Lecture 11 141116

Il pdf delle lezioni puo' essere scaricato da

http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1617/

Connection with thermodynamics: Ideal Gas Law

Isotropic gas of point particles in Thermodynamic Equilibrium:

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_\mathrm{b}T$$

Temperature is defined in terms of kinetic energy of the thermal motions!

Connection with thermodynamics: Ideal Gas Law

Isotropic gas of point particles in Thermodynamic Equilibrium:

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\rm b}T$$

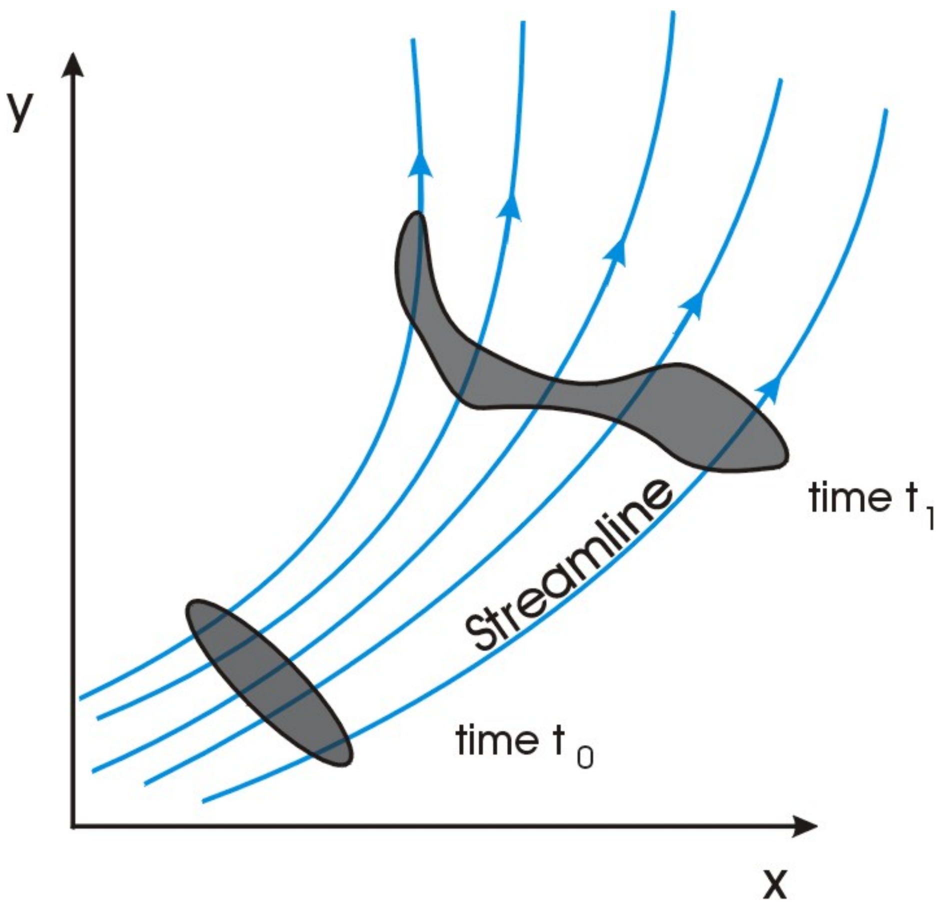
Ideal Gas Law: in terms of temperature T and number-density n:
($\rho = nm = \mu n m_H$, $R = k_b / m_H$)

$$p = \frac{1}{3}\rho\sigma^2$$

$$P(\rho\,,\,T) = nk_{\rm b}T = \frac{\rho\mathcal{R}T}{\mu}$$

Digressione: fluidi viscosi

Density Changes and Mass Conservation

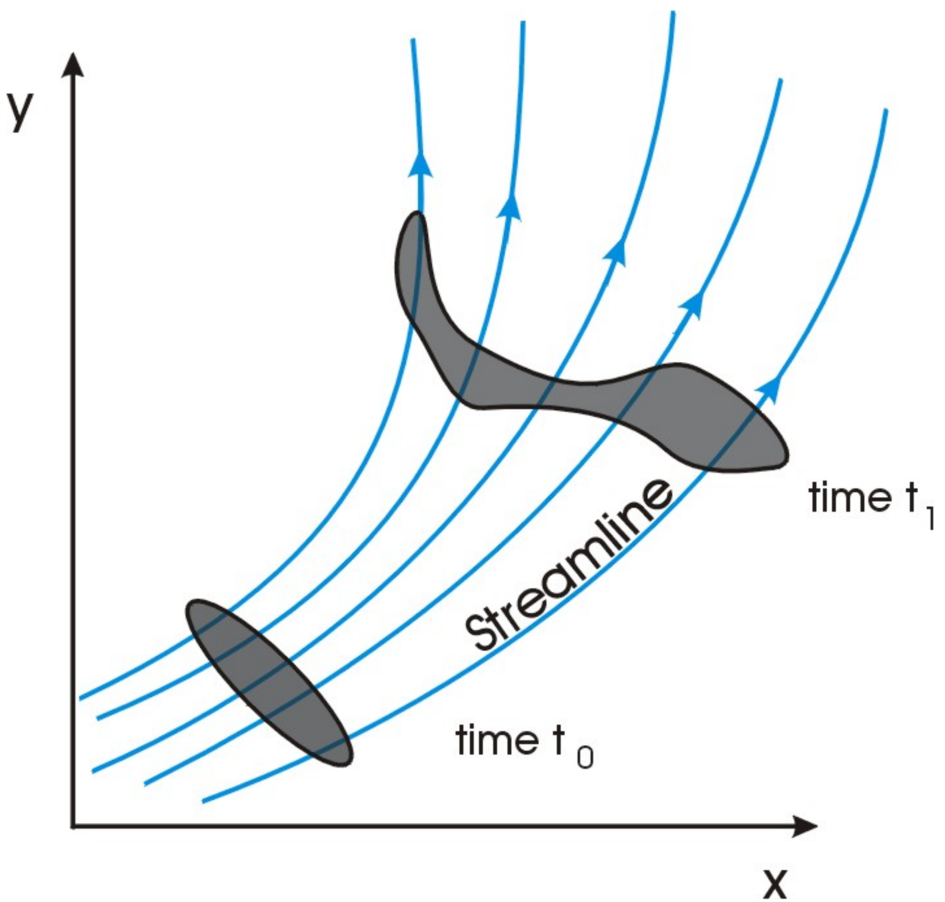


Two-dimensional example:

A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

Density Changes and Mass Conservation



Two-dimensional example:

A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

<u>So:</u> the mass density must change in response to the flow!

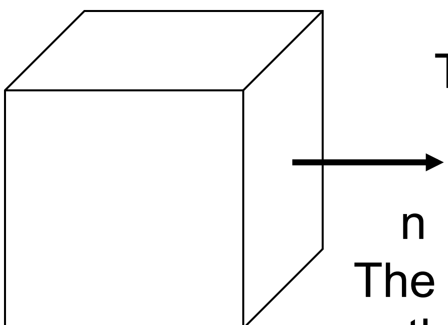
Mass conservation law

The mass cannot be created nor destroyed (in non-relativistic classical dynamics)

Therefore in a volume V the mass can change only because some of it leaves or enters the volume

The amount of mass through d**A** per time unit is $dF = \rho\vec{v} \cdot d\vec{A}$

Convention: dF>0 if outgoing



The flux $(\frac{dM}{dt})_{out} = \int_S \rho\vec{v} \cdot d\vec{A}$

The outgoing flux must be balanced by the change of mass in the volume $\frac{\partial M_V}{\partial t} = -(\frac{dM}{dt})_{out}$

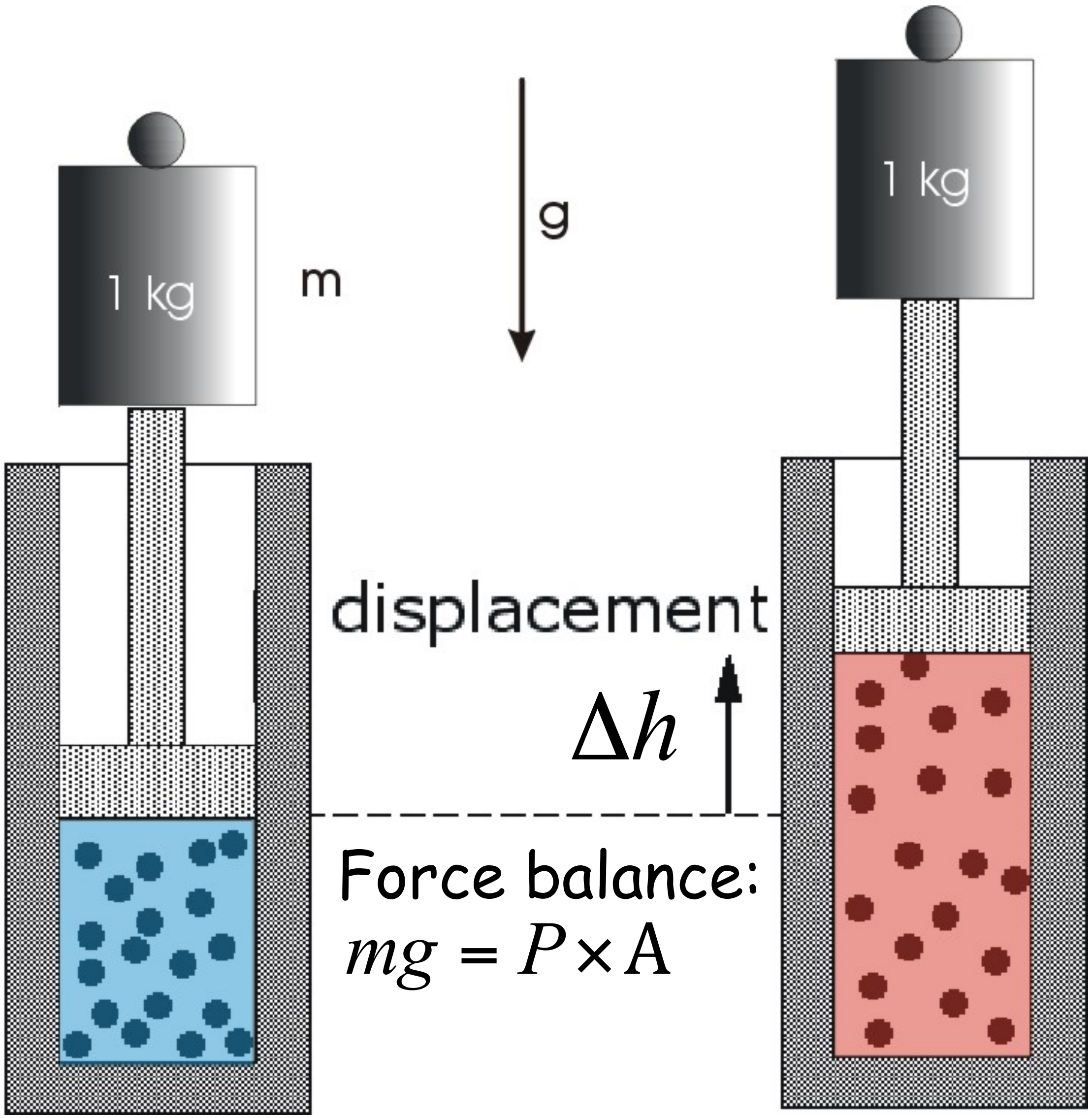
$$\frac{\partial M_V}{\partial t} = \frac{\partial}{\partial t}(\int_V \rho dV)$$

$$\frac{\partial}{\partial t}(\int_V \rho dV) = -\int_S \rho\vec{v} \cdot d\vec{A} \quad = -\int_V \nabla \cdot (\rho\vec{v}) dV$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{v})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0}$$

Thermodynamics



$$\Delta W = mg\Delta h$$

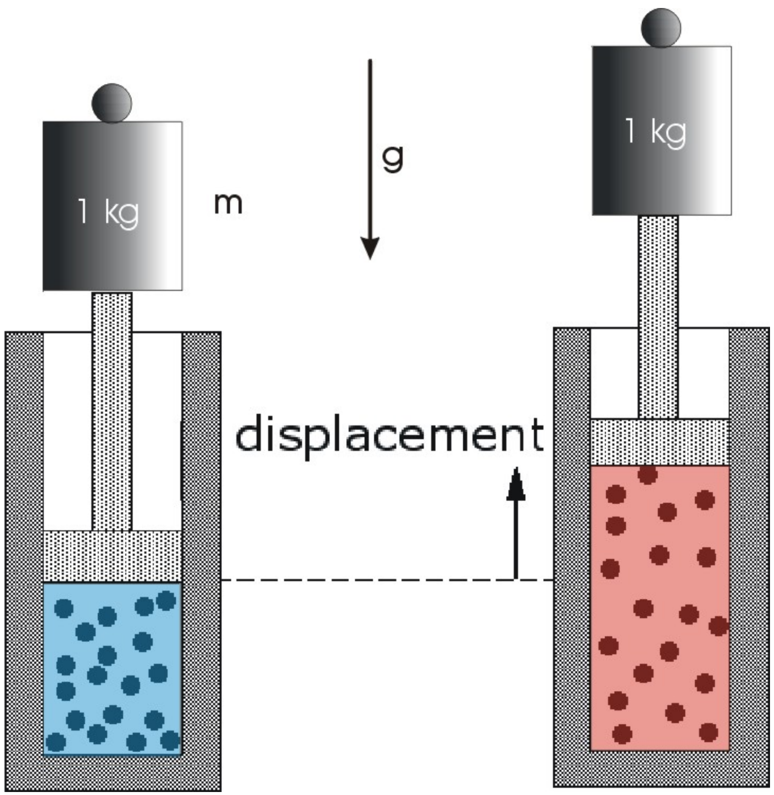
$$\Delta U_{gas} = \Delta Q - \Delta W$$

$$= \Delta Q - mg\Delta h$$

$$= \Delta Q - PA\ \Delta h$$

$$= \Delta Q - P\ \Delta V$$

First law of thermodynamics:



$$dU_{gas} = dQ - pdV$$

Change in internal energy
=
heat added by <u>external</u> sources
-
work done by gas

Entropy:

A measure of 'disorder'

Second Thermodynamic law:

$$dQ = TdS$$

$$dS \geq 0$$

$$\frac{dS}{dt} \geq 0$$

For an isolated system

The Adiabatic Gas Law: the behaviour of pressure

Thermodynamics:

$$\mathrm{d}Q \equiv T\,\mathrm{d}S = \mathrm{d}U + P\,\mathrm{d}\mathcal{V}$$

Special case: adiabatic change

$$\mathrm{d}Q = T\,\mathrm{d}S = 0$$

U = internal energy, T = temperature, S = entropy and V = volume

The Adiabatic Gas Law: the behaviour of pressure

Thermodynamics:

$$\mathrm{d}Q \equiv T\,\mathrm{d}S = \mathrm{d}U + P\,\mathrm{d}\mathcal{V}$$

Special case: adiabatic change

$$\mathrm{d}Q = T\,\mathrm{d}S = 0$$

Gas of point particles of mass *m* :

Internal energy: $U = n\,\mathcal{V} \times \left[\frac{1}{2}m\left(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}\right)\right] = \frac{1}{2}\,\rho\,\mathcal{V}\overline{\sigma^2}$

Pressure: $P = \frac{1}{3}nm\,\left(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}\right) = \frac{1}{3}\,\rho\,\overline{\sigma^2}$

Thermal equilibrium: $\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\rm b}T$

$$\rightarrow \quad P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \frac{3}{2}\,\frac{\rho \mathcal{R} T\,\mathcal{V}}{\mu}$$

Thermal equilibrium:

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{6}\,m\overline{\sigma^2} = \frac{1}{2}\,k_{\mathrm{b}}T$$

$$P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \frac{3}{2}\,\frac{\rho \mathcal{R} T\,\mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P\,\mathrm{d}\mathcal{V} = 0 \quad \longrightarrow \quad \mathrm{d}\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\mathcal{V} = 0$$

Adiabatic Gas Law: a polytropic relation

Thermal equilibrium:

$$\tfrac{1}{2}m\overline{\sigma_x^2} = \tfrac{1}{2}m\overline{\sigma_y^2} = \tfrac{1}{2}m\overline{\sigma_z^2} = \tfrac{1}{6}\,m\overline{\sigma^2} = \tfrac{1}{2}\,k_{\rm b}T$$

$$P = \frac{\rho \mathcal{R} T}{\mu} \quad , \quad U = \tfrac{3}{2}\,\frac{\rho \mathcal{R} T\,\mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P\,\mathrm{d}\mathcal{V} = 0$$

$$\mathrm{d}\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\mathcal{V} = 0$$

Chain rule for 'd'-operator:

$$\mathrm{d}(f\,g) = (\mathrm{d}f)\,g + f\,(\mathrm{d}g)$$

$$\tfrac{5}{3}\,P\,\mathrm{d}\mathcal{V} + \mathcal{V}\,\mathrm{d}P = 0\,.$$

(just like differentiation!)

$$\frac{\mathrm{d}P}{P} + \tfrac{5}{3}\,\frac{\mathrm{d}\mathcal{V}}{\mathcal{V}} = \mathrm{d}\log\left(P\,\mathcal{V}^{5/3}\right) = 0$$

Adiabatic Gas Law: a polytropic relation

Adiabatic pressure change:

$$P \times \mathcal{V}^{5/3} = \text{constant}$$

For small volume:
mass conservation!

$$M = \rho\,\mathcal{V} = \text{constant}$$

$$\rightarrow P\,\rho^{-5/3} = \text{constant}$$

Specific Heat and Entropy

Specific Volume:
contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics of unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific Heat and Entropy

Specific Volume
contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics
of a unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific energy e
and pressure P :

$$e \equiv \tfrac{3}{2}\frac{\mathcal{R}T}{\mu} = \tfrac{3}{2}\frac{k_{\mathrm{b}}T}{m} \ , \quad P = \frac{\rho\,\mathcal{R}\,T}{\mu}$$

Specific heat coeff.
at constant volume

$$c_{\mathrm{v}} = \frac{\partial e}{\partial T} = \tfrac{3}{2}\frac{k_{\mathrm{b}}}{m}$$

ρ is kept constant!
d(1/ρ) =0

Specific Heat and Entropy

Specific Volume contains unit mass

$$\overline{\mathcal{V}} \equiv \frac{1}{\rho}$$

Thermodynamics of a unit mass:

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

Specific energy *e* and pressure *P* :

$$e \equiv \frac{3}{2}\frac{\mathcal{R}T}{\mu} = \frac{3}{2}\frac{k_{\mathrm{b}}T}{m} \;, \quad P = \frac{\rho\,\mathcal{R}\,T}{\mu}$$

Specific heat coeff. at constant volume

$$c_{\mathrm{v}} = \frac{\partial e}{\partial T} = \frac{3}{2}\frac{k_{\mathrm{b}}}{m}$$

$$\mathrm{d}q = \mathrm{d}\left(e + \frac{P}{\rho}\right) - \frac{\mathrm{d}P}{\rho}$$

Specific heat coeff. at constant pressure: d*P* =0

$$c_{\mathrm{p}} = \frac{\partial(e + P/\rho)}{\partial T} = \frac{5}{2}\frac{k_{\mathrm{b}}}{m}$$

$$c_{\mathrm{p}} - c_{\mathrm{v}} = \frac{k_{\mathrm{b}}}{m} = \frac{\mathcal{R}}{\mu}$$

$$\mathrm{d}q = c_\mathrm{v}\,\mathrm{d}T + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

$$= c_\mathrm{v}\,\mathrm{d}T - \left(\frac{\mathcal{R}T}{\rho\mu}\right)\,\mathrm{d}\rho$$

$$= c_\mathrm{v}T\,\left[\frac{\mathrm{d}T}{T} - \boxed{\left(\frac{c_\mathrm{p}}{c_\mathrm{v}} - 1\right)}\,\frac{\mathrm{d}\rho}{\rho}\right]$$

Thermodynamic law for a unit mass,
rewritten in terms of specific heat coefficients

Definition specific entropy s

$$\mathrm{d}q = c_{\mathrm{v}}\,\mathrm{d}T + \left(\frac{\rho \mathcal{R} T}{\mu}\right)\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

$$= c_{\mathrm{v}}\,\mathrm{d}T - \left(\frac{\mathcal{R} T}{\rho \mu}\right)\,\mathrm{d}\rho$$

$$= c_{\mathrm{v}} T \left[\frac{\mathrm{d}T}{T} - \left(\frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} - 1\right)\frac{\mathrm{d}\rho}{\rho}\right]$$

$$T\,\mathrm{d}s = c_{\mathrm{v}} T \left[\frac{\mathrm{d}T}{T} - (\gamma - 1)\frac{\mathrm{d}\rho}{\rho}\right]$$

$$s = c_{\mathrm{v}}\,\log\left(\frac{T}{\rho^{\gamma-1}}\right) + \text{constant}$$

$$s = c_{\mathrm{v}}\,\log\left(P\,\rho^{-\gamma}\right) + \text{constant}$$

γ is the specific heat ratio
= 5/3 for ideal gas of point particles

Definition specific entropy s

$$
\begin{aligned}
dq &= c_v\, dT + \left(\frac{\rho \mathcal{R} T}{\mu}\right) d\left(\frac{1}{\rho}\right) \\
&= c_v\, dT - \left(\frac{\mathcal{R} T}{\rho \mu}\right) d\rho \\
&= c_v T \left[\frac{dT}{T} - \left(\frac{c_p}{c_v} - 1\right) \frac{d\rho}{\rho}\right]
\end{aligned}
$$

$$T\, ds = c_v T \left[\frac{dT}{T} - (\gamma - 1) \frac{d\rho}{\rho}\right]$$

$$s = c_v \log\left(\frac{T}{\rho^{\gamma-1}}\right) + \text{constant}$$

$$s = c_v \log\left(P\, \rho^{-\gamma}\right) + \text{constant}$$

$$\log T - (\gamma - 1) \log \rho = \text{constant}$$

with

$$\gamma \equiv \frac{c_p}{c_v} = \frac{5}{3}\,.$$

Case of constant entropy (adiabatic gas) : $ds = 0$

$$T\rho^{-(\gamma-1)} = \text{constant} \quad , \quad P\rho^{-\gamma} = \text{constant}$$

(Self-)gravity

$$\boldsymbol{f}_{\rm gr} = \rho\, \boldsymbol{g} = -\rho\, \boldsymbol{\nabla} \Phi$$

$$\boldsymbol{g}(\boldsymbol{x}, t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x}, t) = -\begin{pmatrix} \dfrac{\partial \Phi}{\partial x} \\ \\ \dfrac{\partial \Phi}{\partial y} \\ \\ \dfrac{\partial \Phi}{\partial z} \end{pmatrix}$$

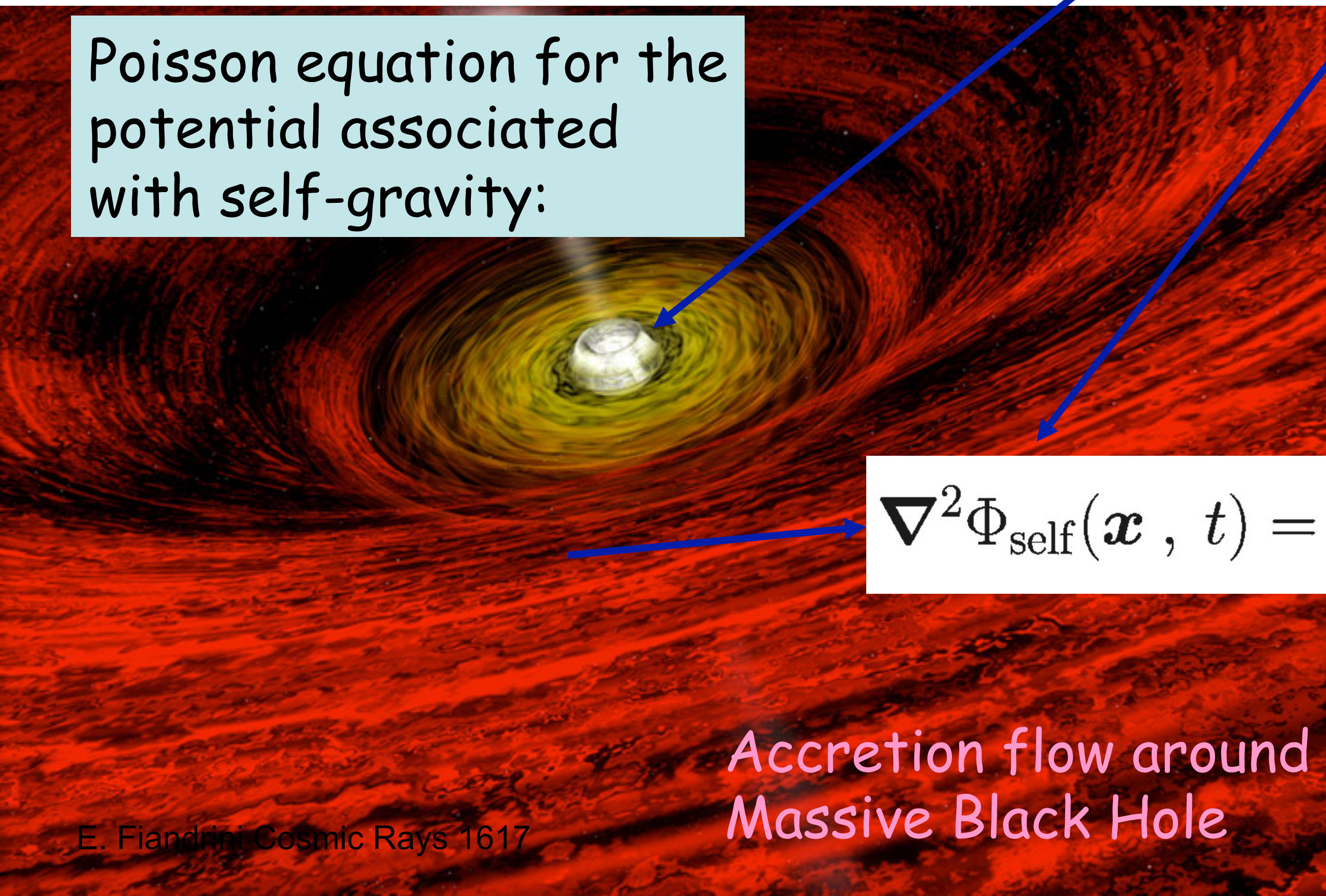
$$\rho\, \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla} \Phi$$

Self-gravity and Poisson's equation

Potential: two contributions!

$$\Phi(\boldsymbol{x}\,,t) = \Phi_{\rm ext}(\boldsymbol{x}\,,t) + \Phi_{\rm self}(\boldsymbol{x}\,,t)$$

Poisson equation for the potential associated with self-gravity:



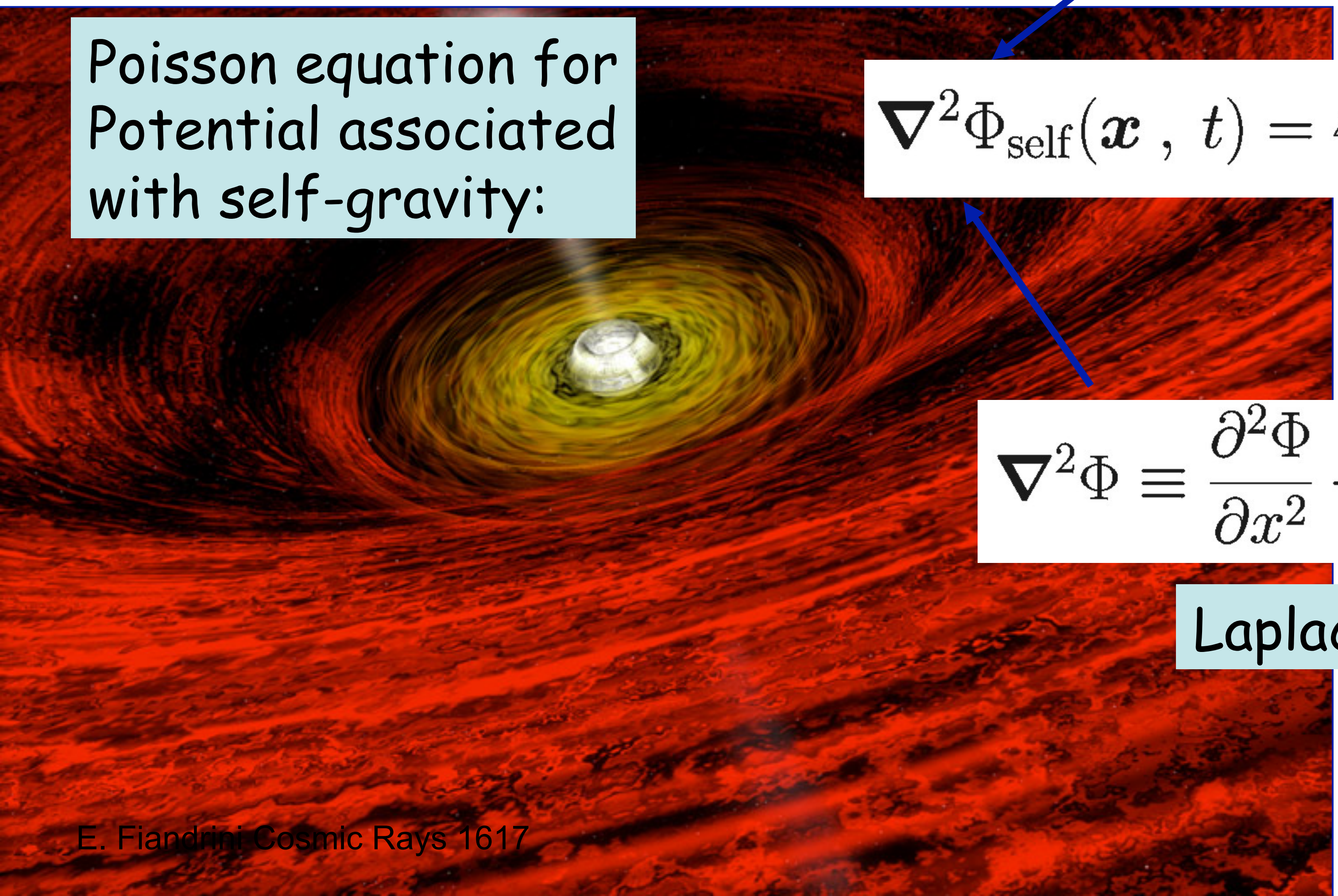
$$\boldsymbol{\nabla}^2\Phi_{\rm self}(\boldsymbol{x}\,,\ t) = 4\pi G\,\rho(\boldsymbol{x}\,,\ t)$$

Accretion flow around Massive Black Hole

Self-gravity and Poisson's equation

Potential: two contributions!

$$\Phi(\boldsymbol{x}\,,t) = \Phi_{\rm ext}(\boldsymbol{x}\,,t) + \Phi_{\rm self}(\boldsymbol{x}\,,t)$$



Poisson equation for Potential associated with self-gravity:

$$\boldsymbol{\nabla}^2\Phi_{\rm self}(\boldsymbol{x}\,,\ t) = 4\pi G\,\rho(\boldsymbol{x}\,,\ t)$$

$$\boldsymbol{\nabla}^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Laplace operator

Summary: Equations describing ideal (self-)gravitating fluid

Equation of Motion:

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla} \Phi$$

Continuity Equation:
behavior of mass-density

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V}) = 0$$

Ideal gas law
&
Adiabatic law:
Behavior of pressure
and temperature

$$P(\rho\,,\,T) = n k_{\rm b} T = \frac{\rho \mathcal{R} T}{\mu}$$

$$P\, \rho^{-5/3} = \text{constant}$$

Poisson's equation: self-gravity

$$\boldsymbol{\nabla}^2 \Phi_{\rm self}(\boldsymbol{x}\,,\; t) = 4\pi G\, \rho(\boldsymbol{x}\,,\; t)$$

Conservative Form of the Equations

Aim: To cast all equations in the same *generic form:*

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Reasons:

1. Allows quick identification of conserved quantities
2. This form works best in constructing numerical codes for *Computational Fluid Dynamics*
3. Shock waves are best studuied form a conservative point of view

Generic Form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix} + \nabla\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix} = \begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Transported quantity is a scalar *S*, so flux **F** must be a vector!

$$\frac{\partial S}{\partial t} + \nabla\cdot\boldsymbol{F} = q(\boldsymbol{x}\ ,\ t)$$

Component form:

$$\frac{\partial S}{\partial t} + \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = q$$

Generic Form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix} + \nabla\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix} = \begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Transported quantity is a vector *M* , so the flux must be a tensor *T* .

$$\frac{\partial \boldsymbol{M}}{\partial t} + \nabla\cdot\boldsymbol{T} = \boldsymbol{Q}(\boldsymbol{x}\,,\,t)$$

Component form:

$$\frac{\partial}{\partial t}\begin{pmatrix}M_x\\ \\ M_y\\ \\ M_z\end{pmatrix} + \begin{pmatrix}\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{yx}}{\partial y}+\frac{\partial T_{zx}}{\partial z}\\ \\ \frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{zy}}{\partial z}\\ \\ \frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{yz}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\end{pmatrix} = \begin{pmatrix}Q_x\\ \\ Q_y\\ \\ Q_z\end{pmatrix}$$

The fact the the flux of a vector field is a rank 2 tensor can be understood as follows: the transported quantity is a vector with 3 arbitrary components, each of them can be transported in 3 indipendent directions → so there are 3x3 indipendent quantitites...exactly the nbr of components of a rank 2 tensor

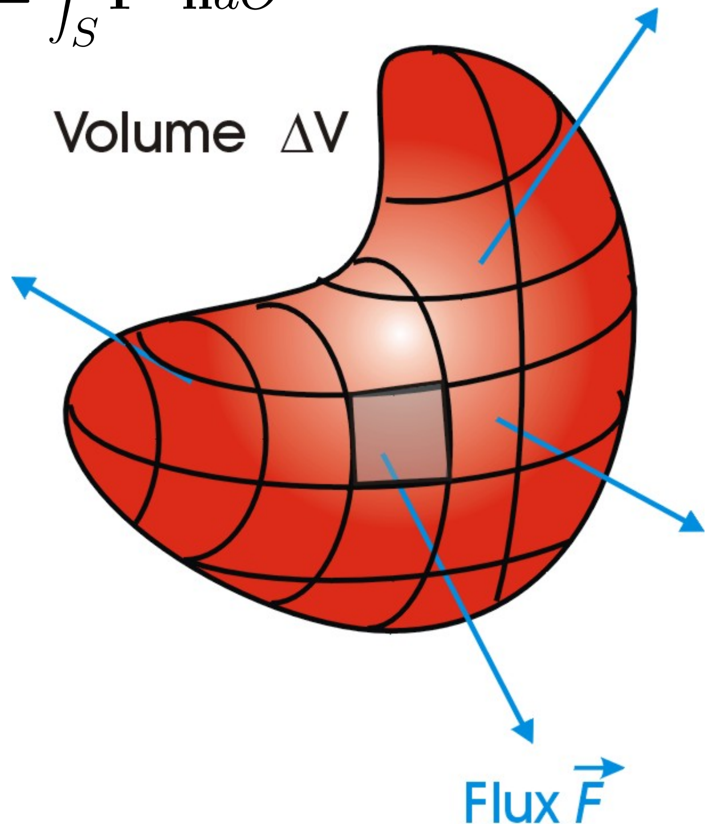
Integral properties: Stokes Theorem

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{density of}\\ \\ \text{quantity}\end{pmatrix}+\boldsymbol{\nabla}\cdot\begin{pmatrix}\text{flux of that}\\ \\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{external sources}\\ \\ \text{per unit volume}\end{pmatrix}$$

Let integrate the equation over a volume V and use the Stokes theorem

$$\int_V \nabla\cdot\mathbf{F}dV=\int_S \mathbf{F}\cdot\mathbf{n}dO$$

$$\frac{\partial}{\partial t}\left(\int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, S\right)=\int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, q(\boldsymbol{x}\,,\,t)-\oint_{\partial\mathcal{V}} \mathrm{d}\boldsymbol{O}\cdot\boldsymbol{F}$$



The integral relation states the amount of quantity S in a volume can change only due to sources in V or by a flux of S into or out from V

Examples: mass and momentum conservation

Mass conservation: already in conservation form!

Continuity Equation:
transport of the scalar ρ

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Excludes 'external mass sources' due to processes like two-photon pair production etc.

Examples: mass- and momentum conservation

Mass conservation: already in conservation form!

Continuity Equation:
transport of the scalar ρ

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{V}) = 0$$

Momentum conservation: transport of a vector!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\, \boldsymbol{V} \right] = -\boldsymbol{\nabla} P - \rho\, \boldsymbol{\nabla} \Phi$$

Algebraic Manipulation

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\, \boldsymbol{I} \right) = -\rho \boldsymbol{\nabla} \Phi$$

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right] = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi$$

Starting point: Equation of Motion

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V}) = 0$$

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} = \frac{\partial(\rho\boldsymbol{V})}{\partial t} - \boldsymbol{V}\frac{\partial \rho}{\partial t}$$

$$= \frac{\partial(\rho\boldsymbol{V})}{\partial t} + \boldsymbol{V}\,(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))$$

Use:
1. chain rule for differentiation
2. continuity equation for density

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right] = -\boldsymbol{\nabla}P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V}) = 0$$

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} = \frac{\partial(\rho\boldsymbol{V})}{\partial t} - \boldsymbol{V}\frac{\partial\rho}{\partial t} = \frac{\partial(\rho\boldsymbol{V})}{\partial t} + \boldsymbol{V}\left(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V})\right)$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t} + (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi$$

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\mathbf{V}\right] = -\boldsymbol{\nabla}P - \rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho\mathbf{V})}{\partial t} + (\boldsymbol{\nabla}\cdot(\rho\mathbf{V}))\mathbf{V} + \rho(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi$$

$$(\boldsymbol{\nabla}\cdot(\rho\mathbf{V}))\,\mathbf{V} + \rho\,(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} = \boldsymbol{\nabla}\cdot\,(\rho\mathbf{V}\otimes\mathbf{V})$$

Use divergence chain rule for dyadic tensors

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\,\mathbf{V}\right]=-\nabla P-\rho\,\nabla\Phi$$

$$\frac{\partial(\rho\mathbf{V})}{\partial t}+(\nabla\cdot(\rho\mathbf{V}))\mathbf{V}+\rho(\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla P-\rho\nabla\Phi$$

$$(\nabla\cdot(\rho\mathbf{V}))\,\mathbf{V}+\rho\,(\mathbf{V}\cdot\nabla)\mathbf{V}=\nabla\cdot\,(\rho\mathbf{V}\otimes\mathbf{V})$$

$$\nabla P=\nabla\cdot(P\,\mathbf{I})$$

Rewrite pressure gradient as a divergence

As advertised: Algebraic Manipulation!

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V}\right] = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + (\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\boldsymbol{V} + \rho(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi$$

$$(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}))\, \boldsymbol{V} + \rho\, (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V})$$

$$\boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot (P\, \boldsymbol{I})$$

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\, \boldsymbol{I}) = -\rho \boldsymbol{\nabla} \Phi$$

As advertised: Algebraic Manipulation!

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\right]=-\boldsymbol{\nabla}P-\rho\,\boldsymbol{\nabla}\Phi$$

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t}+(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}))\boldsymbol{V}+\rho(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=-\boldsymbol{\nabla}P-\rho\boldsymbol{\nabla}\Phi$$

Momentum density

Stress tensor = momentum flux

Momentum source: gravity

$$\frac{\partial(\rho\boldsymbol{V})}{\partial t}+\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{V}\otimes\boldsymbol{V}+P\,\boldsymbol{I}\right)=-\rho\boldsymbol{\nabla}\Phi$$

The tensor $\mathbf{R}_{ik}=\rho V_i V_k+p\delta_{ik}$ is the Reynolds stress tensor for an ideal fluid and represents the momentum flux