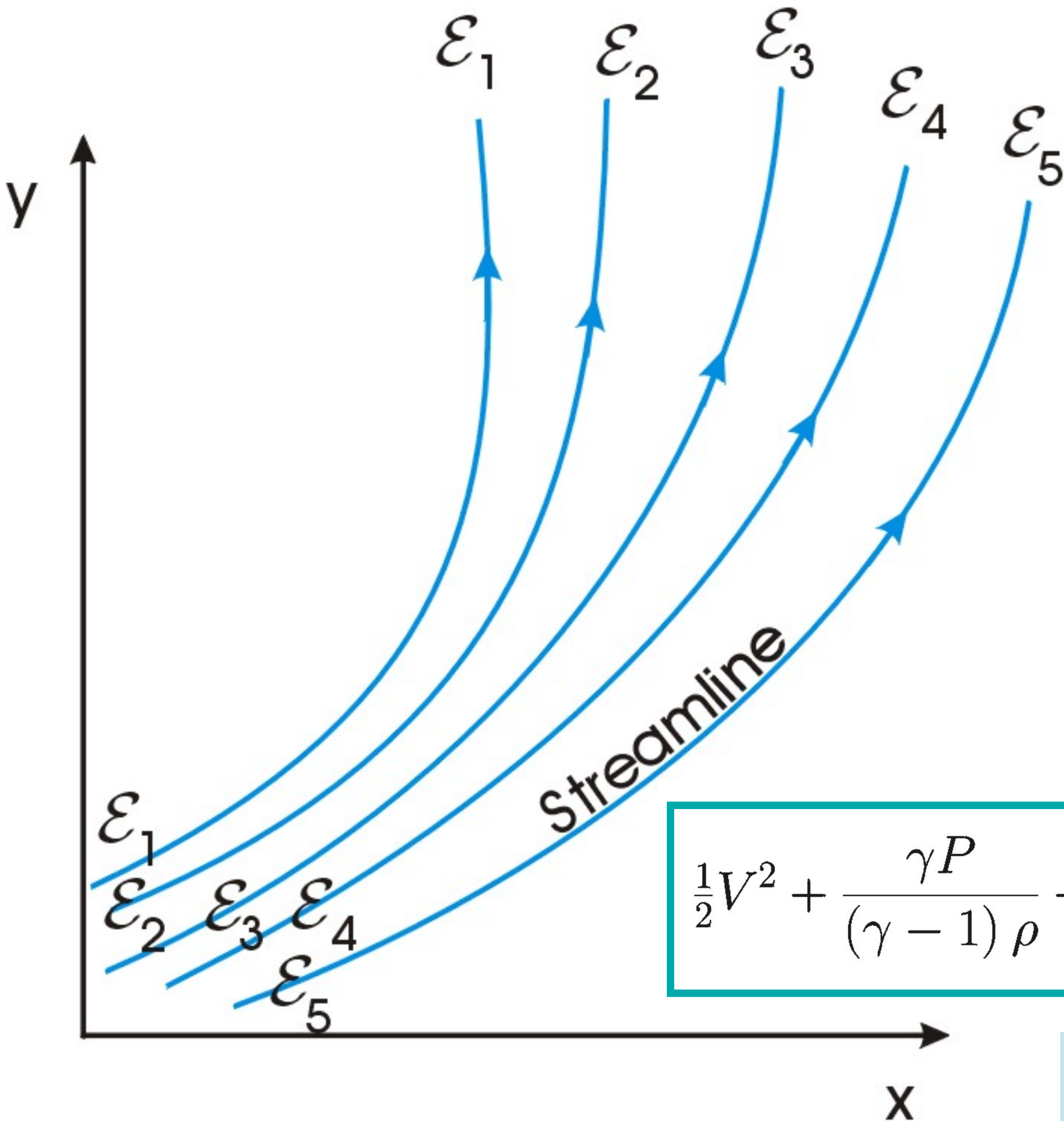
Bernouilli's Law for steady flows:

$$(\boldsymbol{V} \cdot \boldsymbol{\nabla}) \left(\tfrac{1}{2}V^2 + h + \Phi\right) = 0 \longrightarrow \tfrac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

NB: bernoulli law dont say anything about the variation of E_{spec} across the flow lines

In general the constant may differ from flowline to flowline → it is not a global constant over all space

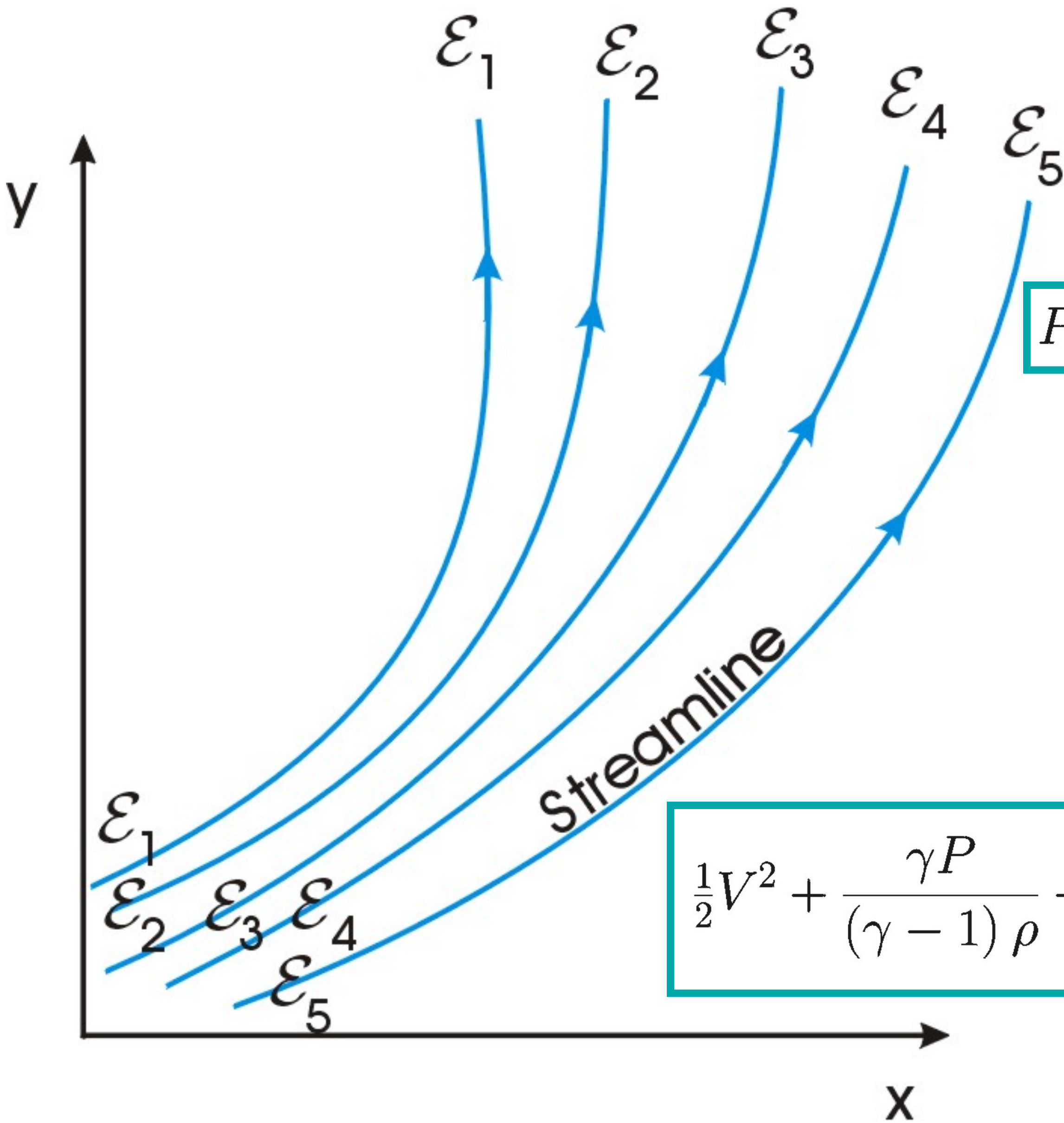




Bernoulli's Law

$$\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

(Energy conservation)



Adiabatic fluid

$$P\rho^{-\gamma} = \text{constant along flowlines}$$

(Entropy conservation)

Bernoulli's Law

$$\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma - 1)\,\rho} + \Phi = \text{constant along flowlines}$$

(Energy conservation)

Stevino's law in astrophysics

Static case: V=0

A short digression: isothermal sphere and globular clusters

Isothermal sphere

The isothermal sphere is a spherically symmetric, self-gravitating system

It is a crude model for a globular cluster, for the quasi-spherical region ("bulge") of a disk galaxy or for the nucleus of an elliptical galaxy

Consider a large number of star with number density distribution n=n(r) only, r is the distance form the center of the sphere and with a mass density $\rho=m_*n(r)$, where m_* is the mass of the stars (supposed to be the same)

If the number of stars is large enough we can describe it as a "gas" of stars with a "temperature" T determined by the velocity dispersion (i.e. energy equipartition)

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 \equiv \sigma^2 = \frac{kT}{m_*}$$

In the isothermal sphere model, the cluster is treated as a self-gravitating ball of gas→ the pressure is then $p(r) = n(r)kT = \rho(r)\sigma^2$

Typically a globular cluster contains 100.000 stars with a mass between $10^4 – 10^6$ solar masses and an average of $10^5 M_{sun}$

Governing Equations:

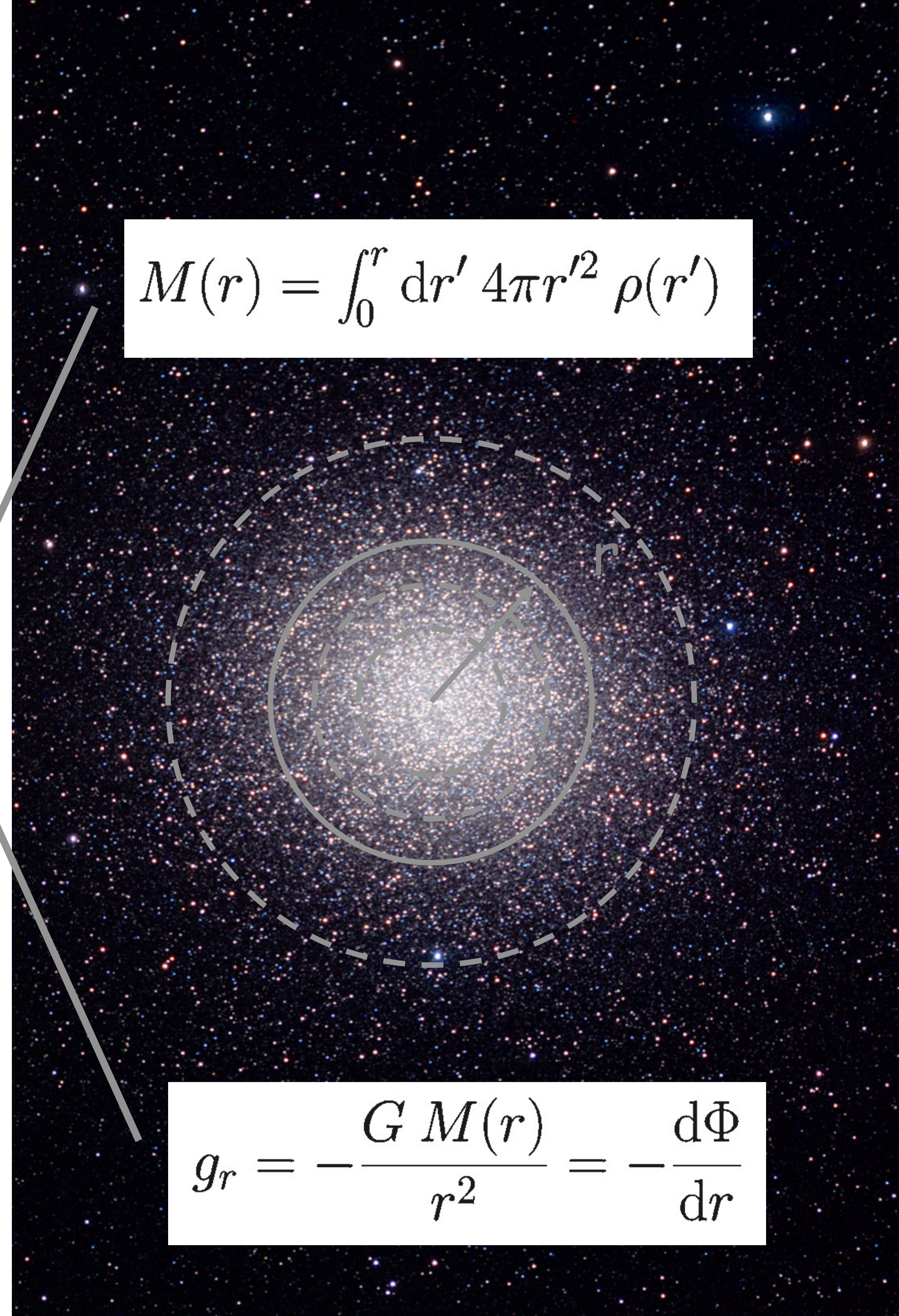
Equation of Motion: <u>no</u> bulk motion, only pressure!
→Hydrostatic Equilibrium!

Isothermal sphere means that the velocity dispersion does not depend on the radius r

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \tilde{\sigma}^2 \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\rho \frac{G\,M(r)}{r^2}$$

↓

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$



$$M(r) = \int_0^r \mathrm{d}r'\, 4\pi r'^2\, \rho(r')$$

$$g_r = -\frac{G\,M(r)}{r^2} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Where ρ_o is the mass density at r=0, assuming $\Phi(0)=0$

'Down to Earth' Analogy: the Isothermal Atmosphere

Low density & low pressure

Constant temperature

High density & high pressure

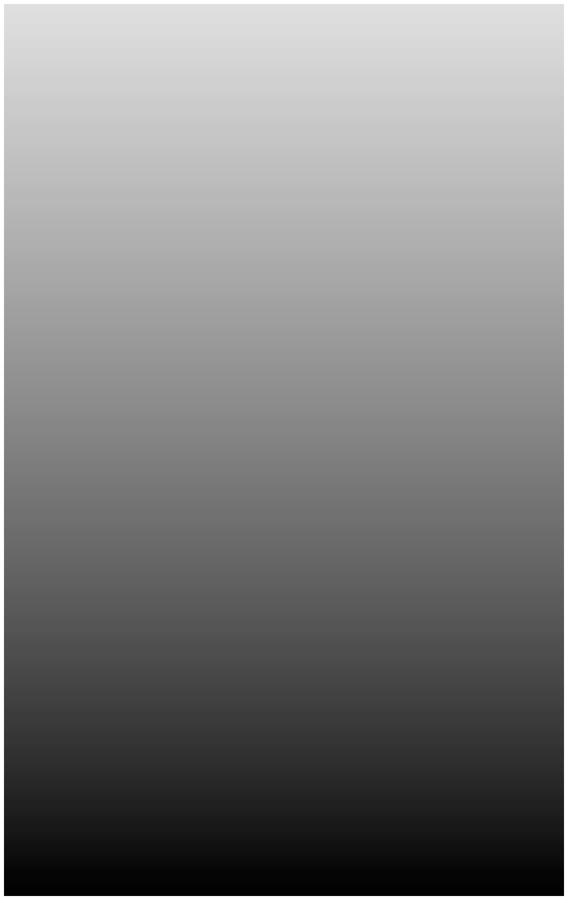
$$\boldsymbol{g} = -\nabla\Phi = -g\hat{\boldsymbol{e}}_z \quad \Leftrightarrow \quad \Phi(z) = gz$$

$$\nabla P = \left(\frac{dP}{dz}\right)\hat{\boldsymbol{e}}_z = \frac{RT}{\mu}\frac{d\rho}{dz}\hat{\boldsymbol{e}}_z$$

Force balance:

$$0 = -\nabla P + \rho \boldsymbol{g} = -\left(\frac{RT}{\mu}\frac{d\rho}{dz} + \rho g\right)\hat{\boldsymbol{e}}_z$$

‘Down to Earth’ Analogy: the Isothermal Atmosphere



$$0 = -\nabla P + \rho \boldsymbol{g} = -\left(\frac{RT}{\mu} \frac{d\rho}{dz} + \rho g \right) \hat{\boldsymbol{e}}_z$$

'Down to Earth' Analogy: the Isothermal Atmosphere

$$0 = -\nabla P + \rho \boldsymbol{g} = -\left(\frac{RT}{\mu} \frac{d\rho}{dz} + \rho g \right) \hat{\boldsymbol{e}}_z$$

Set to zero!

$$\frac{1}{\rho} \frac{d\rho}{dz} = -\frac{\mu g}{RT} \quad \Leftrightarrow$$

$$\rho(z) = \rho(0) \exp(-z/H)$$

$$P(z) = P(0) \exp(-z/H)$$

$$\rho(z) = \rho(0) \exp(-\frac{\mu g z}{RT}) = \rho(0) \exp(-\frac{\mu \Phi(z)}{RT})$$

$$= \rho(0) \exp(-z/H) \quad , \quad H \equiv \frac{RT}{\mu g}$$

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

The gravitational potential is described by the Poisson's equation

$$\nabla^2\Phi(r) = 4\pi G\rho(r)$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\,\rho(r) = 4\pi G\rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Spherically symmetric Laplace Operator

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\,\rho(r) = 4\pi G\rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Poisson Eqn.

Spherically symmetric Laplace Operator

$$\xi = \frac{r}{r_\mathrm{K}} \quad , \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_*\Phi}{k_\mathrm{b}T}$$

$$r_\mathrm{K} = \left(\frac{\tilde{\sigma}^2}{4\pi G\rho_0}\right)^{1/2} = \left(\frac{k_\mathrm{b}T}{4\pi G m_*\rho_0}\right)^{1/2}$$

King radius

Scale Transformation

Density law and Poisson's Equation

Hydrostatic Eq.

$$\tilde{\sigma}^2 \left(\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Exponential density law

$$\rho(r) = \rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\,\rho(r) = 4\pi G\rho_0\, e^{-\Phi(r)/\tilde{\sigma}^2}$$

Poisson Eqn.

Spherically symmetric Laplace Operator

$$\xi = \frac{r}{r_\mathrm{K}} \quad , \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_*\Phi}{k_\mathrm{b}T}$$

$$r_\mathrm{K} = \left(\frac{\tilde{\sigma}^2}{4\pi G\rho_0}\right)^{1/2} = \left(\frac{k_\mathrm{b}T}{4\pi G m_*\rho_0}\right)^{1/2}$$

Scale Transformation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right) = e^{-\Psi}$$

Density law and Poisson's Equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right) = e^{-\Psi}$$

$$\xi = \frac{r}{r_\mathrm{K}} \quad , \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_*\Phi}{k_\mathrm{b}T}$$

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Scale Transformation

This dimensionless form displays NO explicit information about the properties of the cluster

In particular all the reference to the central density ρ_o and velocity dispersion σ^2 has disappeared

➔ this means that all the isothermal are self-similar

If one plots the density relative to the central value ρ/ρ_o as function of $\xi=r/r_K$, all isothermal spheres have exactly the same density profile

The boundary conditions are: $\Phi(0) = 0$ and $(d\Psi/d\xi)_{\xi=0} = 0$

The 1st is possible because potential is defined up to a costant, while the 2nd is a consequence of the spherical symmetry: at the center the net force is zero

Solution: $$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Psi}{d\xi}\right)=e^{-\Psi}$$

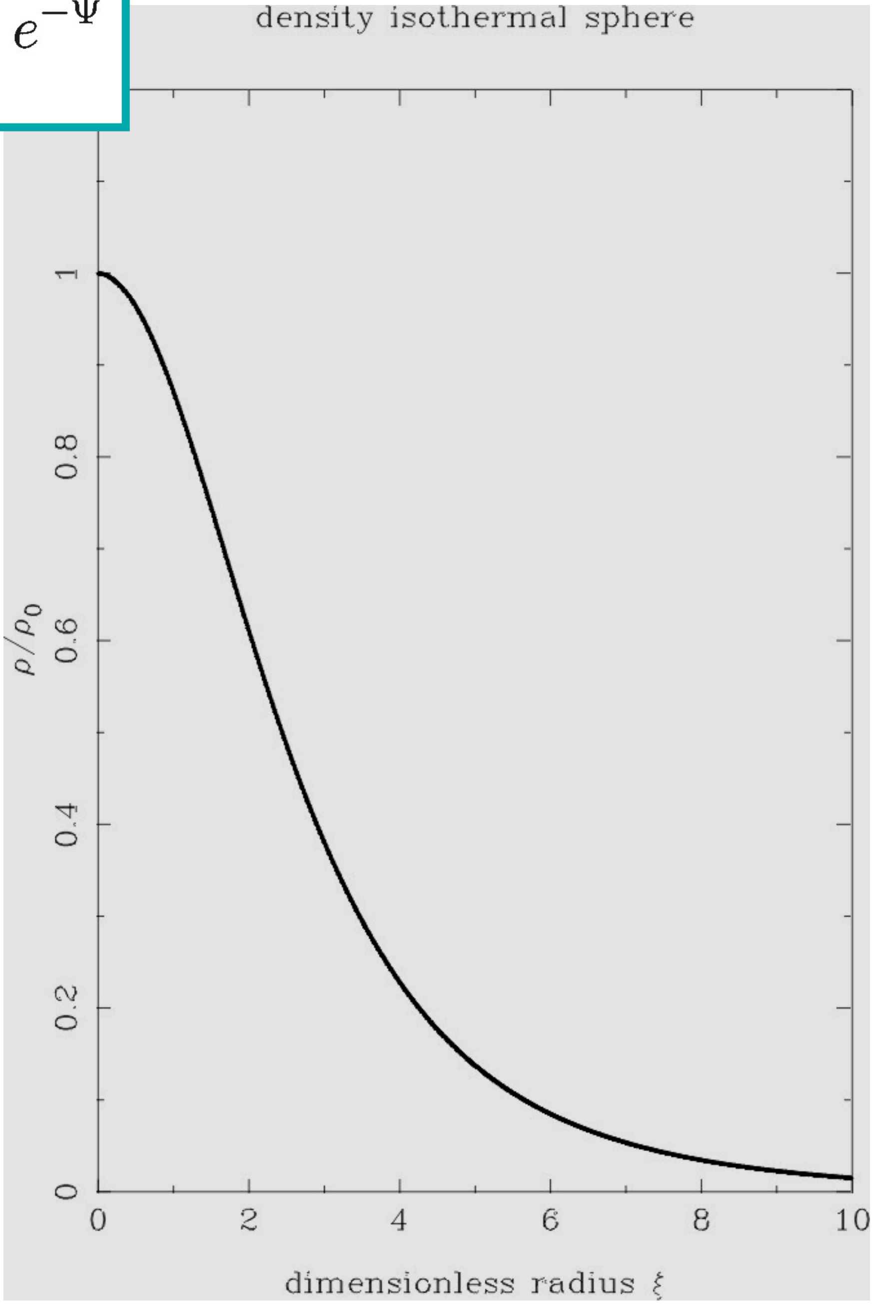
There is no analytical solution

Near ξ=0 one con solve by a power series, using the fact that for Ψ<< 1 so that the exp on RHS can be expanded

$$\text{For } \xi = r/r_K \ll 1: \begin{cases} \rho \approx \rho_0\left(1-\dfrac{\xi^2}{6}+\dfrac{\xi^4}{45}\right) \\ \Psi \approx \dfrac{\xi^2}{6}-\dfrac{\xi^4}{120} \end{cases}$$

For large ξ, the solution goes asymptotically to $\Psi \sim \log(\xi^2/2)$

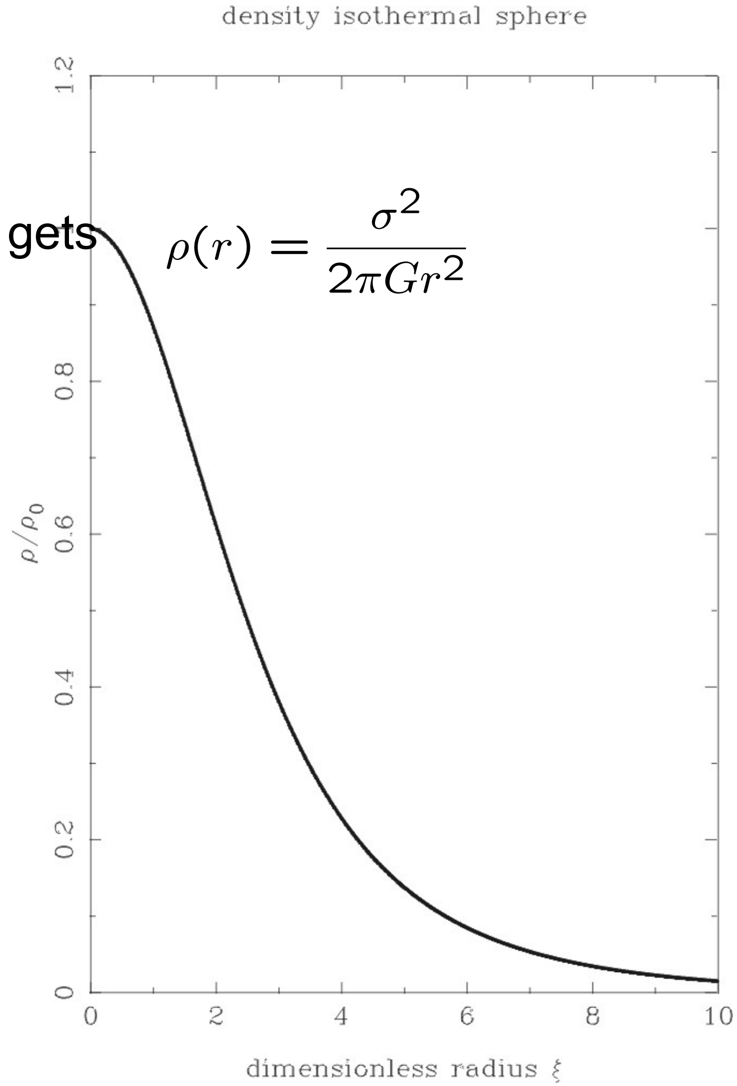
$$\text{For } \xi = r/r_K \gg 1: \begin{cases} \rho \approx \dfrac{2\rho_0}{\xi^2}=\dfrac{\widetilde{\sigma^2}}{2\pi G r^2} \\ \Psi \approx \log\left(\dfrac{\xi^2}{2}\right) \end{cases}$$



Singular Solution

Expressing the density in terms of the radius one gets

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$



Known as the "singular isothermal sphere" solution as the density goes to ∞ as r→0

Infact this is the ONLY analytic solution known to the isothermal sphere equation, as can be checked by substitution

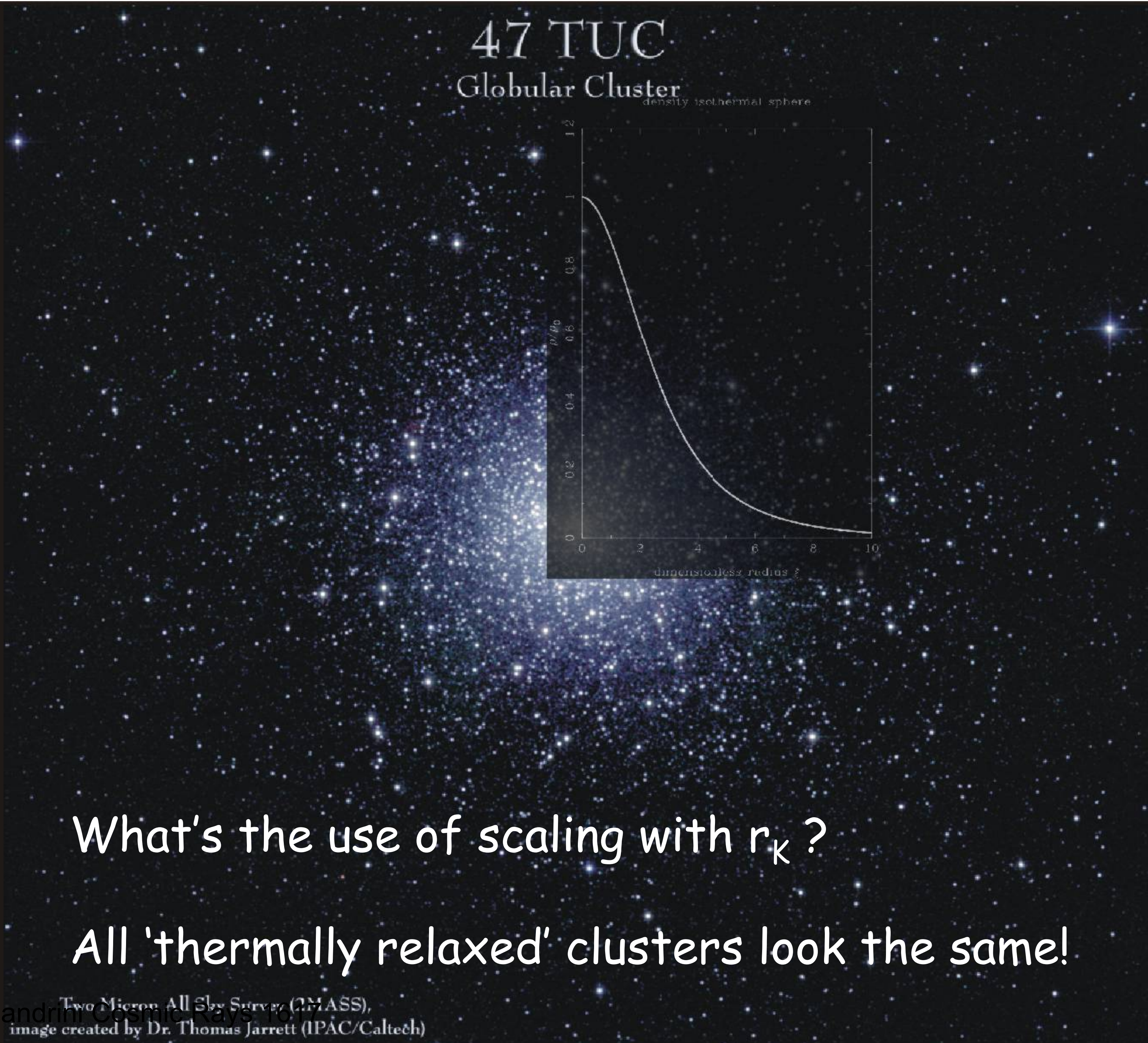
Notice that ρ depends only on dispersion velocity and radius but not on central density ρ_o

For $\xi = r / r_K \gg 1$:

$$M(r) \simeq \int_0^r dr\ 4\pi r^2 \left(\frac{\tilde{\sigma}^2}{2\pi G r^2} \right) = \frac{2\tilde{\sigma}^2 r}{G}$$

$$= 8\pi \rho_0 r_K^2 r$$

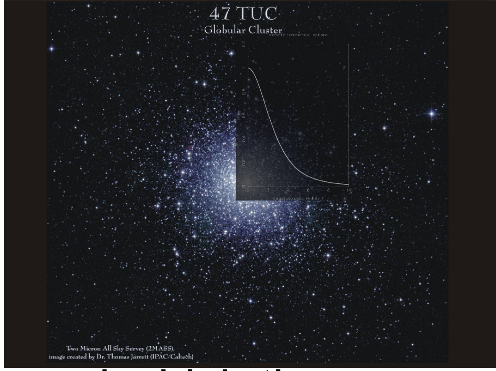
Such behavior is clearly unacceptable for a real <globular cluster because m→∞ as r→∞ ➔ isothermal sphere can only be an approximate model which fails at large r



What's the use of scaling with r_K ?

All 'thermally relaxed' clusters look the same!

Tidal radius



Observations show that clusters have a well-defined edge beyond which the stellar density rapidly goes to zero

This can be explained if the tidal forces are taken into account: the variation of the gravitational pull of the galaxy across the globular cluster

If the cluster has a radius r_t and is located at a distance R from galactic center, the typical magnitude of the tidal acceleration is for $r_t \ll R$

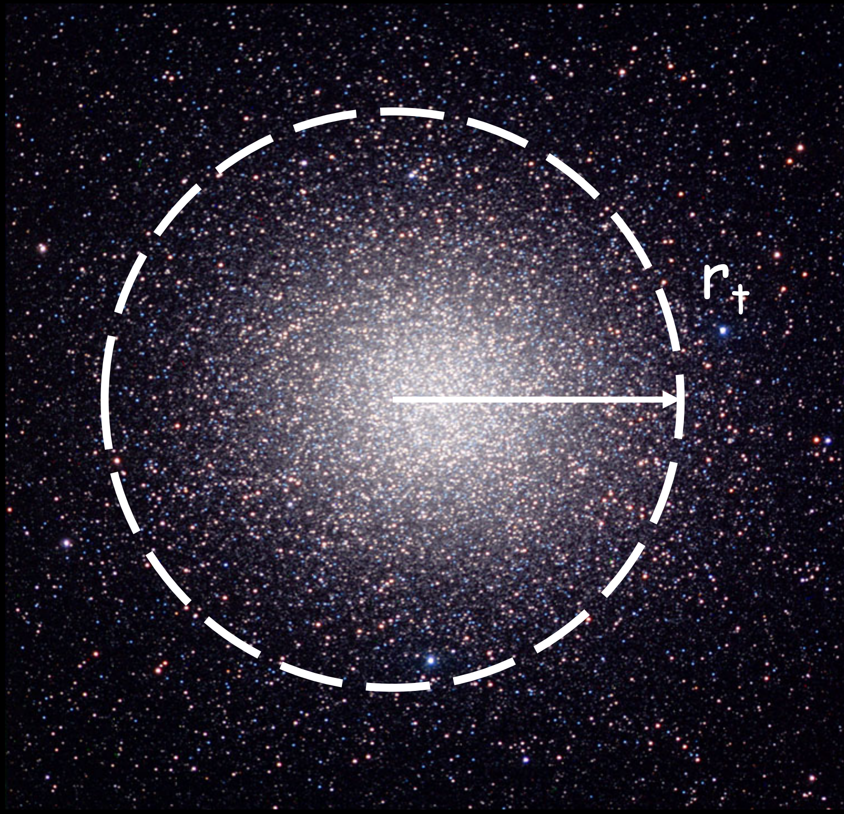
$$g_t \approx r_t \frac{\partial}{\partial r}\left(-\frac{GM_{gal}}{R^2}\right) \quad = \frac{2GM_{gal} r_t}{R^3}$$

This is essentially the difference between the galactic gravitational force at the center and the outer edge of the globular cluster

Tidal Radius

The value of r_t, the so-called tidal radius can be evaluated equating the tidal force to the self-gravitational force of cluster

This defines the maximum size of the cluster where stars in the clusters are still marginally bound by the gravitational pull of the cluster mass



$$\frac{GM_{cl}}{r_t^2} \approx r_t \frac{\partial}{\partial R}\left(-\frac{GM_{gal}}{R^2}\right) = \frac{2GM_{gal} r_t}{R^3}$$

$$\Leftrightarrow$$

$$r_t \approx \left(\frac{M_{cl}}{2M_{gal}}\right)^{1/3} R$$

$$M_{cl} \approx 2.5\times10^6 \left(\frac{\tilde{\sigma}}{5\ \mathrm{km/s}}\right)^3 \left(\frac{R}{10\ \mathrm{kpc}}\right)^{3/2} M_\odot$$

$$M_{cl} \approx 8\pi\rho_0 r_K^2 r_t$$

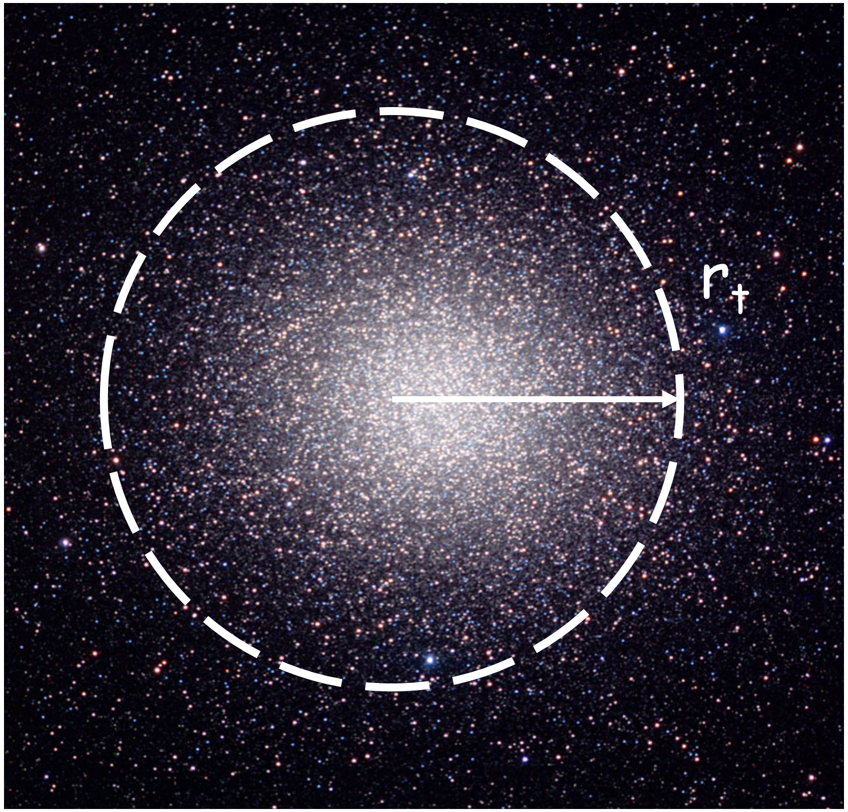
Tidal Radius

If one uses the relation $M=8\pi\rho_o r^2_K r_t$ for the mass contained within r_t we obtain

$$M_C \approx 8\pi\rho_o r_K^2 r_t$$

And from $$g_t \approx r_t \frac{\partial}{\partial r}(-\frac{GM_{gal}}{R^2}) = \frac{2GM_{gal}r_t}{R^3}$$

$$r_t = (\frac{4\pi\rho_o R^3}{M_{gal}})^{1/2} \qquad r_K = (\frac{\sigma^2 R^3}{GM_{gal}})^{1/2}$$



Using typical values for distances, observed velocity dispersion and central mass of glubular clusters and for the mass of our galaxy

$\sigma \sim 5$km/s, $\rho_o \sim 10^4$ M_o pc^{-3}, R~ 10 kpc, $M_{gal}=10^{11}$ M–

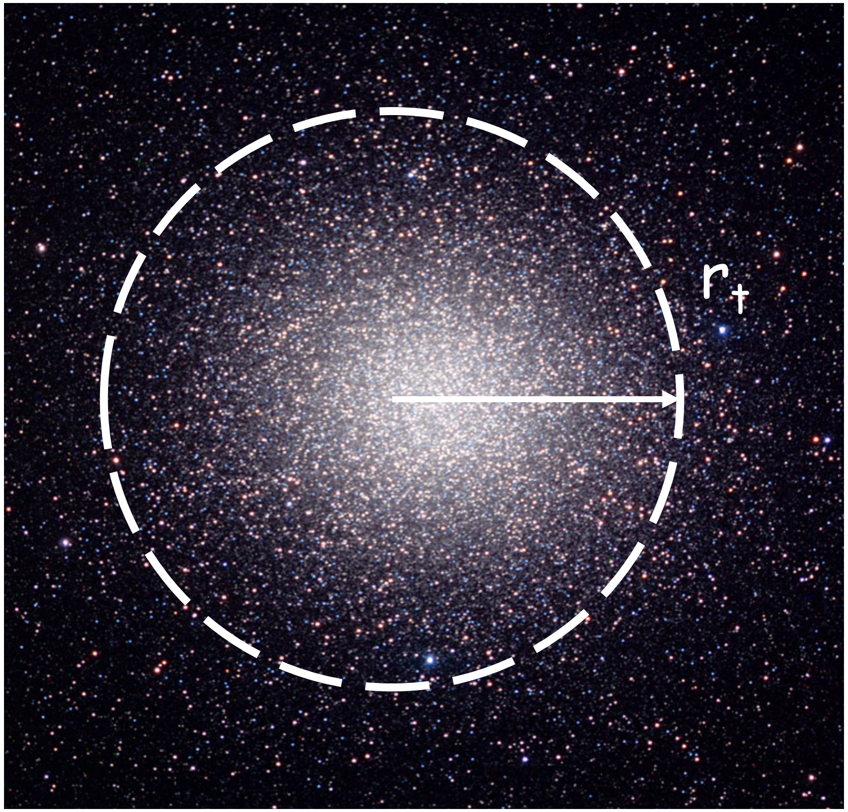
The tidal radius is $$r_t = 200(\frac{\sigma}{5\ km/s})(\frac{R}{10\ jps})^{3/2}$$

It is much larger than the King radius $$r_K \approx 0.2(\frac{\sigma}{5\ km/s})(\frac{\rho_o}{10^4 M_s\ pc^{-3}})^{-1/2}\ pc$$

$$r_K = \left(\frac{\sigma^2 R^3}{G M_{gal}}\right)^{1/2}$$

$$r_K \approx 0.2\left(\frac{\sigma}{5\ km/s}\right)\left(\frac{\rho_o}{10^4 M_s\ pc^{-3}}\right)^{-1/2}\ pc$$

The King radius yields a good estimate for the size of the dense central core of the cluster: the density in an isothermal sphere drops to $\rho_0/2$ at r~ $3r_K$ ~ 1pc



From these estimates, using

$$M_C \approx 8\pi\rho_o r_K^2 r_t$$

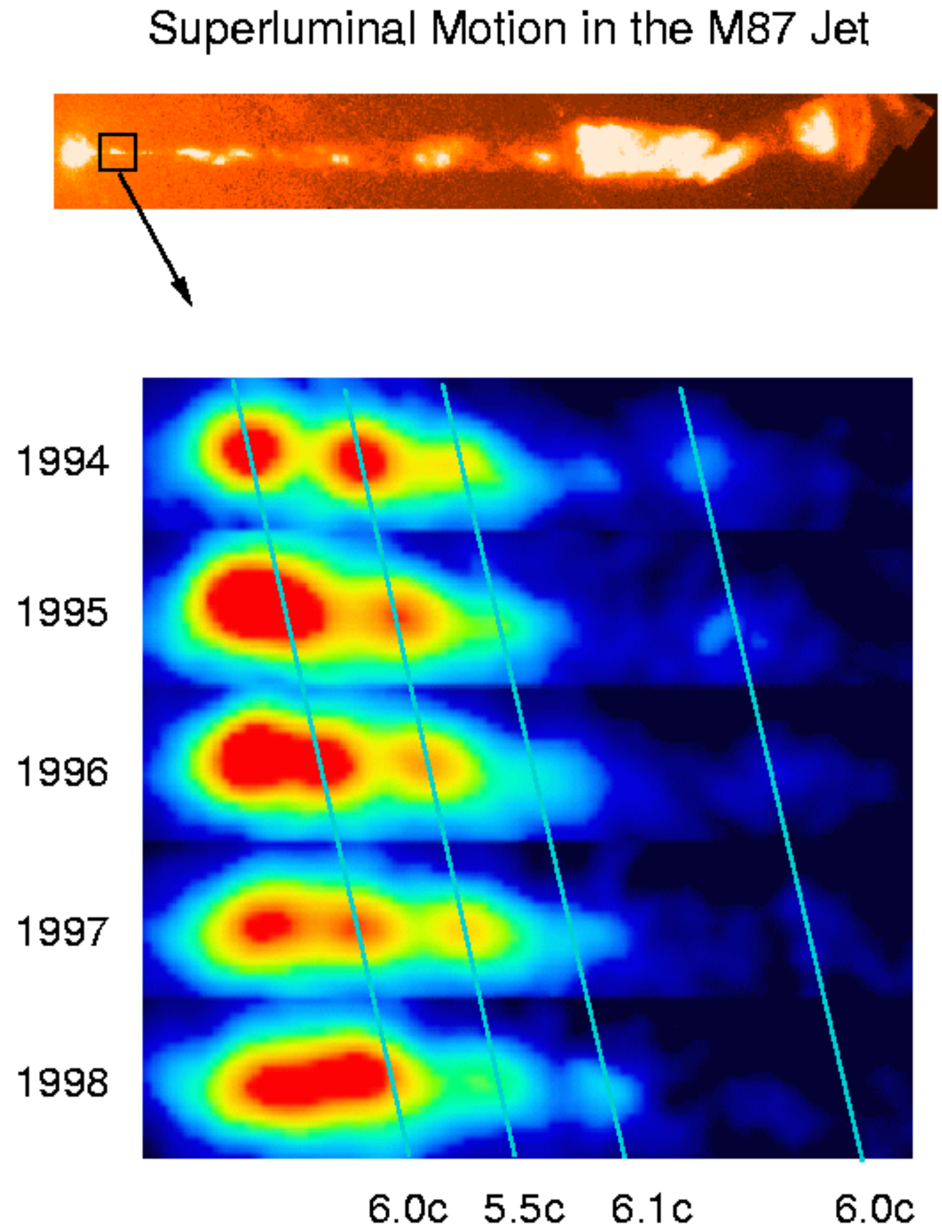
We can obtain the typical mass of a globular cluster

$$M_C \sim \frac{2\sigma^2}{G}\frac{\sigma^2 R^3}{G M_{gal}} \qquad \approx 2.5 \times 10^6\left(\frac{\sigma}{5\ km/s}\right)\left(\frac{R}{10\ kpc}\right)^{3/2}\ M_s$$

Which compares well with the masses of globular clusters inferred by observations

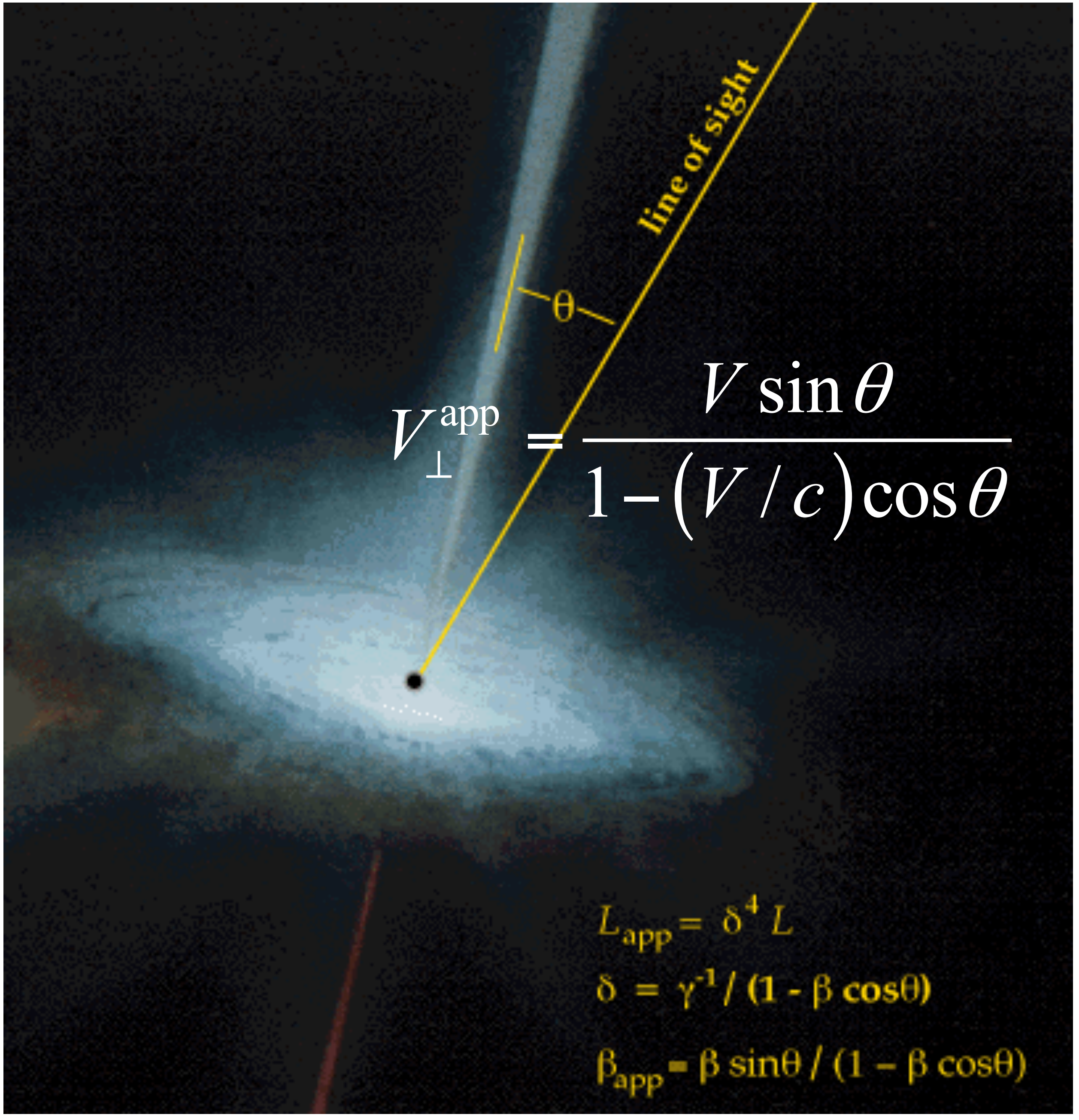
In the case of micro-quasars and powerful radio galaxies, the flow speeds are estimated to be close to the light speed

The consequence is that the apparent speed on the celestial sphere can be greater than c!

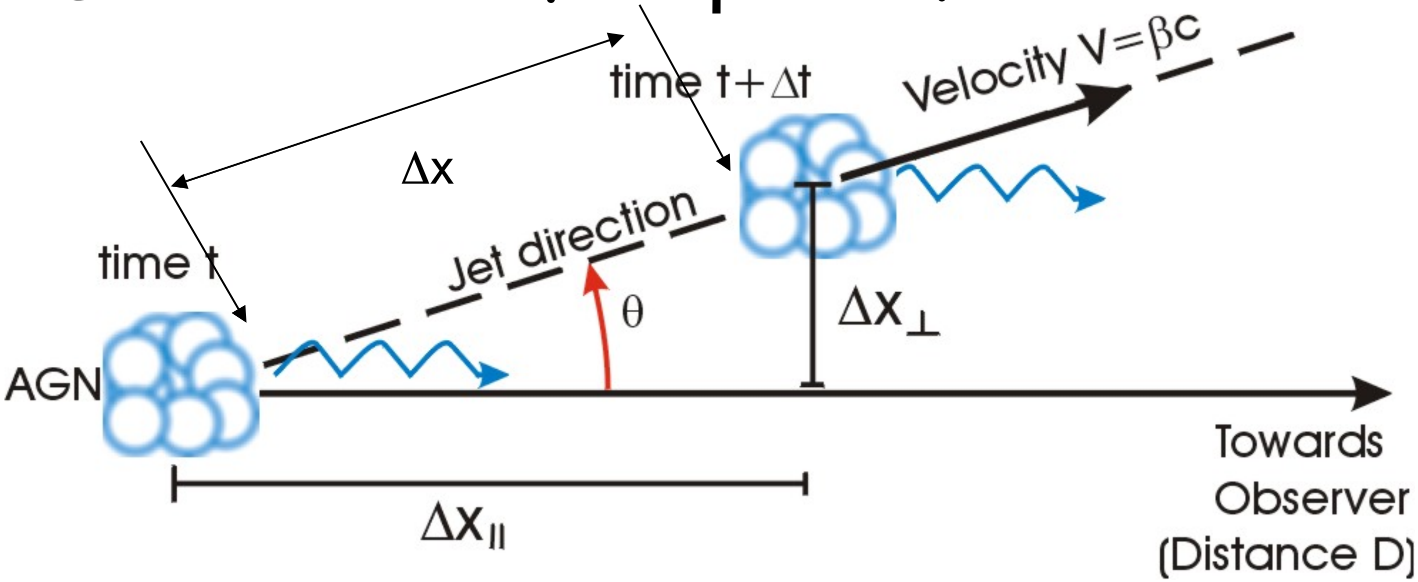


Observational clue:

Superluminal Motion: a relativistic illusion



Derivation of Superluminal Motion

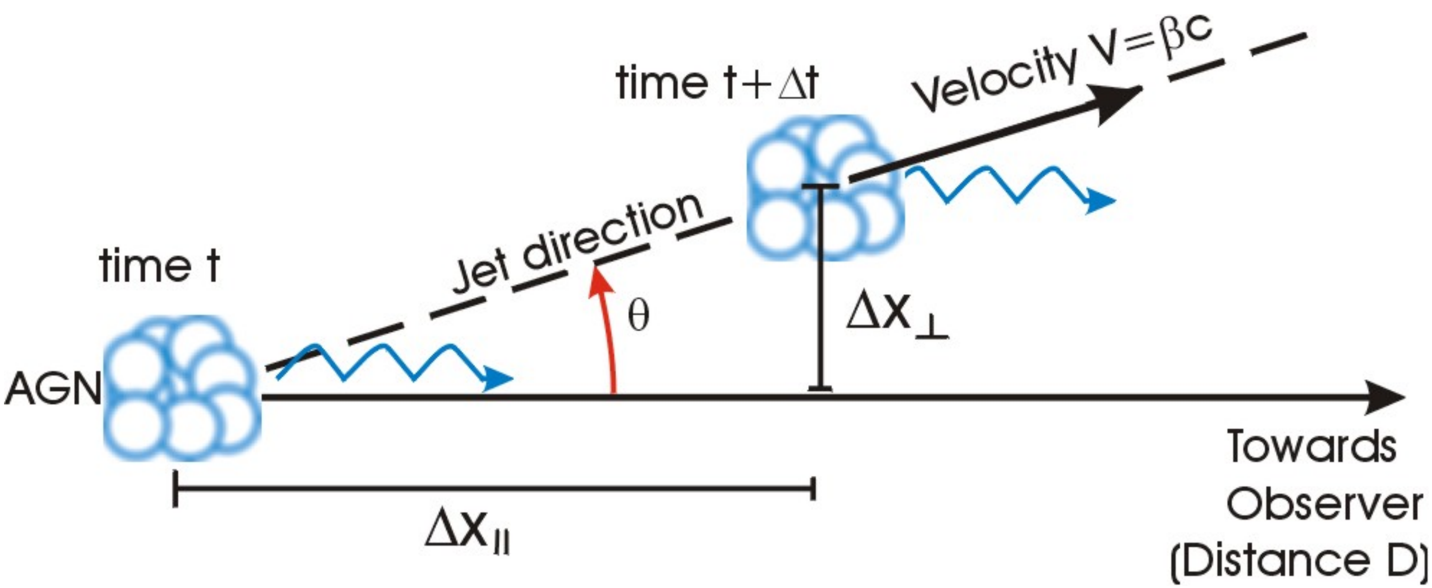


Let the source starts to emit at t → an observer on Earth receives the wave packet after a time $t_1 = t+D/c$

Let the source stop the emission after a time Δt, as measured at the source→ the observer receives the photon after a time $t_2 = t+\Delta t +(D-\Delta x)/c$, being Δx the distance covered in Δt by the emitting blob

The observer at Earth measures a time duration of the emission of

$$t_2 - t_1 = \Delta t - \Delta x_{\|}/c$$

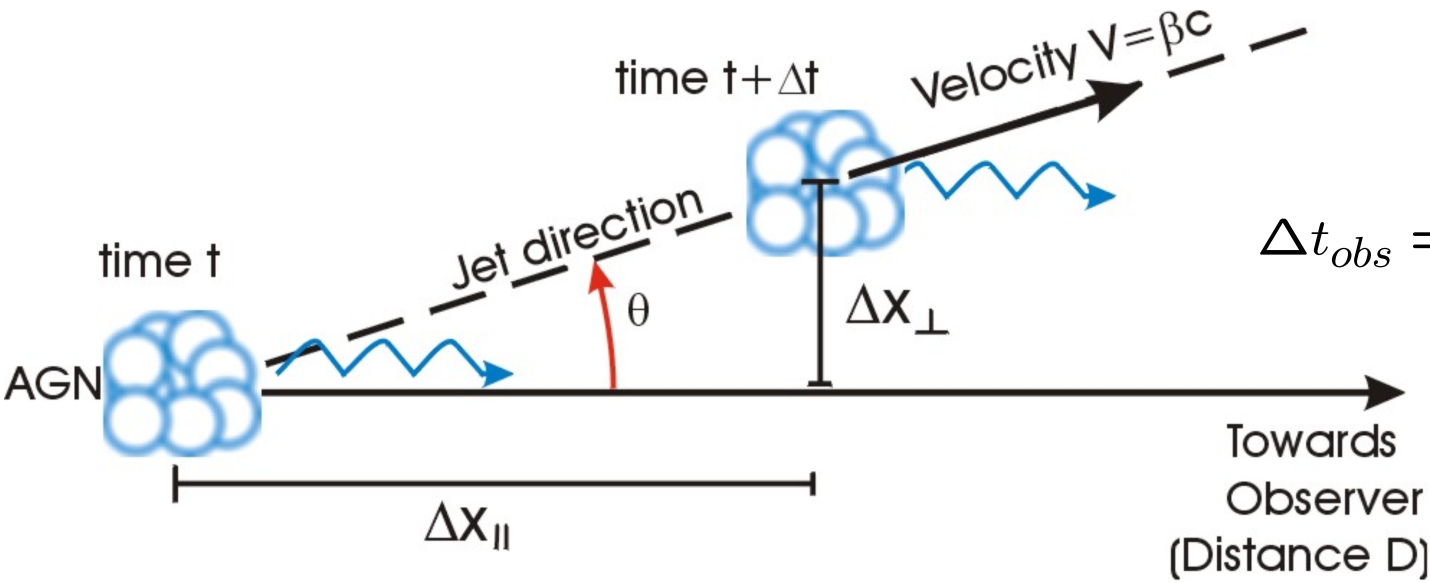


$$\Delta t_{obs} = t_2 - t_1 = \Delta t - \Delta x_{||}/c \qquad \Delta x_{||} = v\Delta t cos\theta = \beta c \Delta t cos\theta$$

$$\Delta t_{obs} = \Delta t(1 - \beta cos\theta)$$

If β =v/c ~ 1, the source "almost" catches up the emitted light, as a consequence the duration of emission measured at Earth is shorter than the duration at source (this is a consequence of the relativity of simultaneity due to the fact that the 2 observers are in different places→ the observer at rest in the source and at earth measure different durations)

$\Delta t_{obs} = \Delta t_{source}$ only if c = inif (as in newtonian mechanics)



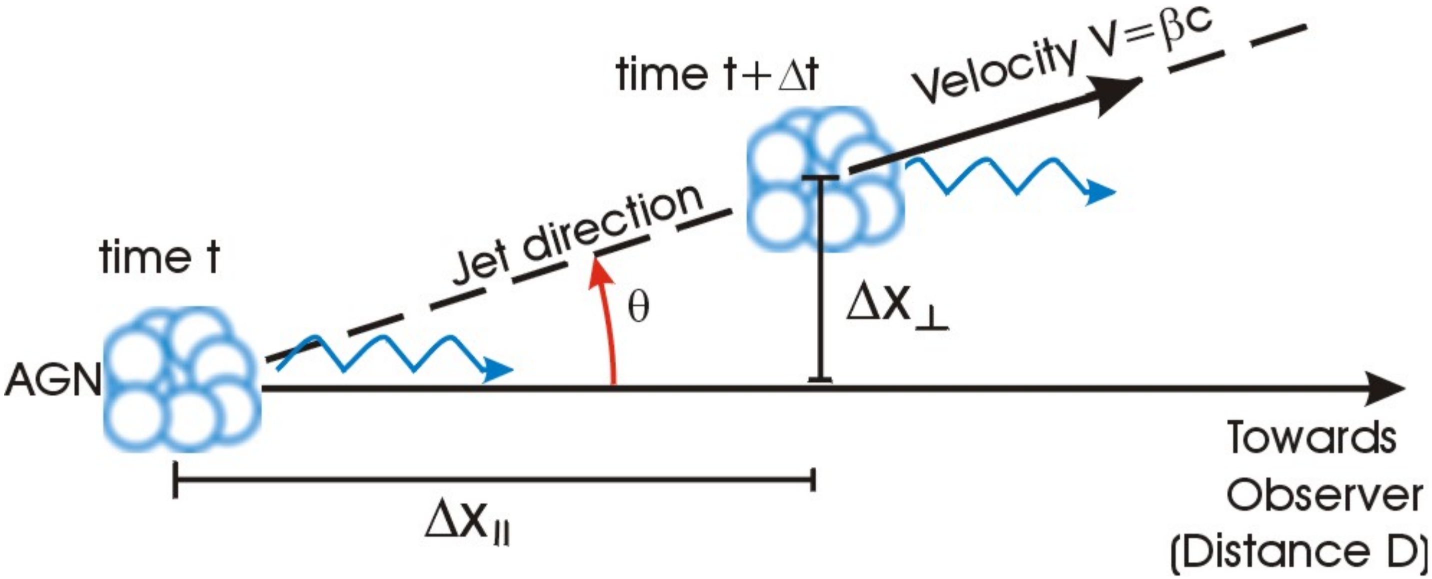
$$\Delta t_{obs} = \Delta t(1 - \beta cos\theta)$$

What we measure is the projection on the celestial sphere of the source motion, or more precisely the motion component orthogonal to the sight line, Δx_n

$$\Delta x_{\perp} = v\Delta tsin\theta = \beta c\Delta tsin\theta$$

The measured apparent speed from Earth is then

$$v_{app} = \Delta x_{\perp}/\Delta t_{obs} = \beta c\Delta tsin\theta/\Delta t(1 - \beta cos\theta) = \beta csin\theta/(1 - \beta cos\theta)$$



$$v_{app} = \beta c sin\theta / (1 - \beta cos\theta)$$

It is easy to show that v_{app} has a maximum when

$$dv_{app}/d\theta = (cos\theta - \beta)/(1 - \beta cos\theta)^2 = 0$$

This occurs when $\cos\theta = \beta$

At maximum the apparent speed is $v_{app}^{max} = \beta c(1 - \beta^2)^{1/2}/(1 - \beta^2)$

$$v_{app}^{max} = \beta c \gamma \qquad \gamma = (1-\beta^2)^{1/2}$$

It is clear that for $\beta \sim 1$ (that is relativistic source motion) $\gamma >> 1$ and therefore $v_{app} > c$

Small perturbations

The hydrodynamical equations have a very huge number of solutions

Because of non linearity of hydrodynamical equations is very difficult to find exact solutions

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P \qquad \frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V})=0$$

In the case of small perturbations it is possible to find solutions: sound waves

Let assume for simplicity to have a fluid at rest and with uniform pressure and density:

$V=0$, $\rho=\rho_o$ and $p=p_o$

All their derivatives are zero, so that the equations are trivially satisfied

Small perturbations

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V}) = 0$$

Consider now "small" arbitrary deviations from equilibrium, in which
$\mathbf{V}(\mathbf{x})=\delta\mathbf{V}(\mathbf{x})$, $p(\mathbf{x})=p_o+\delta p(\mathbf{x})$. $\rho(\mathbf{x})=\rho_o+\delta\rho(\mathbf{x})$

Where "small" means that $\delta V<<1$ $\delta p<<p_o$ $\delta\rho<<\rho_o$
(ideally they are infinitesimal)

Substituting into the motion equations we can find the equations for the perturbations

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_o \nabla \cdot (\delta\vec{v}) + \nabla \cdot (\delta\rho\delta\vec{v}) = 0$$

$$\frac{\partial}{\partial t}(\delta\vec{v}) + (\delta\vec{v} \cdot \nabla)\delta\vec{v} = -\frac{\nabla p}{\rho_0 + \delta\rho}$$

There are clearly three non linear terms involving variables

Small perturbations

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_o \nabla \cdot (\delta\vec{v}) + \nabla \cdot (\delta\rho\delta\vec{v}) = 0 \qquad \frac{\partial}{\partial t}(\delta\vec{v}) + (\delta\vec{v} \cdot \nabla)\delta\vec{v} = -\frac{\nabla\delta p}{\rho_0 + \delta\rho}$$

Using tha fact that $\delta V \ll 1$ $\delta p \ll p_o$ $\delta\rho \ll \rho_o$ we can neglect the non linear terms in the equations

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_o \nabla \cdot (\delta\vec{v}) \approx 0 \qquad \frac{\partial}{\partial t}(\delta\vec{v}) \approx -\frac{\nabla\delta p}{\rho_0}$$

Now we need to know the thermodynamical properties of the fluid, that is an equation of state which describes the fluid transformations

Usually, this equation expresses a TD variable as function of other two, for istance **p=p(ρ,s)**, where s is the specific entropy of the system

The latter is not contained in the fluid motion equations, so that we have to make some hypothesis on the fluid tranformations: let suppose then that the transformations are <u>adiabatic and reversible</u>, so that **δs=0** (ie constant entropy) and **p=aρ**$^{\gamma}$

Small perturbations

$\delta V << 1$ $\delta p << p_o$ $\delta\rho << \rho_o$ $$\frac{\partial}{\partial t}(\delta\rho) + \rho_o \nabla \cdot (\delta\vec{v}) \approx 0 \qquad \frac{\partial}{\partial t}(\delta\vec{v}) \approx -\frac{\nabla\delta p}{\rho_0}$$

p=p(ρ,s) **δs=0** (ie constant entropy) ➔**$p=a\rho^\gamma$**

In such a case, $\delta p = (\frac{\partial p}{\partial \rho})_s \delta\rho = (\frac{\gamma p}{\rho})\delta\rho = c_s^2 \delta\rho$ That is the sound speed describes as the fluid pressure responds to density variations

With the aid of this relation we can eliminate the speed from equations

$$\frac{\partial}{\partial t}(\delta\vec{v}) \approx -\frac{\nabla c_s^2 \delta\rho}{\rho_0} \qquad \frac{\partial}{\partial t}(\delta\rho) + \rho_o \nabla \cdot (\delta\vec{v}) \approx 0$$

Take the div of the first and the time derivative of the second

$$-\rho_o \frac{\partial}{\partial t}(\nabla \cdot \delta\vec{v}) \approx \nabla^2 c_s^2 \delta\rho \qquad \frac{\partial^2}{\partial t^2}(\delta\rho) \approx -\rho_o \frac{\partial}{\partial t}\nabla \cdot (\delta\vec{v}) \quad \Rightarrow \quad \nabla^2 c_s^2 \delta\rho \approx \frac{\partial^2}{\partial t^2}(\delta\rho)$$

For small pertubations c_s is constant ➔ $$c_s^2 \nabla^2 \delta\rho \approx \frac{\partial^2}{\partial t^2}(\delta\rho)$$ Wave equation!

→ solutions are, for istance $$\frac{\delta\rho}{\rho_o} = Ae^{k\cdot x \pm \omega t}$$

With wave amplitude arbitrary but constrained to the condition of linearity **A<<1**

$\omega^2 - \kappa^2 c^2 = 0$

Le onde sono longitudinali

In a moving medium the propagation speed of the wave and the frequency are found by applying for istance Galilei transformation between the reference frame where the fluid is at rest, where the wave propagates with speed c_s, and the "laboratory" frame where the fluid is moving with speed V

$$\boldsymbol{x}_{\text{lab}} = \boldsymbol{x}_{\text{fluid}} + \boldsymbol{V}t$$

The speed is simply obtained as $\mathbf{c_{s,lab}} = \mathbf{c_s} + \mathbf{V}$.
Ie in a moving medium the sound speed depends on the fluid speed
(this made difficult for '800 physicists to believe that light speed is constant for *any* observer)

Wave phase:

$$S_{\text{fluid}} = \boldsymbol{k} \bullet \boldsymbol{x}_{\text{fluid}} - \omega_{\text{fluid}} t = S_{\text{lab}} = \boldsymbol{k} \bullet \boldsymbol{x}_{\text{lab}} - \omega_{\text{lab}} t$$

$$\boldsymbol{k} \bullet \boldsymbol{x}_{\text{fluid}} - \omega_{\text{fluid}} t = \boldsymbol{k} \bullet \left(\boldsymbol{x}_{\text{fluid}} + \boldsymbol{V}t \right) - \omega_{\text{lab}} t$$

$$\Leftrightarrow$$

$$\omega_{\text{lab}} - \boldsymbol{k} \bullet \boldsymbol{V} = \omega_{\text{fluid}}$$

"Doppler shift"

with $\omega^2 - \kappa^2 c^2 = 0$