Lecture 10 091116

Il pdf delle lezioni puo' essere scaricato da

http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1617/

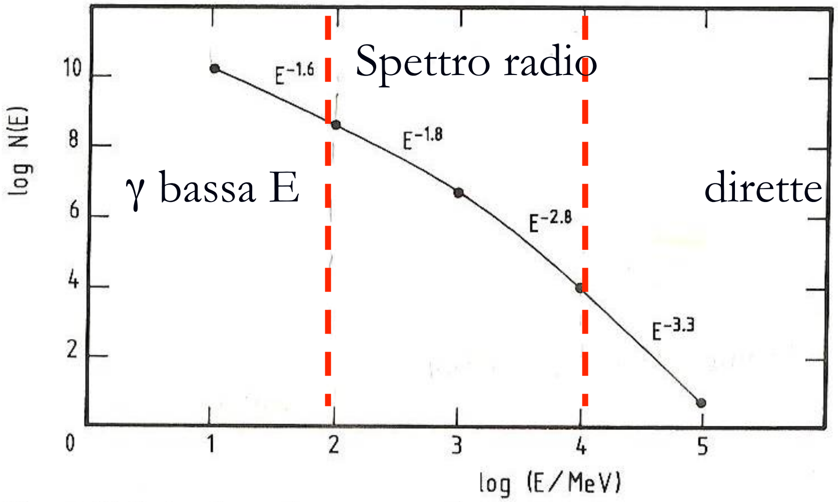
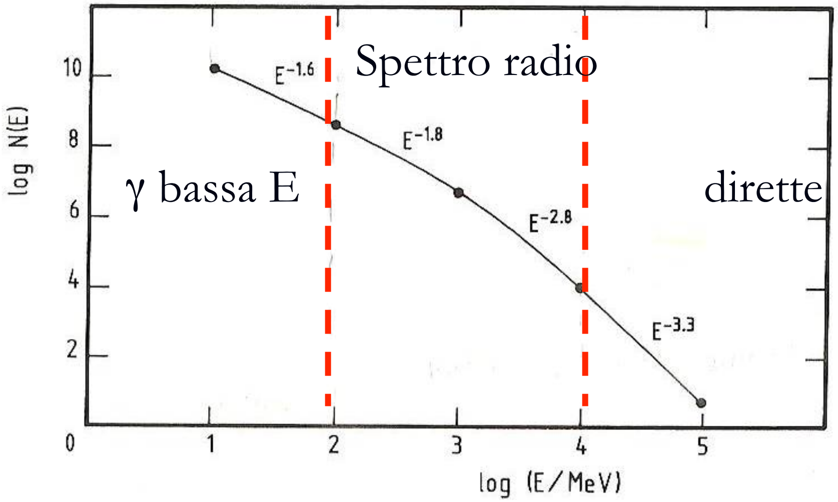


Figure 19.3. A schematic representation of the electron energy spectrum in the local interstellar medium from the data discussed in Sections 18.2 and 18.3. This spectrum has been subject to energy losses at high and low energies during propagation of the electrons through the interstellar medium. The units on the ordinate are relative units (see Fig. 18.14 for physical units).

Potremmo risolvere l'equ di diffusione completa...ma sarebbe piuttosto lungo...

Usiamo invece il modello Leaky Box in cui il termine di diffusione e' rimpiazzato da un term N(E)/T(E), T tempo di fuga

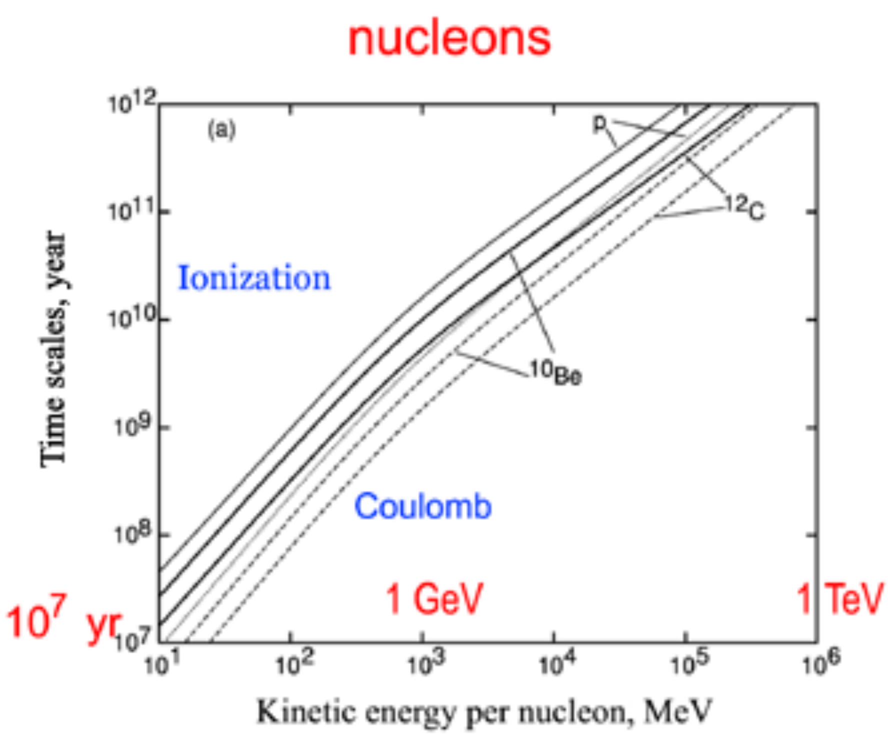
- Distr gaussiana di path length→ simula la diffusione
- Distr exp per il caso in cui non c'e' diffusione ma solo prop nel volume

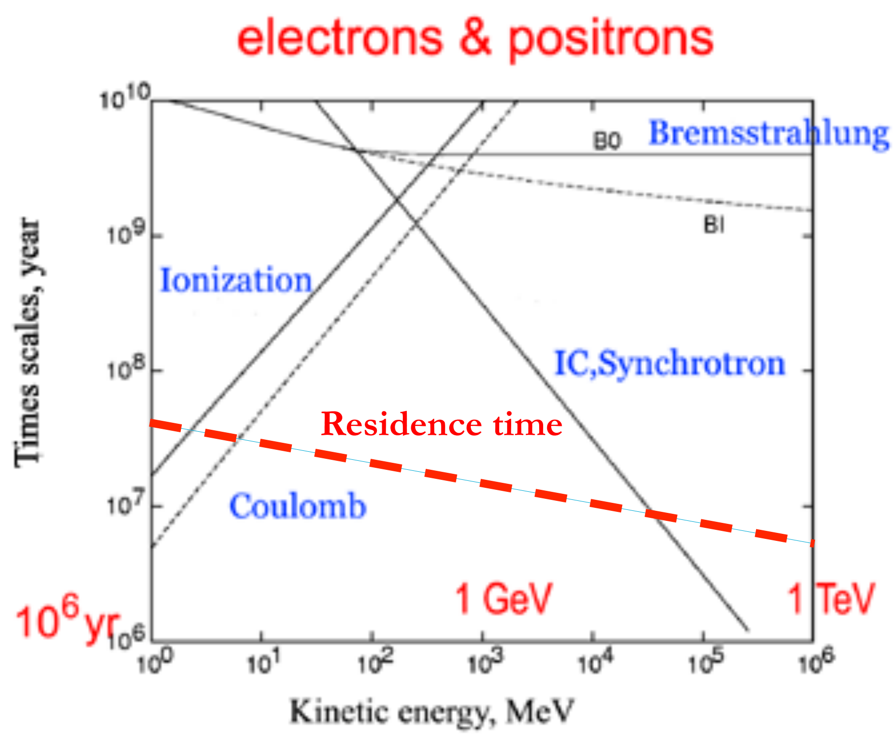


Abbiamo visto che i tempi di confinamento dei RC sono dell'ordine di T(E) ≈ 1-3 x 10^7 anni

Se gli e- hanno tempi simili, il tempo scala per le perdite di energia **τ** e' < T(E) per differenti ragioni in differenti intervalli di energia:

- Ad E< 1Gev dominano le perdite per ionizz. **τ**(E) ≈ 10^7 anni per e- ad E=300 MeV
- Ad E>10 GeV, IC e sincr dominano ed **τ**(E) **<** 10^7 anni per e-


nucleons
(a)
p
12C
10Be
Ionization
Coulomb
Time scales, year
Kinetic energy per nucleon, MeV
1 GeV
1 TeV
10^7 yr


electrons & positrons
B0
BI
Bremsstrahlung
Ionization
IC,Synchrotron
Residence time
Coulomb
Times scales, year
Kinetic energy, MeV
1 GeV
1 TeV
10^6 yr

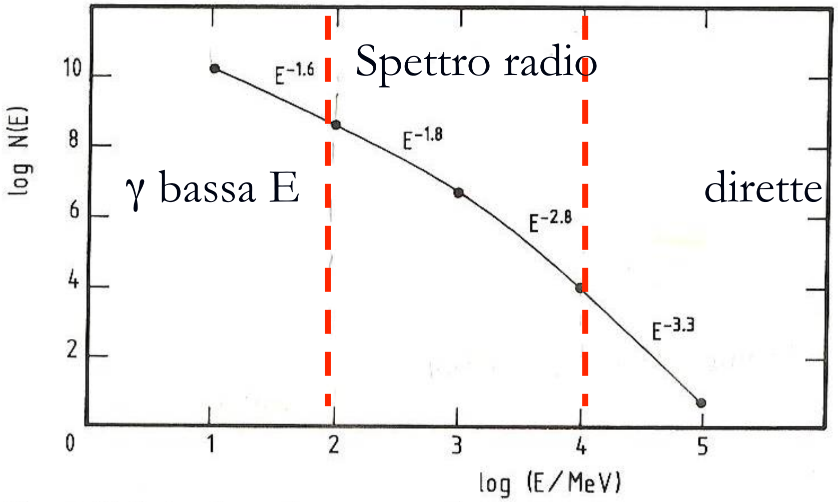
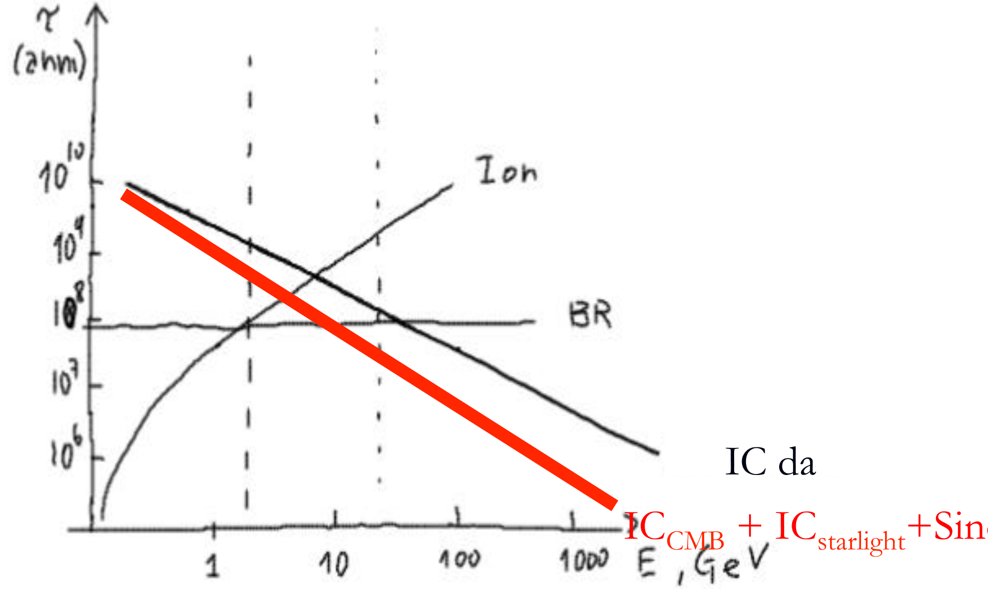


Figure 19.3. A schematic representation of the electron energy spectrum
interstellar medium from the data discussed in Sections 18.2 and 18.3.
has been subject to energy losses at high and low energies during prop



Quindi, se gli e- sono iniettati continuamente nell'ISM, dovrebbero raggiungere una situazione stazionaria di equil fra iniezione e perdite, $dN/dt = 0$

A bassa E, dove la ionizz domina quindi $N(E) \approx E^{-p-1} \approx Q(E)E^{-1} \rightarrow (p-1) = 1.6 \rightarrow Q(E) \approx E^{-2.6}$

Ad alta E, IC e Sincr dominano quindi $N(E) \approx E^{-(p+1)} \rightarrow p+1 = 3.3 \rightarrow Q(E) \approx E^{-2.3}$

I due indici spettrali sono piuttosto simili: cio' suggerisce che lo spettro alla sorgente degli e- sia una legge di potenza con indice spettrale $p \approx 2.5$ fra 100 MeV e 100 GeV

Possibili Break del flusso

- Un processo crea una distr. di e^- alla sorgente di tipo $Q(E) = kE^{-p}$ $[m^{-3}\,GeV^{-1}]$

- Gli e^- possono rimanere nella sorgente per un tempo t_0 prima di sfuggire e propagarsi fino a noi. Per esempio SNR

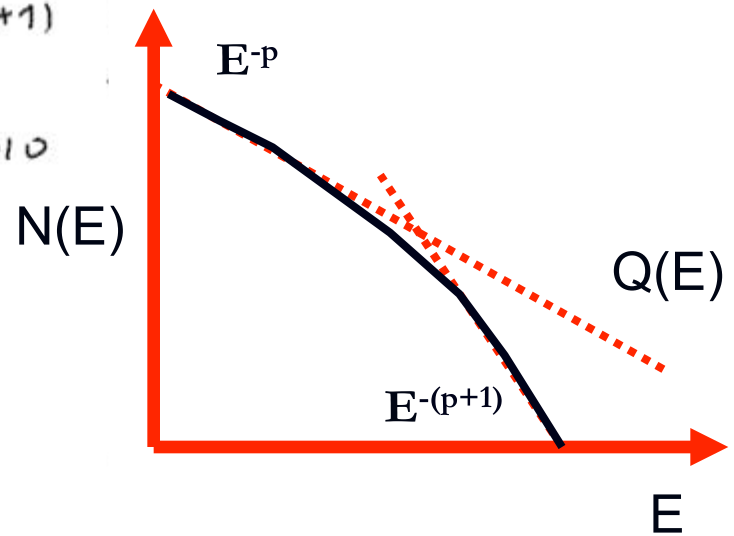
- Durante la permanenza nella sorgente possono perdere energia, per es. per sincrotrone

$\Rightarrow$ e^- con $E \geq E_0 = \frac{1}{k t_0} = \frac{0.125}{B^2 t}$ eV $[T]$ $[anni]$ hanno perso una fraz. significativa di E <u>prima</u> lasciare la sorgente

- per $E < E_0$, gli e^- non hanno perso frazioni importanti di E
$\Rightarrow$ lo spettro degli e^- che lasciano la sorgente dopo t_0 è lo stesso di $Q(E) \Rightarrow N(E) = k E^{-p}$

- per $E > E_0$, osserviamo solo gli elettroni prodotti nell'intervallo τ precedente t_0, cioè $\tau = 1/E$
$\Rightarrow N(E) = Q(E) \cdot \tau(E) \propto E^{-(p+1)}$

- All'energia E_0 c'è un cambio di pendenza nello spettro (un "break")



Electron mass small compared to protons and heavy nuclei, ➔ lose energy more rapidly

Lifetimes are short, ➔ electron sources are **Galactic**.

Observed energy density $\sim 4x10^3$ eV m^{-3} (total for cosmic rays $\sim 10^6$ eV m^{-3})

Assuming Crab pulsar-like sources…

can Galactic pulsars source CR electrons?

Need first to calculate how many electrons produced by the Crab nebula.

Observed synchrotron X-rays from SNR,

$$\nu = 4 \times 10^{36}\, E^2 B \text{ Hz}$$

@ $\nu \sim 10^{18}$ Hz (X-ray) assume $B_{SNR} = 10^{-8}$ Tesla

$$\rightarrow E_e \sim 5 \times 10^{-6} \text{ J} = \boxed{3 \times 10^{13} \text{ eV}}$$

$P = 2.4 \times 10^{12}\, E^2 B^2$ J/s

$= 2.4 \times 10^{12} \times 2.5 \times 10^{-11} \times 10^{-16}$ J/s

$= 6 \times 10^{-15}$ J/s

Observed flux $= 1.6 \times 10^{-11}$ J $m^{-2}s^{-1}$

Distance = 1kpc = 3×10^{19} m

Total luminosity, $L = 1.6 \times 10^{-11} \times 4\pi d^2$ J/s $= 1.6 \times 10^{-10} \times 10^2 \times 10^{38}$ J/s $= 1.6 \times 10^{30}$ J/s

Number of electrons = luminosity/power per e^- $= 1.6 \times 10^{30} / 6 \times 10^{-15} = 2.6 \times 10^{44}$

Synchrotron lifetime, $\tau_{sinc} = 5 \times 10^{-13} B^{-2} E^{-1}$ s $= 30$ years

Thus in 900 yrs since SN explosion, must be 30 replenishments of electrons and these must be produced by the pulsar.

Total no. electrons $= 2.6 \times 10^{44} \times 30 \sim 8 \times 10^{45}$

each with E $= 5 \times 10^{-6}$ J

Total energy is thus 4×10^{40} J

Assume 1 SN every 100 years for 10^{10} years ➔ total energy due to pulsars $4\times10^{40} \times 10^{8}$ J = 4×10^{48} J in a volume of $\sim10^{63}$ m^{-3} (ie. the Galaxy)

➔ energy density of electrons produced by pulsars :

$=4\times10^{48}/10^{63}\ Jm^{-3} = 4\times10^{-15}\ Jm^{-3}$

$= 4\times10^{-15}/1.6\times10^{-19}\ eVm^{-3} = 2.5\times10^{4}\ eVm^{-3}$

Remember that the observed electron energy density is 4×10^{3} eV/m^{3}, so electron emissions from pulsars are sufficient to produce observed levels, given the assumptions made.

SNRs vicine

- Vicino al noi ci sono numerose potenziali sorgenti di elettroni

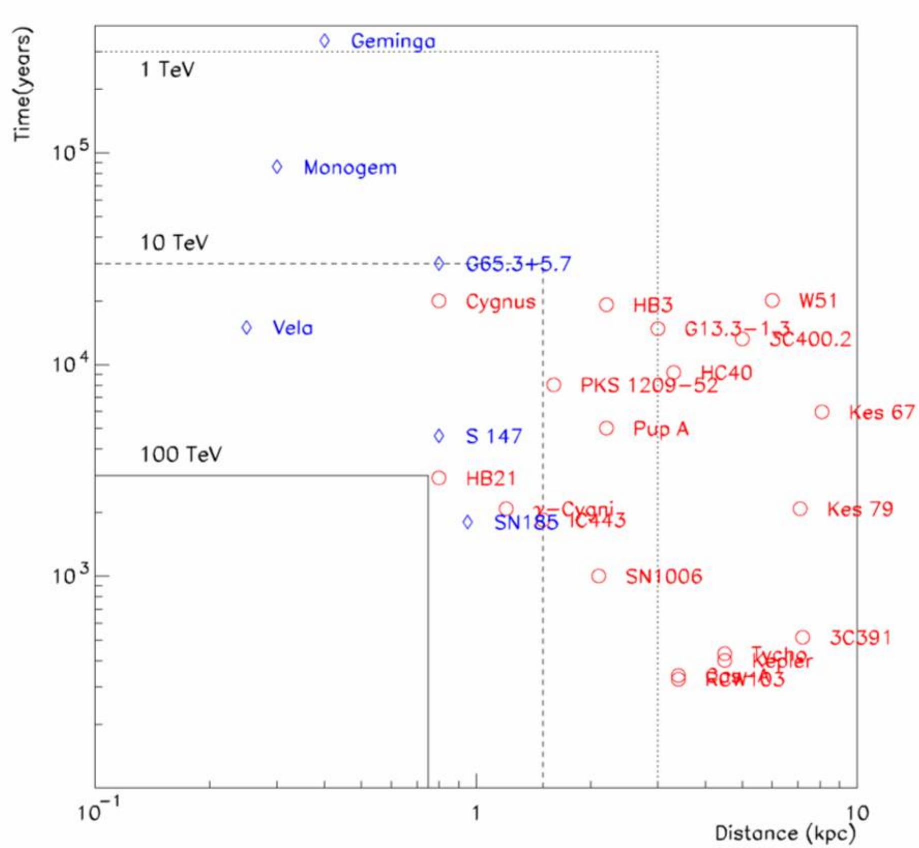


TABLE 1
List of Nearby SNRs

SNR	Distance (kpc)	Age (yr)	E_{max}[a] (TeV)	Reference
SN 185	0.95	1.8×10^3	1.7×10^2	1
S147	0.80	4.6×10^3	63	2
HB 21	0.80	1.9×10^4	14	3, 4
G65.3+5.7	0.80	2.0×10^4	13	5
Cygnus Loop.................	0.44	2.0×10^4	13	6, 7
Vela	0.30	1.1×10^4	25	8
Monogem	0.30	8.6×10^4	2.8	9
Loop1	0.17	2.0×10^5	1.2	10
Geminga	0.4	3.4×10^5	0.67	11

[a] Maximum energy limited by the propagation of electrons in the case of the prompt release after the explosion. The delay of the release time gives the larger value.

References.—(1) Strom 1994; (2) Braun et al. 1989; (3) Tatematsu et al. 1990; (4) Leahy & Aschenbach 1996; (5) Green 1988; (6) Miyata et al. 1994; (7) Blair et al. 1999; (8) Caraveo et al. 2001; (9) Plucinsky et al. 1996; (10) Egger & Aschenbach 1995; (11) Caraveo et al. 1996.

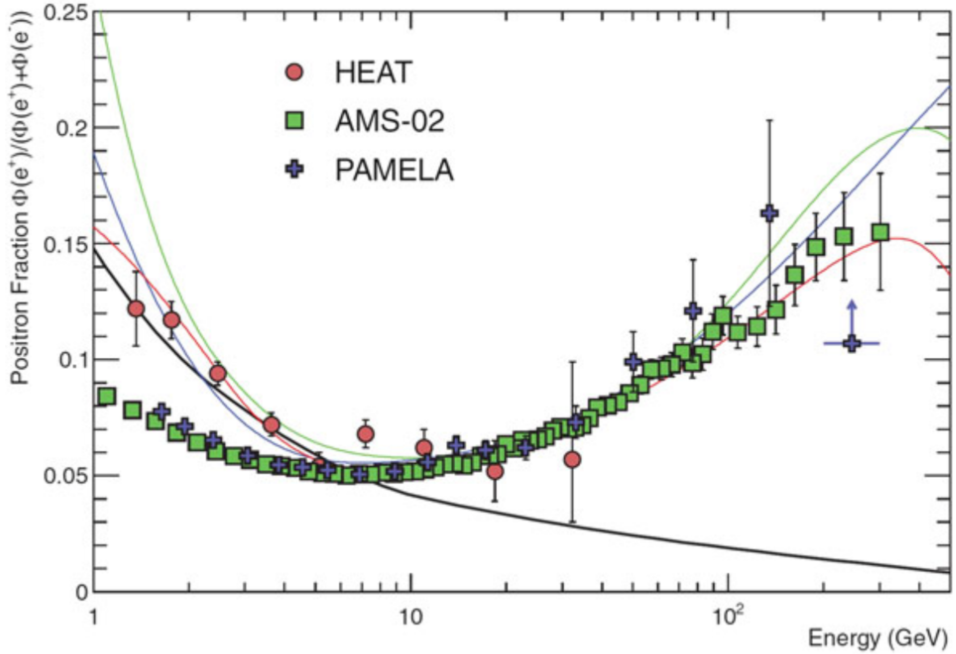


Fig. 3.14 The positron fraction (ratio of the flux of e^+ to the total flux of $e^+ + e^-$) as a function of the energy measured HEAT, PAMELA, and AMS-02. The heavy *black line* is a model of pure secondary production using a detailed propagation model of CRs (Sect. 5.4). The *three thin lines* show three representative attempts to model the positron excess with different phenomena discussed in Sect. 13.9.3: dark matter decay (*green*); propagation physics (*blue*); production in pulsars (*red*). The ratio below 10 GeV is dependent on the polarity of the solar magnetic field. Figure from the Sect. 27. Cosmic Rays of Beringer et al. (2012)

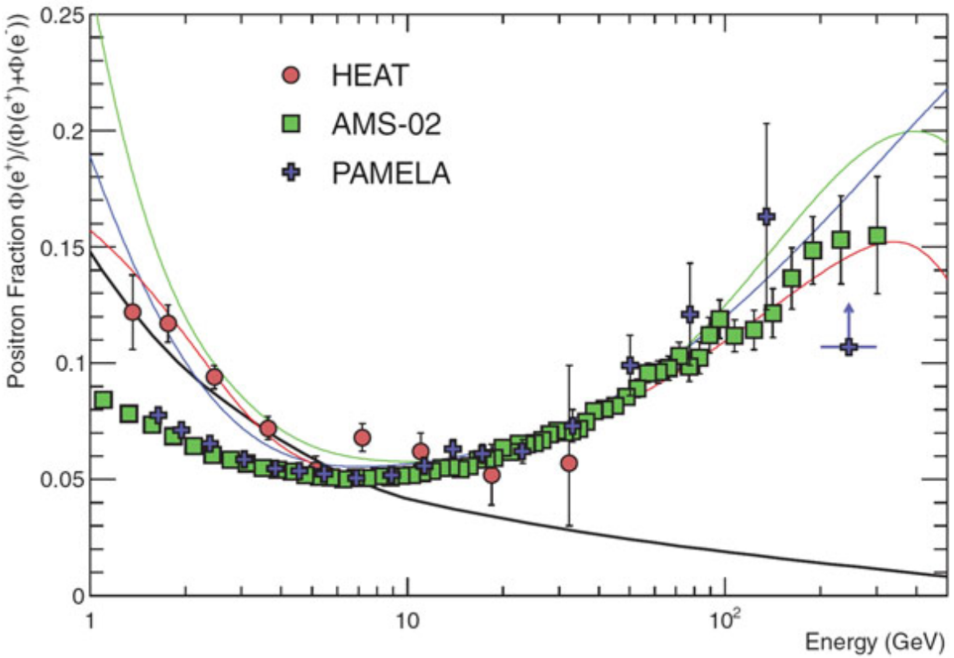
- The positron-fraction spectrum does not exhibit fine structures and steadily increases in the region between 10 and 250 GeV. In the high statistics AMS-02 sample the ratio is of the order of ~10 % above a few tens of GeV. As a consequence, since positrons are always created in pair with an electron *about 90 % of the observed electrons must be of primary origin*.

Eccesso rispetto a cosa?

Produzione secondaria nell'ISM di e+ ed e-

Sorgenti primarie di e-: $n_-(E)=K_{ep} N_{cr}(E) + \sigma_{pe-} n_{ISM} c\, n_{cr}(E)$, $K_{ep} \approx 10^{-3}$

No sorgenti primarie di e+: $n_+(E) = \sigma_{pe+} n_{ISM} c\, n_{cr}(E)$



Leaky box equilibrium density $n_{cr}(E) = N_{cr}(E)\tau_{esc}(E)$, $\tau_{esc}(E) = \tau_0 E^{-\delta}$

$$n_-(E)/n_+(E) = (K_{ep} N_{cr}(E) + \sigma_{pe-} n_{ISM} c\, N_{cr}(E)\tau_{esc}(E))/\sigma_{pe+} n_{ISM} c\, N_{cr}(E)\tau_{esc}(E) =$$

$$= \sigma_{pe-}/\sigma_{pe+} + K_{ep}/\tau_{esc}(E)$$

$$n_+(E)/(n_+(E) + n_-(E)) = 1/(1 + n_-(E)/n_+(E)) = 1/(1 + \sigma_{pe-}/\sigma_{pe+} + K_{ep}/\tau_{esc}(E))$$

$$\sigma_{pe-}/\sigma_{pe+} \approx 0.5 - 0.3 \rightarrow n_+(E)/(n_+(E) + n_-(E)) \approx 1/(1.5 + (K_{ep}/\tau_0)E^{\delta})$$

→ Senza produzione primaria di e+ alle sorgenti la frazione decresce con l'energia

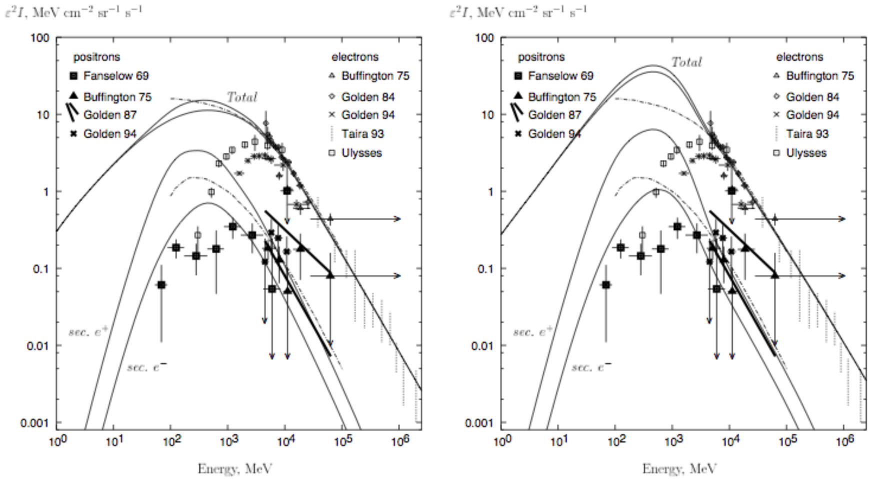


Fig. 5.— Spectra of secondary positrons and electrons, and of primary electrons. *Left panel*: model with no reacceleration (08–005). Electron injection spectrum index 2.1, 2.4 above and below 10 GeV respectively. Model: upper curves: primary electrons and primary+secondary electrons and positrons. Lower curves: secondary positrons, electrons. Lower dashed-dot line: Protheroe (1982) leaky-box prediction. Data: electrons: Buffington, Orth, & Smoot (1975), Golden et al. (1984), Golden et al. (1994), Taira et al. (1993), Ulysses (Ferrando et al. 1996), upper dashed-dot line: Protheroe (1982); positrons: Fanselow et al. (1969), Buffington, Orth, & Smoot (1975), Golden et al. (1987), Golden et al. (1994). *Right panel*: same, model with reacceleration (08–006).

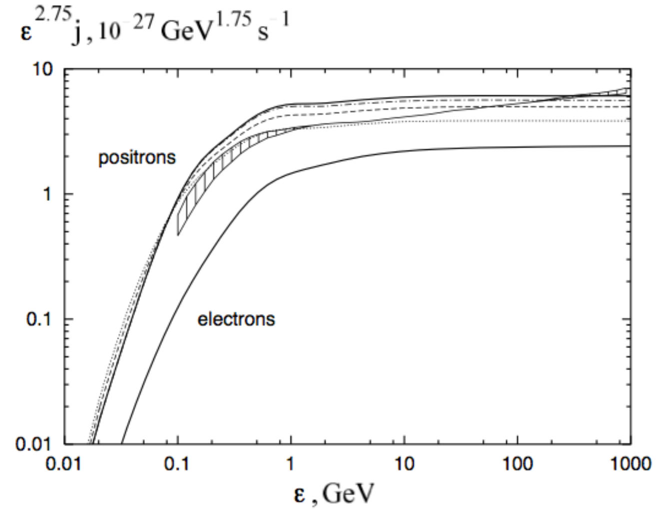


Fig. 4.— Illustration of the effect of the different positron distributions in the muon rest system. The thick solid lines show the production spectrum of secondary positrons and electrons per hydrogen atom for the cosmic-ray proton spectrum given by Mori (1997). The positron spectrum with no kaon contribution is shown by the dash-dotted line. The dashed line shows the spectrum calculated assuming an isotropic distribution of e^+ in the muon rest system ($\xi = 0$), the dotted line shows the spectrum calculated for $\xi = -1$. The thin solid line with hatched regions shows

osmic rays in interstellar medium (including contribution of He

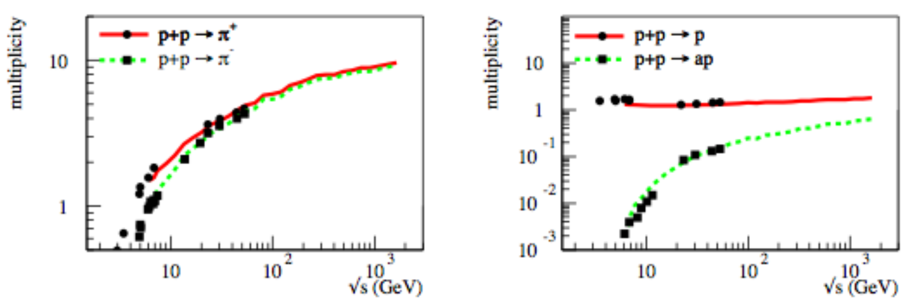


Figure 1: The 4π multiplicities of π^+, π^+, proton, antiproton produced in a pp collision as a function of $\sqrt{s}$. The full and dashed lines show the result of the χ^2 fit. The points are data from [15].

Comparing our data with a minimal model, as an example.

In this model the e^+ and e^- fluxes, Φ_{e+} and Φ_{e-}, are parametrized as the sum of individual diffuse power law spectra and the contribution of a single common source of $e^{\pm}$:

$$\Phi_{e^+} = C_{e^+} E^{-\gamma_{e^+}} + C_s E^{-\gamma_s} e^{-E/E_s} \quad \text{pulsar source} \qquad \text{Eq(1)}$$

$$\Phi_{e^-} = C_{e^-} E^{-\gamma_{e^-}} + C_s E^{-\gamma_s} e^{-E/E_s} \quad (E \text{ in GeV}) \qquad \text{Eq(2)}$$

Coefficients C_{e+} and C_{e-} correspond to relative weights of diffuse spectra for positrons and electrons.

C_s is the weight of the source spectrum.

γ_{e+}, γ_{e-} and γ_s are the corresponding spectral indexes.

E_s is a characteristic cutoff energy for the source spectrum.

With this parametrization the positron fraction depends on 5 parameters.

A fit to the data in the energy range 1 to 350 GeV yields a $\chi^2/d.f. = 28.5/57$ and:

$\gamma_{e-} - \gamma_{e+} = -0.63 \pm 0.03$, *i.e.*, the diffuse positron spectrum is less energetic than the diffuse electron spectrum;

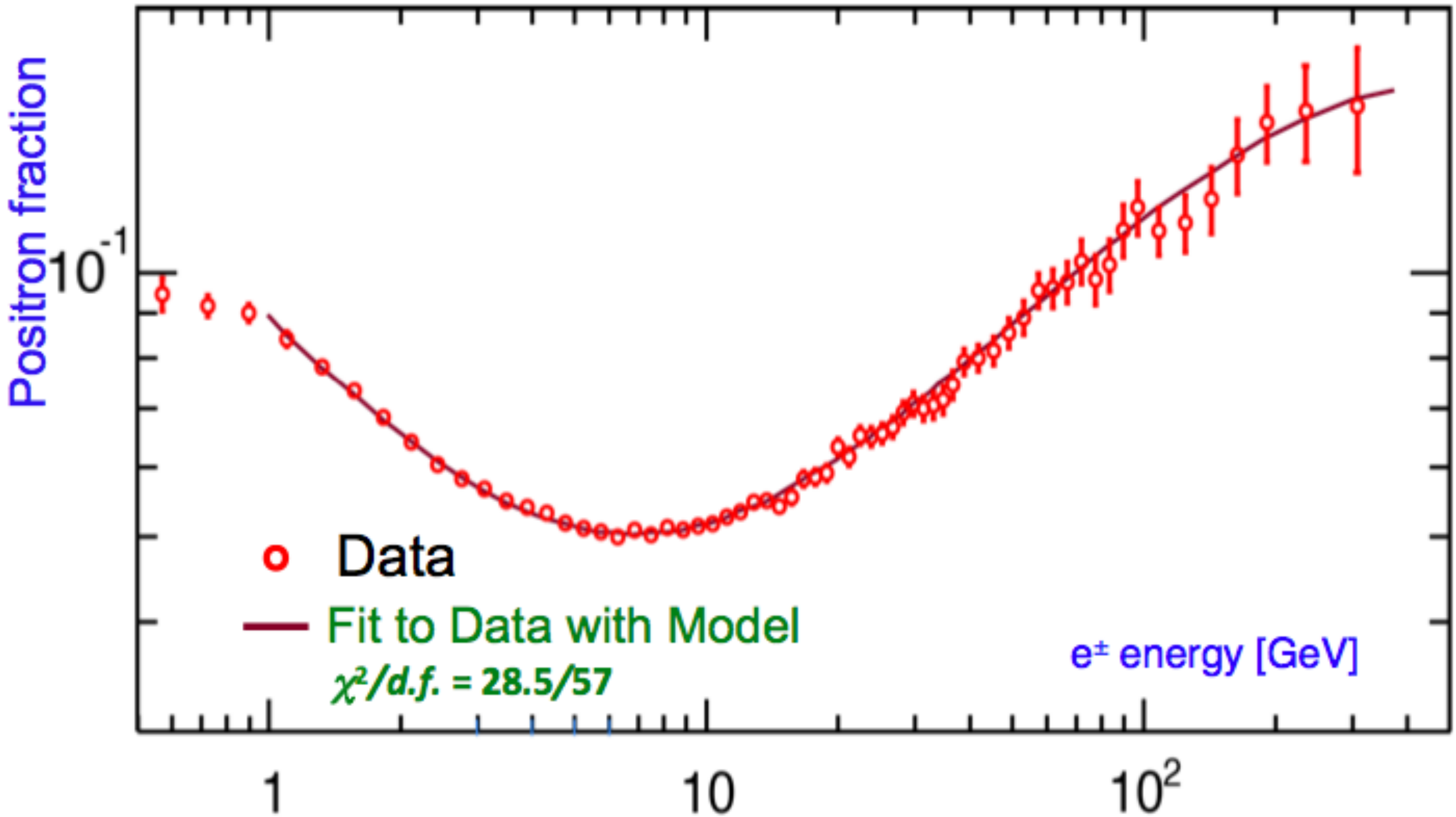
$\gamma_{e-} - \gamma_{S} = 0.66 \pm 0.05$, *i.e.*, the source spectrum is more energetic than the diffuse electron spectrum;

$C_{e+}/C_{e-} = 0.091 \pm 0.001$, *i.e.*, the weight of the diffuse positron flux amounts to ~10% of that of the diffuse electron flux;

$C_S/C_{e-} = 0.0078 \pm 0.0012$, *i.e.*, the weight of the common source constitutes only ~1% of that of the diffuse electron flux;

$1/E_s = 0.0013 \pm 0.0007 \text{ GeV}^{-1}$,

corresponding to a cutoff energy of 760^{+1000}_{-280} GeV.



The agreement between the data and the model shows that the positron fraction spectrum is consistent with e± fluxes each of which is the sum of its diffuse spectrum and a single common power law source.

The origin of the excess …

…is there any privileged arrival direction?

Analysis of possible deviation of the measured ratio as a function of the arrival direction in galactic coordinates (b,l)

$$\frac{r_e(b,l)}{\langle r_e \rangle} - 1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\pi/2 - b, l)$$

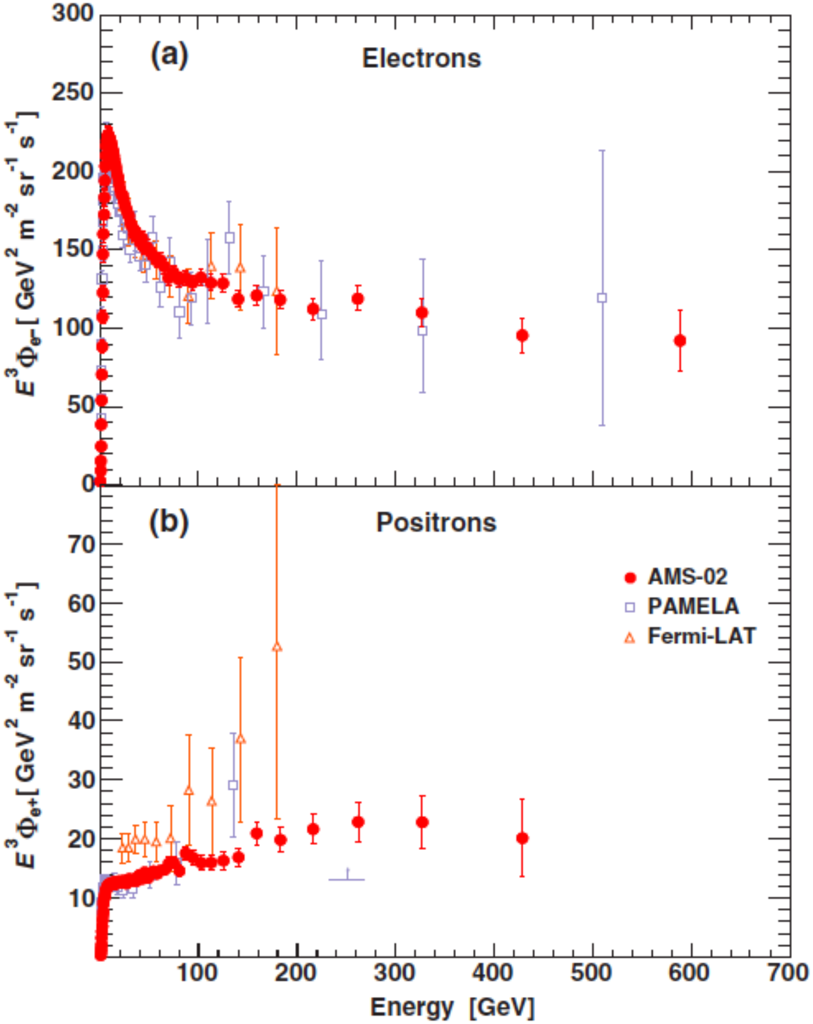
Power spectrum from the coefficient of the spherical armonics:

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2.$$

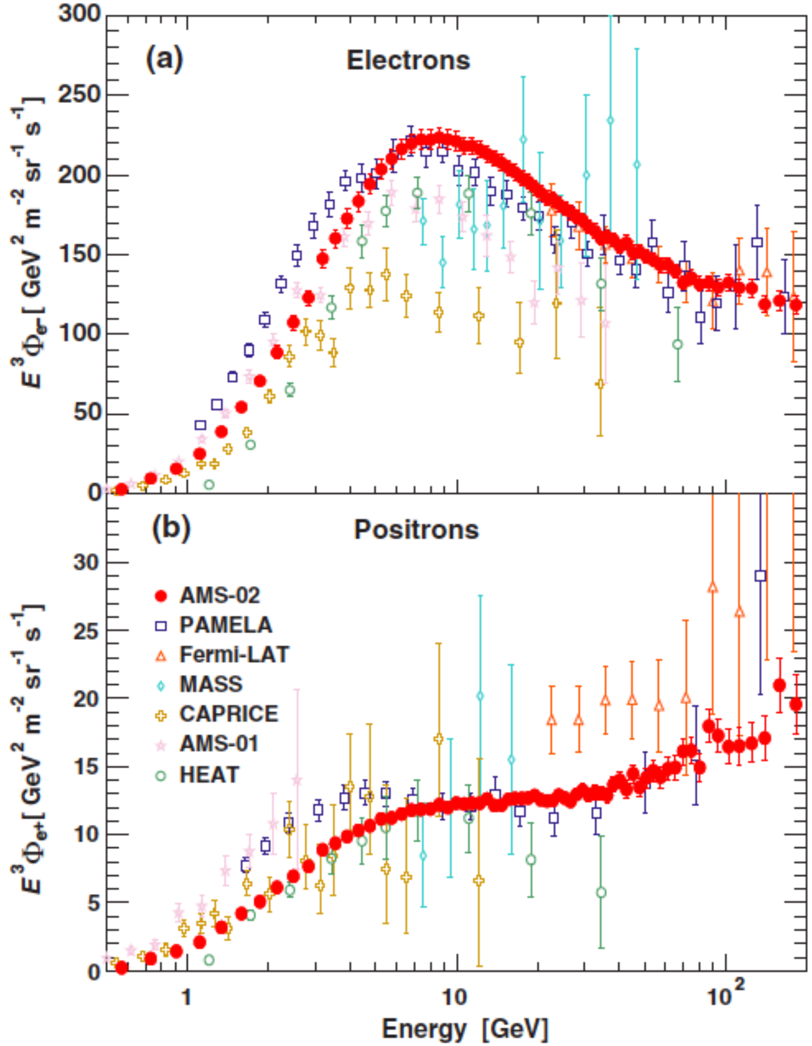
Compatible with isotropy in the dipolar mode $\delta = 3\sqrt{C_1/4\pi}.$

δ ≤ 0.036 at the 95% confidence level

Oggi:



(a)
Electrons
(b)
Positrons
AMS-02
PAMELA
Fermi-LAT
$E^3\Phi_{e-}$[GeV^2 m^{-2} sr^{-1} s^{-1}]
$E^3\Phi_{e+}$[GeV^2 m^{-2} sr^{-1} s^{-1}]
Energy [GeV]



(a)
Electrons
(b)
Positrons
AMS-02
PAMELA
Fermi-LAT
MASS
CAPRICE
AMS-01
HEAT
$E^3\Phi_{e-}$[GeV^2 m^{-2} sr^{-1} s^{-1}]
$E^3\Phi_{e+}$[GeV^2 m^{-2} sr^{-1} s^{-1}]
Energy [GeV]

Possiamo dare un quadro generale degli spettri alle sorgenti:

- Per la componente nucleare $p \approx 2$
- Per la componente e^{+-} $p \approx 2.5$

Fino ad almeno qualche centinaio di GeV

Gli spettri alla sorgente sono differenti per e- e protoni (cum grano salis)

Per entrambe le componenti c'e' evidenza sperimentale di hardening dello spettro a qualche O(100 GeV)

Per e+ ed e- l'evidenza e' netta: c'e' una sorgente che immette e+ ed e- di alta energia nell'ISM

Per i nuclei la situazione e' piu' incerta: nuova classe di sorgenti? Effetto di propagazione? Servono piu' dati.

Le pulsar accelerano e+ ed – ma non protoni (per esempio)

Occorre quindi trovare uno o piu' processi astrofisici che siano in grado di accelerare le particelle fino ad energie ultrarelativistiche con la distribuzione osservata

Idrodinamica (non relativistica)

La maggior parte dei fenomeni astrofisici concerne il rilascio di energia all'interno di stelle, o al loro esterno, nel mezzo interstellare.

In entrambi i casi, il mezzo in cui avviene il rilascio di energia comincia a muoversi, a espandersi o contrarsi, a riscaldarsi o raffreddarsi

Le proprieta' della radiazione emessa (fotoni, particelle cariche), e rivelata a Terra, dipendono in dettaglio dalle condizioni termodinamiche e di moto del fluido in questione

Ne segue che un requisito necessario per l'astrofisica delle alte energie e' lo studio dell'idrodinamica e della magnetoidrodinamica

Idrodinamica (non relativistica)

In idrodinamica il fluido e' considerato un sistema termodinamico macroscopico.
Viene idealizzato come un mezzo continuo

La descrizione del fluido in quiete richiede la conoscenza delle sue proprieta' termodinamiche locali (p, ρ, T) → occorre quindi avere un'equazione di stato che lo caratterizza

Lo stato di moto di un fluido generico, <u>non in quiete</u>, sara' descritto dalla velocita' istantanea dell'elemento di fluido $\mathbf{v}(\mathbf{x},t)$

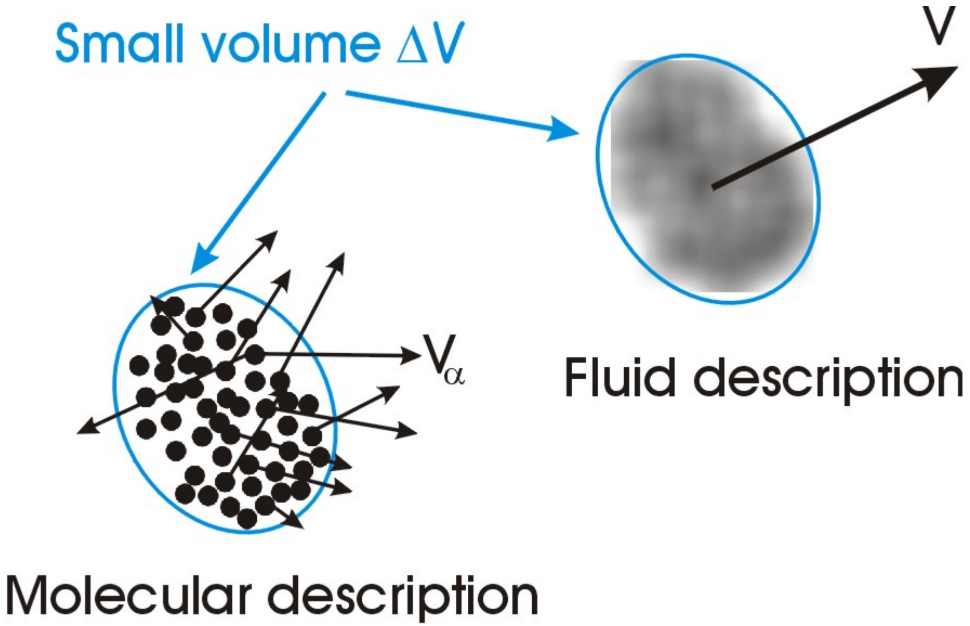
➔ lo stato del fluido e' quindi determinato dalla conoscenza di:

$p(\mathbf{x},t)$, $\rho(\mathbf{x},t)$, $T(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$

Classical Mechanics vs. Fluid Mechanics

Single-particle (classical) Mechanics	Fluid Mechanics
Deals with single particles with a fixed mass	Deals with a continuum with a variable mass-density
Calculates a single particle trajectory	Calculates a collection of flow lines (flow field) in space
Uses a position *vector* and velocity *vector*	Uses a *fields* : Mass density, velocity field..
Deals only with externally applied forces (e.g. gravity, friction etc)	Deals with internal AND external forces
Is formally linear (superposition principle for solutions)	Is intrinsically non-linear No superposition principle

Mass, mass-density and velocity



$$\rho(\boldsymbol{x}\,,\,t) = \lim_{\Delta\mathcal{V}\to 0} \frac{\Delta m}{\Delta\mathcal{V}}$$

$$\Delta m = \sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha$$

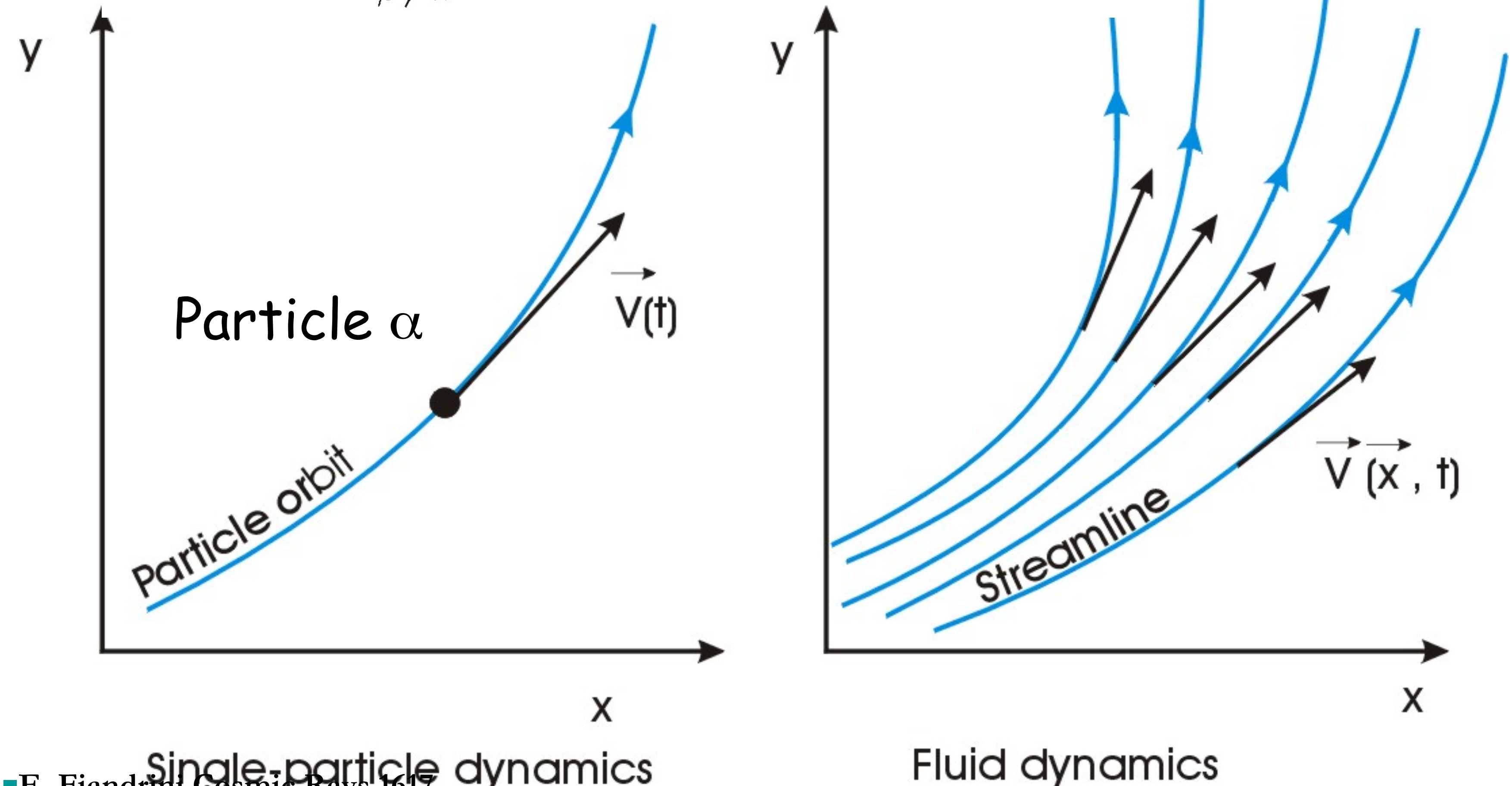
$$\boldsymbol{V} = \frac{\sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha \boldsymbol{V}_\alpha}{\Delta m}$$

Dal punto di vista dei costituenti la velocita' ***V*** di un elemento di fluido e' la velocita' media delle particelle contenute nel volume (che sono tante in modo che la media statistica abbia senso)

Equation of Motion: from Newton to Navier-Stokes/Euler

$$m_\alpha \frac{\mathrm{d}\boldsymbol{V}_\alpha}{\mathrm{d}t} = \sum_{\beta \neq \alpha} \boldsymbol{F}_{\alpha\beta}$$

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$

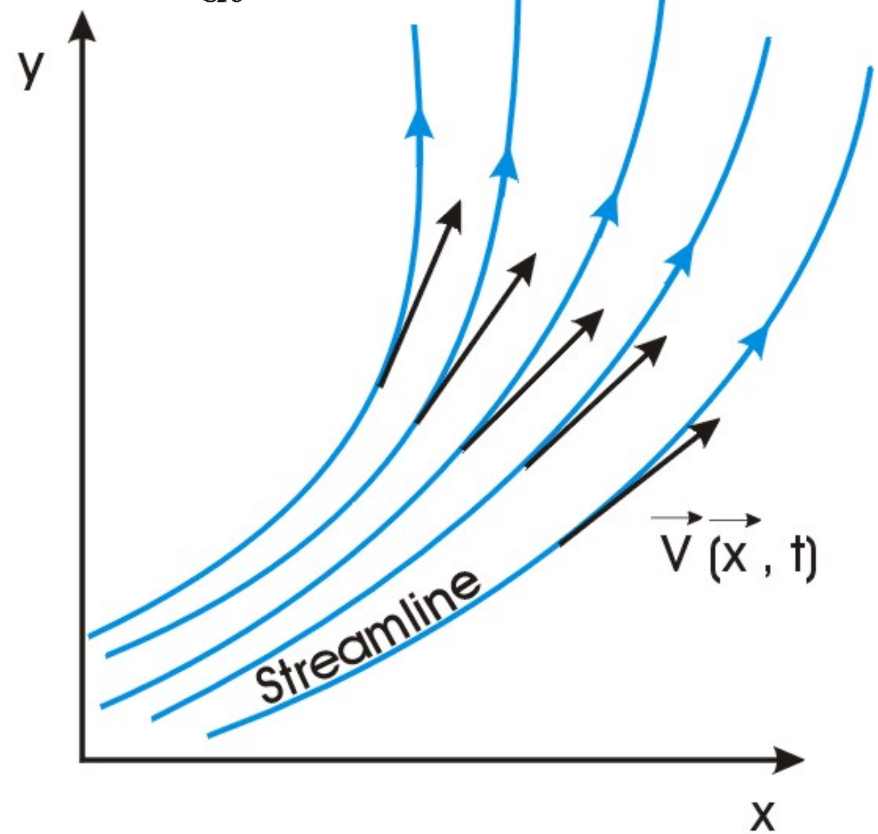


Equation of Motion: from Newton to Navier-Stokes/Euler

You have to work with a velocity field that depends on position *and* time!

$$\boldsymbol{V} = (V_x\,,\,V_y\,,\,V_z) = \boldsymbol{V}(\boldsymbol{x}\,,\,t)$$

$$\rho\,\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$



Fluid dynamics

Derivatives, derivatives…

Eulerian change:
fixed position

$$\delta Q = Q(\boldsymbol{x}\ ,\ t + \Delta t) - Q(\boldsymbol{x}\ ,\ t) \approx \frac{\partial Q}{\partial t}\,\Delta t$$

Lagrangian change:
shifting position

$$\Delta Q = Q(\boldsymbol{x} + \Delta\boldsymbol{x}\ ,\ t + \Delta t) - Q(\boldsymbol{x}\ ,\ t) \approx \frac{\mathrm{d}Q}{\mathrm{d}t}\,\Delta t$$

Shift along streamline:

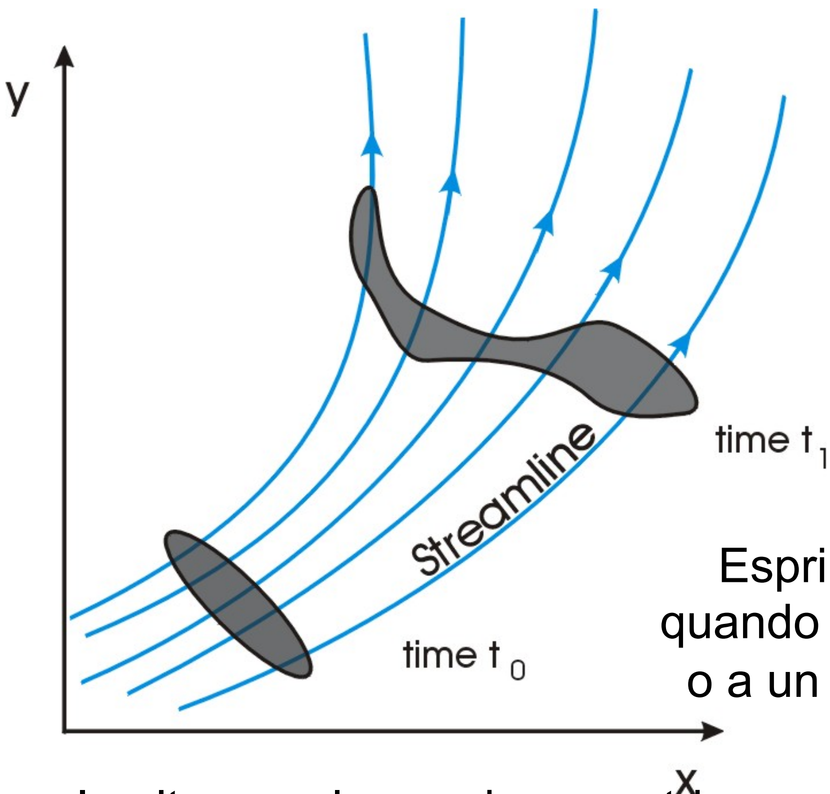
$$\Delta\boldsymbol{x} = \boldsymbol{V}\,\Delta t$$

Comoving derivative d/dt

$$\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t$$

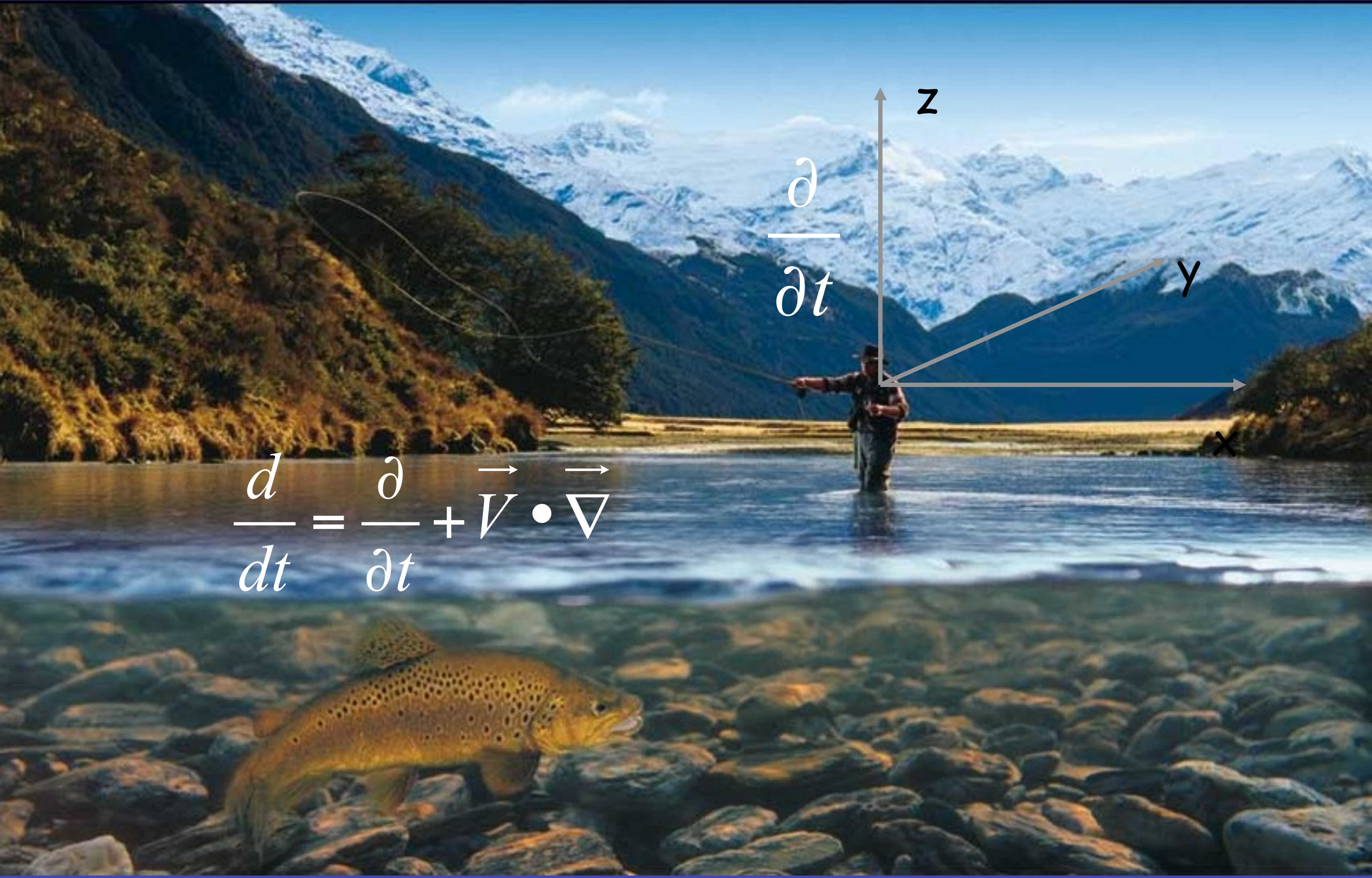
$$\begin{aligned}
\Delta Q &= Q(t+\Delta t\,,\ \boldsymbol{x}+\Delta\boldsymbol{x}) - Q(t\,,\ \boldsymbol{x}) \\
&\approx \frac{\partial Q}{\partial t}\Delta t + (\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla})Q \\
&= \left[\frac{\partial Q}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})Q\right]\Delta t \\
&\equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)\,\Delta t\ .
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})$$



Esprime come varia una quantita' fisica nel tempo quando consideriamo la sua variazione non in un posto o a un tempo fissato ma a elemento di massa fissato

In altre parole, se ci concentriamo su un certo elemento di massa e lo seguiamo nel suo moto, d/dt esprime la variazione col tempo come la vedremmo se ci trovassimo a cavalcioni dell'elemento di massa in questione


z
$\frac{\partial}{\partial t}$
y
x
$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \vec{\nabla}$

Equazione del moto per un fluido

Variazione di p di un elemento = F totale applicata sull'elemento

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}$$

<u>termine non lineare!</u>

Rende molto piu' difficile trovare soluzioni "semplici"

E' il prezzo da pagare per lavorare con un <u>campo</u> di velocita'

$$\boldsymbol{V} = (V_{\mathrm{x}}, V_{\mathrm{y}}, V_{\mathrm{z}}) = \boldsymbol{V}(\boldsymbol{x}, t)$$

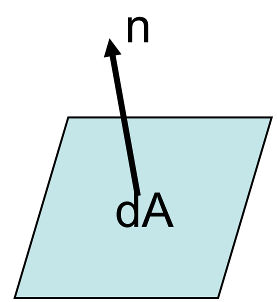
<u>Densita' di forza</u>

Puo' essere:

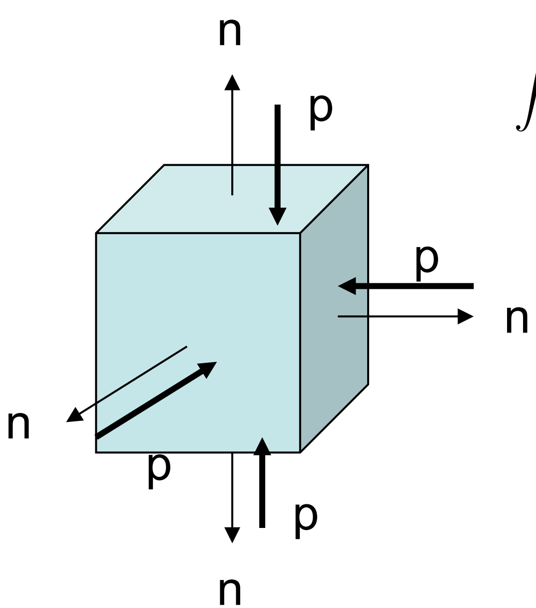
- interna:
  - pressione
  - viscosita' (frizione)
  - self-gravity
- esterna

Equazione del moto per un fluido ideale

In un fluido su un elemento di superficie arbitrario viene esercitata una forza d**F**=pd**A**=pdA**n**



→ la forza a cui e' soggetto un elemento di fluido in un volume V e' $\vec{F} = -\int_S p d\vec{A}$



$$\int_S p d\vec{A} = \int_V \nabla p dV$$

→ la densita' di forza e' $\vec{f} = \nabla p$

$$\rho \frac{d\vec{V}}{dt} = -\nabla p \qquad \rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla}\right)\vec{v} = -\nabla p$$

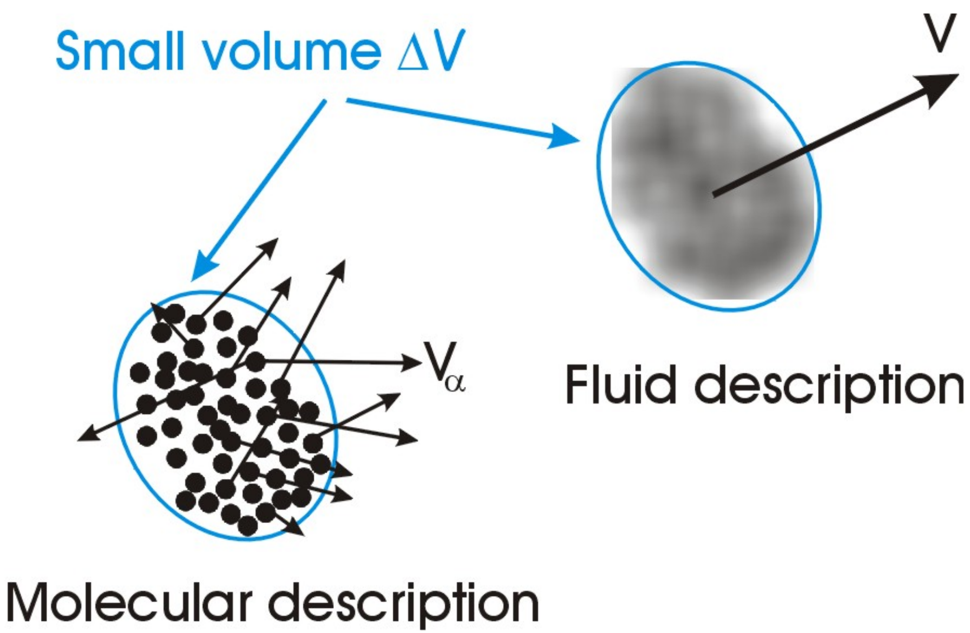
$$\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla}\right)\vec{v} = -\frac{\nabla p}{\rho} + \frac{\vec{f}_{ext}}{\rho}$$

E' l'equazione del moto del fluido ideale (detta di Eulero)

P. Es **f** puo' essere forza di gravita' $\vec{f} = -\rho \nabla \Phi$

Connessione microscopica

La pressione deriva dal fatto che i costituenti (ie particelle) sono in agitazione termica per cui le singole particelle possiedono una distribuzione di velocita' (p. Es. Maxwelliana all'equilibrio) intorno al valore medio V della velocita' dell'elemento di fluido



Questo significa che l'impulso esatto istantaneo di una particella NON coincide con quello medio del fluido

Questa differenza "genera", in ultima analisi, una forza: la pressione

Pressure force and thermal motions

Split velocity into the <u>average velocity</u>

V(**x**, t),

and an

Isotropically distributed deviation from average, the <u>random velocity</u>:

σ(**x**, t)

Individual particle:

$$\boldsymbol{v}_\alpha = \boldsymbol{V}(\boldsymbol{x}\,,\ t) + \boldsymbol{\sigma}_\alpha(\boldsymbol{x}\,,\ t)\ .$$

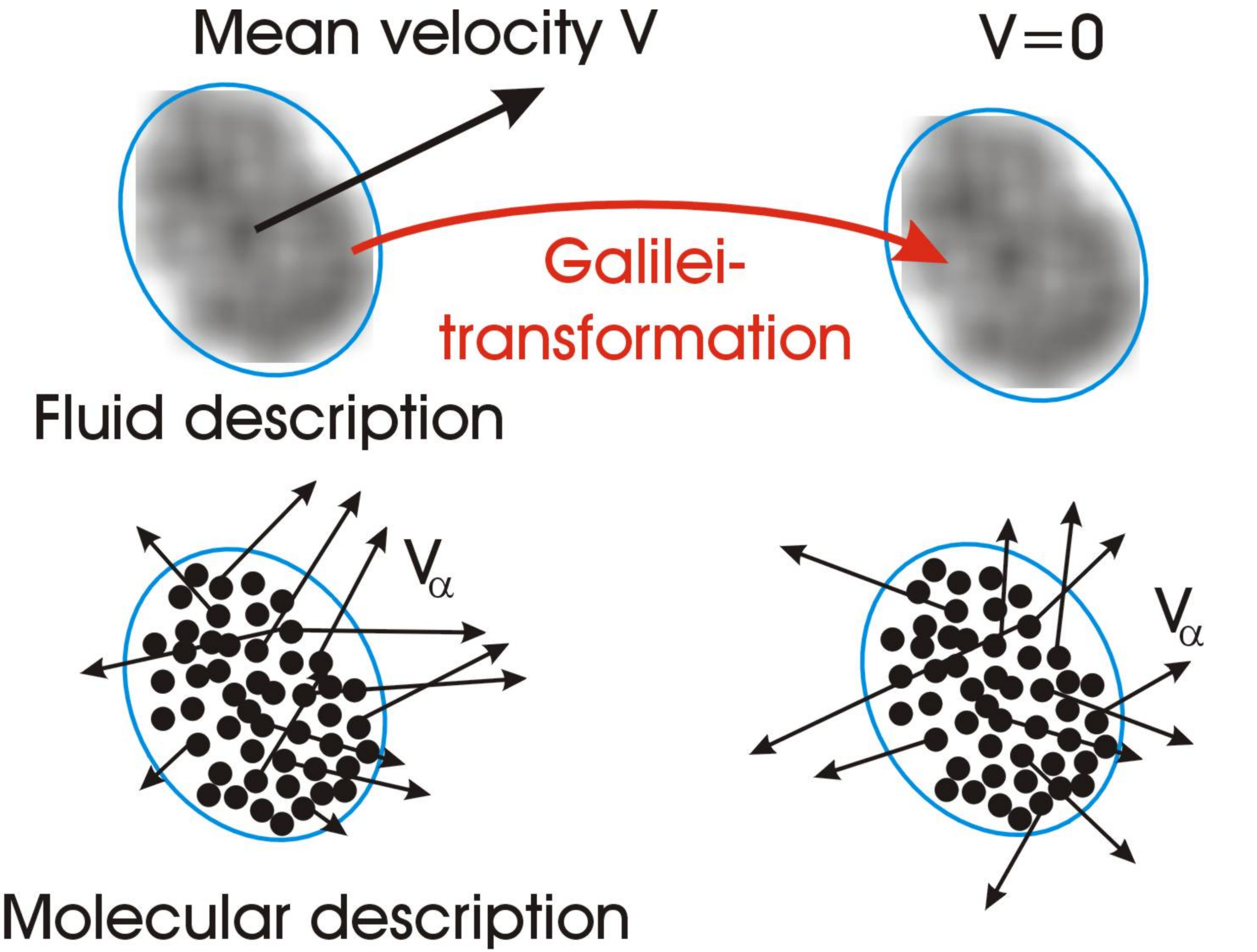
Average properties of random velocity $\boldsymbol{\sigma}$:

$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{v}} - \boldsymbol{V} = \boldsymbol{0}\ ;$$

$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = \tfrac{1}{3}\overline{\sigma^2}\ ,$$

and

$$\overline{\sigma_x\sigma_y} = \overline{\sigma_x\sigma_z} = \overline{\sigma_y\sigma_z} = \cdots = 0\ .$$



Nel sistema di quiete dell'elemento di fluido le molecole sono in moto a causa dell'agitazione termica

Acceleration of particle α

$$\frac{\mathrm{d}\boldsymbol{v}_\alpha}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}_\alpha}{\partial t} + (\boldsymbol{v}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{v}_\alpha$$

$$= \frac{\partial(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)}{\partial t} + ((\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \cdot \boldsymbol{\nabla})\,(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)$$

$$= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\frac{\partial \boldsymbol{\sigma}_\alpha}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}_\alpha}_{\text{linear in } \boldsymbol{\sigma}}} + \boxed{\underbrace{(\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}_\alpha}_{\text{quadratic in } \boldsymbol{\sigma}}}$$

Effect of average over many particles in small volume:

$$\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}}$$

$$= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\right)\overline{\boldsymbol{\sigma}}}_{\text{vanishes: } \overline{\boldsymbol{\sigma}}=0!}} + \boxed{\underbrace{\overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}}}_{\text{remains: quadratic in } \boldsymbol{\sigma}}}$$

Average equation of motion

$$\rho \, \overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\boldsymbol{f}}$$

$$\rho \left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right) = \underbrace{\overline{\boldsymbol{f}}}_{\text{mean ext. force}} - \rho \, \overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \, \boldsymbol{\sigma}}$$

For isotropic fluid: $\rho \, \overline{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \boldsymbol{\sigma})} = \boldsymbol{\nabla} \left(\frac{\rho \overline{\sigma^2}}{3} \right) \equiv \boldsymbol{\nabla} P$

Some tensor algebra:

Vector

$$\boldsymbol{A} \equiv A_i\, \boldsymbol{e}_i = A_x\, \boldsymbol{e}_1 + A_y \boldsymbol{e}_2 + A_z\, \boldsymbol{e}_3 = \begin{pmatrix} A_x \\ \\ A_y \\ \\ A_z \end{pmatrix}$$

Three notations for the same animal!

the divergence of a vector in cartesian coordinates

Scalar

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Rank 2 Tensor

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Rank 2 Tensor and Tensor Divergence

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Vector

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = \left(\frac{\partial T_{ij}}{\partial x_i}\right)\,\boldsymbol{e}_j = \begin{pmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{pmatrix}$$

Special case:
Dyadic Tensor = Direct Product of two Vectors

$$\boldsymbol{A} \otimes \boldsymbol{B} \equiv A_i B_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \, \boldsymbol{B} + (\boldsymbol{A} \cdot \boldsymbol{\nabla}) \boldsymbol{B}$$

Application: Pressure Force

Tensor divergence

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\;\boldsymbol{\sigma}$$

Isotropy of Random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

Second term = scalar x vector!

This <u>must</u> vanish upon averaging!!

Application: Pressure Force

Tensor divergence

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot\,(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\;\boldsymbol{\sigma}$$

Isotropy of Random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

$$\overline{\sigma_i\sigma_j} = \tfrac{1}{3}\overline{\sigma^2}\,\delta_{ij} = \begin{cases} \tfrac{1}{3}\overline{\sigma^2} & \text{when } i=j \\ 0 & \text{when } i\neq j \end{cases}$$

$$\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} = \rho\begin{pmatrix} \tfrac{1}{3}\overline{\sigma^2} & 0 & 0 \\ 0 & \tfrac{1}{3}\overline{\sigma^2} & 0 \\ 0 & 0 & \tfrac{1}{3}\overline{\sigma^2} \end{pmatrix} = \frac{\rho\overline{\sigma^2}}{3}\,\boldsymbol{I}$$

Diagonal Pressure Tensor

Pressure force, continued

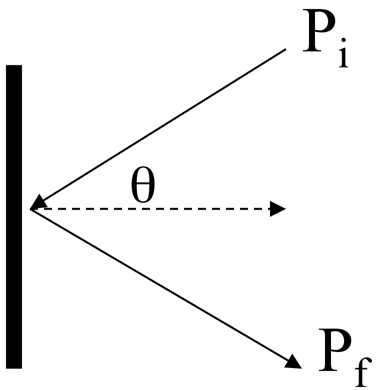
$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\left(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}\right) = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P$$

Equation of motion for frictionless ('ideal') fluid:

$$\boxed{\begin{gathered}\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = -\boldsymbol{\nabla}P + \text{other (external) forces}\\[2em] P(\boldsymbol{x}\,,\,t) \equiv \tfrac{1}{3}\rho\,\overline{\sigma^2}\end{gathered}}$$

Pressure, statistical derivation

We have to find the momentum transfer to a wall from a gas of particles with number density n in the hypothesis of elastic collisions $|p_i|=|p_f|$



A single particle with momentum p arriving from direction θ with respect to the normal n to the wall gets a $\Delta p = 2p\cos\theta$

In the time dt, the # of particles coming from θ direction impinging the wall are $dN=nv\cos\theta\, Sdt$, ie all the particles in the volume $(v\cos\theta)dtS$

Then the net momentum transfer to the wall is dP = (single part Δp)x(# of part imping. the wall) = $2p\cos\theta \times nv\cos\theta\, Sdt = 2nvp\cos^2\theta\, Sdt$

Pressure

The force is F=dP/dt =2nvpcos^{2}θ S The pressure is p=dF/dS

So p(θ)=2nvpcos^{2}θ is the pressure due to to particles arriving from direction θ

If the distribution of directions is isotropic, then the probability for a particle to arrive in a solid angle dΩ along θ is dΩ/4π, so the total pressure is

$$P_{tot} = \int p(\theta)\frac{d\Omega}{4\pi} \quad = \int_0^1 p(\theta)\frac{2\pi d cos\theta}{4\pi} \quad = nvp\int_0^1 \cos^2\theta d\cos\theta = nvp/3$$

non relativistic particle p=mv

$$P_{tot} = nmv^2/3 \quad = \frac{2}{3}u \quad u = \frac{1}{2}nmv^2$$

From equipartition theorem <E> = 3kT/2

Energy density

$$P_{tot} = nkT$$

Ideal gas law

Relativistic particle (as photons) E_{tot}' cp and v' c

$$P_{tot} = ncp/3 \quad = u/3 \qquad u = ncp$$

Energy density

Summary:

- We know how to treat the time-derivative
- We know what the equation of motion looks like
- We know where the pressure term comes from

- We <u>still</u> need:

  - A way to link the pressure to density and temperature
  - A way to calculate how the density of the fluid changes