Lecture 5 171016

- Il pdf delle lezioni puo' essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1617/

Richiamo: moto di un RC nel campo magnetico Galattico

$$mv^2/r = pv/r = ZevB/c$$

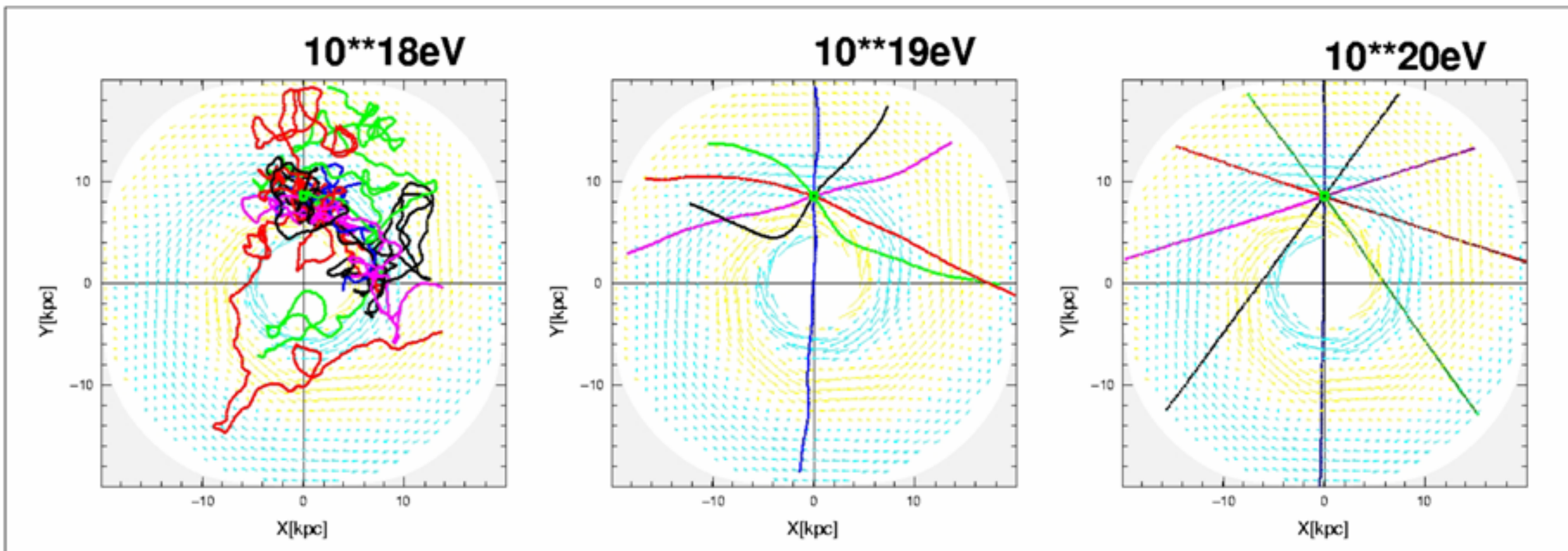
$$r = pc/ZeB$$

$$r(cm) = \frac{1}{300}\frac{E(eV)}{ZB(G)}$$

$$(10^{12}eV) = 10^{15}cm = 3\times10^{-4}\,pc$$

$$r = (10^{15}eV) = 10^{18}cm = 3\times10^{-1}\,pc$$

$$(10^{18}eV) = 10^{21}cm = 300\,pc$$



Motion in non-uniform B fields: guiding center approximation

The fundamental equation describing the motion of a charged particle in magnetic and electric fields is the Lorentz equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E}) \tag{2.1}$$

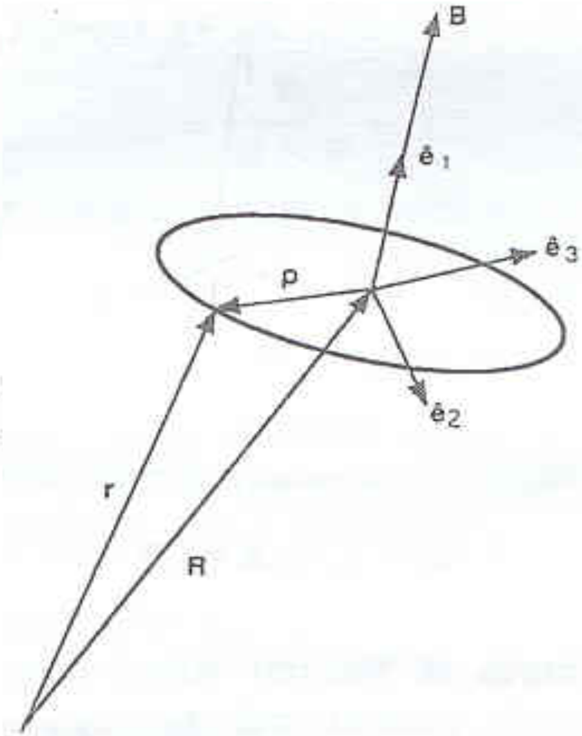
Express the position $\mathbf{r}$ of a particle in terms of its instantaneous gyroradius $\boldsymbol{\rho}$ and the center of gyration $\mathbf{R}$. Thus $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$. Expand the magnetic field in the vicinity of $\mathbf{R}$ in a Taylor series about $\mathbf{R}$

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla\mathbf{B}(\mathbf{R}) + \ldots$$

where

$$\boldsymbol{\rho} \cdot \nabla\mathbf{B} = \left(\rho_x \frac{\partial}{\partial x} + \rho_y \frac{\partial}{\partial y} + \rho_z \frac{\partial}{\partial z}\right)\mathbf{B}$$

$$m(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}) = q(\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \times [\mathbf{B}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla\mathbf{B}(\mathbf{R}) + \ldots]$$



$$\rho \left|\nabla B / B\right| << 1$$

Motion in non-uniform B fields

$$m(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}) = q(\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \times [\mathbf{B}(\mathbf{R}) + \boldsymbol{\rho}\cdot\nabla\mathbf{B}(\mathbf{R})$$

In the plane perp. to **B** we can write:

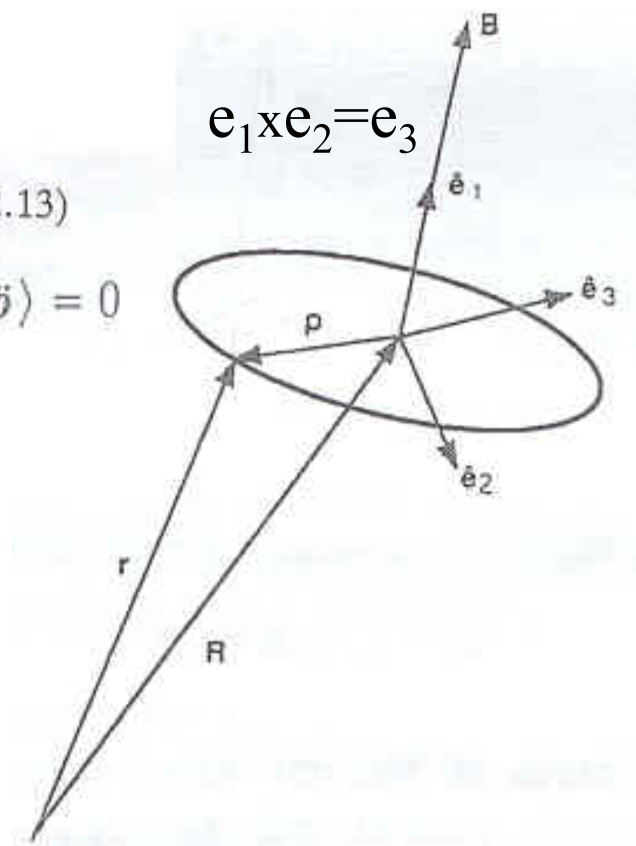
$$\boldsymbol{\rho} = \rho(\hat{\mathbf{e}}_2 \sin\Omega t + \hat{\mathbf{e}}_3 \cos\Omega t) \qquad (2.11)$$

Repeated differentiations with respect to time give

$$\dot{\boldsymbol{\rho}} = \Omega\rho(\hat{\mathbf{e}}_2 \cos\Omega t - \hat{\mathbf{e}}_3 \sin\Omega t) + \sin\Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho\hat{\mathbf{e}}_2) + \cos\Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho\hat{\mathbf{e}}_3) \qquad (2.12)$$

$$\ddot{\boldsymbol{\rho}} = \Omega^2\rho(-\hat{\mathbf{e}}_2 \sin\Omega t - \hat{\mathbf{e}}_3 \cos\Omega t) + \dot{\Omega}\rho(\hat{\mathbf{e}}_2 \cos\Omega t - \hat{\mathbf{e}}_3 \sin\Omega t)$$
$$+ 2\Omega\cos\Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho\hat{\mathbf{e}}_2) - 2\Omega\sin\Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho\hat{\mathbf{e}}_3) + \sin\Omega t \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho\hat{\mathbf{e}}_2)$$
$$+ \cos\Omega t \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho\hat{\mathbf{e}}_3) \qquad (2.13)$$

average over a Larmor period.



All the terms with only **sinωt** or **cosωt** give zero $\langle \boldsymbol{\rho} \rangle = \langle \dot{\boldsymbol{\rho}} \rangle = \langle \ddot{\boldsymbol{\rho}} \rangle = 0$

The term giving a non zero contribution (because it contains $\cos^2\omega t$ and $\sin^2\omega t$ terms) is

$$\vec{\rho}\times(\vec{\rho}\cdot\vec{\nabla})\vec{B} = \frac{1}{2}\omega\rho^2\left[\hat{e}_2\times(\hat{e}_3\cdot\vec{\nabla}) - \hat{e}_3\times(\hat{e}_2\cdot\vec{\nabla})\right]\hat{e}_1 B$$
$$= \frac{1}{2}\omega\rho^2\left[(\hat{e}_3\cdot\vec{\nabla}B)(\hat{e}_2\times\hat{e}_1) - (\hat{e}_2\cdot\vec{\nabla}B)(\hat{e}_3\times\hat{e}_1)\right]$$
$$= \frac{1}{2}\omega\rho^2\left[-(\hat{e}_3\cdot\vec{\nabla}B)\hat{e}_3 - (\hat{e}_2\cdot\vec{\nabla}B)\hat{e}_2\right] = -\frac{1}{2}\omega\rho^2\vec{\nabla}_n B$$

$$m\ddot{\mathbf{R}} = q[\dot{\mathbf{R}} \times \mathbf{B}(\mathbf{R})] - q\frac{\rho^2\Omega}{2}\nabla B + \ldots$$

Motion in non-uniform B fields

Now separate normal and parallel comp.: for the normal we take the vector product with $\mathbf{e_1}$

$$m\ddot{\mathbf{R}} \times \hat{\mathbf{e}}_1 = q(\dot{\mathbf{R}} \times \mathbf{B}) \times \hat{\mathbf{e}}_1 - \frac{q\rho^2\Omega}{2}\nabla B \times \hat{\mathbf{e}}_1$$

$$= q\{(\hat{\mathbf{e}}_1 \cdot \dot{\mathbf{R}})\hat{\mathbf{e}}_1 B - B\dot{\mathbf{R}}\} - \frac{q\rho^2\Omega}{2}\nabla B \times \hat{\mathbf{e}}_1$$

or

$$Bq\{\dot{\mathbf{R}} - (\hat{\mathbf{e}}_1 \cdot \dot{\mathbf{R}})\hat{\mathbf{e}}_1\} = Bq\dot{\mathbf{R}}_\perp = m(\hat{\mathbf{e}}_1 \times \ddot{\mathbf{R}}) + \frac{q\rho^2\Omega}{2}\hat{\mathbf{e}}_1 \times \nabla B$$

Hence

$$\dot{\mathbf{R}}_\perp = \frac{m}{Bq}(\hat{\mathbf{e}}_1 \times \ddot{\mathbf{R}}) + \frac{\rho^2\Omega}{2B}\hat{\mathbf{e}}_1 \times \nabla B \qquad (2.16)$$

To the approximation required here,

$$\ddot{\mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{R}}_\perp + \dot{\mathbf{R}}_\parallel) \approx \frac{\mathrm{d}\dot{\mathbf{R}}_\parallel}{\mathrm{d}t} = \hat{\mathbf{e}}_1\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + v_\parallel^2\frac{\partial \hat{\mathbf{e}}_1}{\partial s} \qquad (2.17)$$

Because the average normal accel. is small wrt the the parallel one (not influenced by the time average)

where s is the distance measured along the field line, which need not be straight. With this expression for $\ddot{\mathbf{R}}$ inserted into (2.16) the perpendicular velocity then becomes

$$\dot{\mathbf{R}}_\perp = \hat{\mathbf{e}}_1 \times \left(\frac{\rho^2\Omega}{2B}\nabla B + \frac{m}{Bq}v_\parallel^2\frac{\partial \hat{\mathbf{e}}_1}{\partial s}\right)$$

$$= \hat{\mathbf{e}}_1 \times \left(\frac{mv_\perp^2}{2qB^2}\nabla B + \frac{m}{Bq}v_\parallel^2\frac{\partial \hat{\mathbf{e}}_1}{\partial s}\right) \qquad (2.18)$$

For obvious reasons the first term in (2.18) is called the gradient drift and the second term the curvature drift. A more transparent interpretation of

For curl-free fields, there is a useful relation between field line curvature and the perpendicular gradient $\nabla_\perp B$, deduced in Appendix I:

$$\nabla_\perp B = -\frac{B}{R_c}\,\boldsymbol{n} = -B\,\frac{\boldsymbol{R}_c}{R_c^2} = B\,\frac{\partial \boldsymbol{e}}{\partial s}. \tag{1.25}$$

For the definition of the normal vector $\boldsymbol{n}$, and of $\boldsymbol{R}_c$, see Fig. AI.1 in Appendix I. Using this expression, we can convert (1.24) into the following:

$$\boldsymbol{V}_c = \frac{m v_\parallel^2}{q B^2}\,\boldsymbol{e} \times \nabla_\perp B. \tag{1.26}$$

The curvature drift (1.26) and the gradient drift (1.23) both point in the same direction. Since both drifts always appear together, they can be added to form the combined gradient-curvature drift $\boldsymbol{V}_{CG}$, written here in several equivalent ways:

$$\begin{aligned} \boldsymbol{V}_{CG} &= \frac{m}{2qB^2}\,(v_\perp^2 + 2v_\parallel^2)\,\boldsymbol{e} \times \nabla_\perp B \\ &= \frac{m v^2}{2qB^2}\,(1+\cos^2\alpha)\,\boldsymbol{e} \times \nabla_\perp B \\ &= -\frac{m v^2}{2qBR_c}\,(2-\sin^2\alpha)\,\boldsymbol{e} \times \boldsymbol{n}. \end{aligned} \tag{1.27}$$

In the non-relativistic case the magnitude of $\boldsymbol{V}_{CG}$ is:

$$V_{CG} = \frac{T}{qBR_c}\,(1+\cos^2\alpha).$$

Motion in non uniform fields

This is the drift velocity normal to the field and to the gradient normal to the field

Relation between Field Line Curvature and $\nabla_\perp B$

Consider a curved field line (Fig. AI.1) and a local coordinate system x, y, z centered at point P. We take the z axis parallel to e, and (y, z) lying in the field line's osculating plane through P. We shall assume that *there are no currents* in the neighborhood of P. At P we have $B_x = 0$, $B_y = 0$, $B_z = B$ and

$$\nabla \times B = 0 \qquad \frac{\partial B_x}{\partial z} = 0 .$$

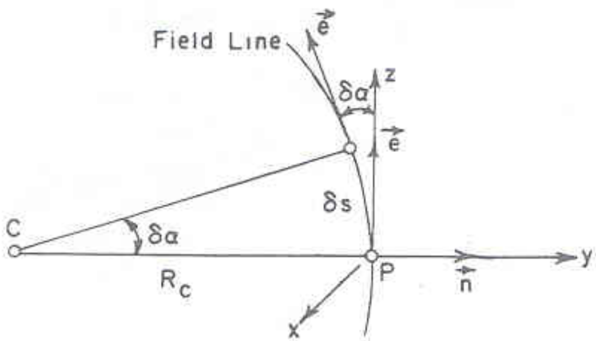


Fig. A I. 1

This latter equality is a direct consequence of (y, z) being the osculating plane. From the above relations:

$$\frac{\partial B_z}{\partial x} = 0$$

For the components of ∇B we thus obtain

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}\left[(B_x^2 + B_y^2 + B_z^2)^{\frac{1}{2}}\right] = \frac{\partial B_z}{\partial x} = 0$$

$$\frac{\partial B}{\partial y} = \frac{\partial B_z}{\partial y}$$

$$\frac{\partial B}{\partial z} = \frac{\partial B}{\partial s} = \nabla_{\|} B .$$

These relations with (1.20) show that the transverse gradient is directed along the y axis and that

$$\frac{\partial B}{\partial y} = \nabla_\perp B$$

We also have

$$\frac{\partial B_y}{\partial z} \cong B \frac{\delta \alpha}{\delta s} \cong \frac{B}{R_c} ,$$

where $R_c = \delta s / \delta \alpha$ is the field line's radius of curvature. Thus

$$\nabla_\perp B = \frac{\partial B}{\partial y} \cong \frac{\partial B_z}{\partial y} = \frac{\partial B_y}{\partial z} \cong \frac{B}{R_c} .$$

This can be written in vector form (Fig. A I. 1)

$$\nabla_\perp B = -\frac{B}{R_c} \boldsymbol{n} = -B \frac{\boldsymbol{R}_c}{R_c^2} = B \frac{\partial \boldsymbol{e}}{\partial s} \qquad \text{q.e.d.}$$

Motion in non-uniform B fields

The parallel component of equation (2.15) is extracted by forming the scalar product with $\hat{\mathbf{e}}_1$.

$$m\ddot{\mathbf{R}} \cdot \hat{\mathbf{e}}_1 = -\frac{q\rho^2\Omega}{2}(\nabla B) \cdot \hat{\mathbf{e}}_1$$

or

$$\boxed{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\frac{1}{2}\frac{v_\perp^2}{B}(\nabla B)_\parallel} \tag{2.19}$$

Equation (2.19) shows that for motion parallel to **B** the guiding center of a particle is accelerated in a direction opposite to the gradient of the magnetic field. If the particle is moving into a stronger field, it will be repelled, regardless of the sign of the particle's charge or the direction of the magnetic field.

Equations (2.8), (2.18) and (2.19) give the guiding center drifts of primary interest to radiation belt physics. As mentioned before, they contain approximations which may become important as the particle energy and gyration radius increases. In particular, the equation for parallel motion (2.19) is less exact than the equation for guiding center motion perpendicular to the magnetic field (2.18). Whenever these equations are used together when numerically tracking a particle trajectory, it is necessary to use a more accurate version of (2.19).

Motion in B fields: classical approach

Guiding center decomposition:

Parallel and normal components to the field line: $\mathbf{V=V_p + V_n}$ and

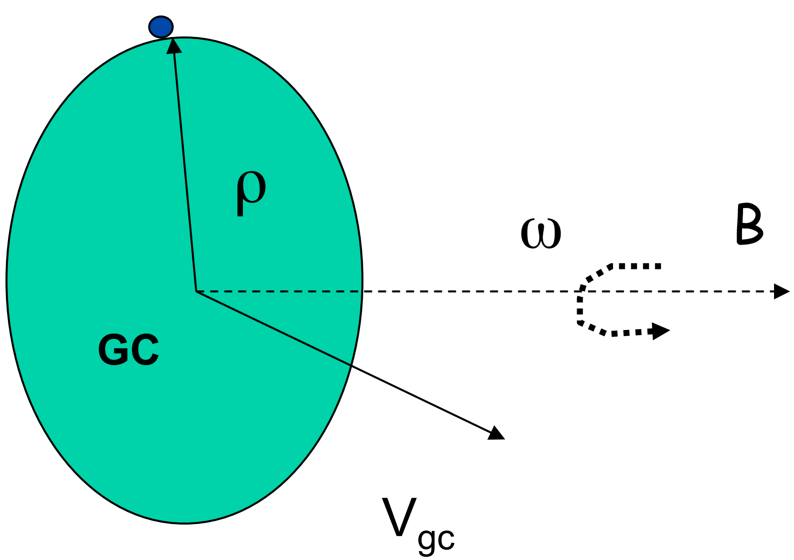
$\mathbf{V_n}$ is decomposed in a drift and a gyration with Larmor radius $\boldsymbol{\rho=P_n/Bq}$ and frequency

$$\boldsymbol{\omega = qB/m \rightarrow V = V_p + V_D + \omega x \rho = V_{gc} + \omega x \rho}$$

The motion is then described by a traslation of a point, the Guiding Center, plus a gyration around GC normal to B

<u>Parallel and normal components are decoupled</u>

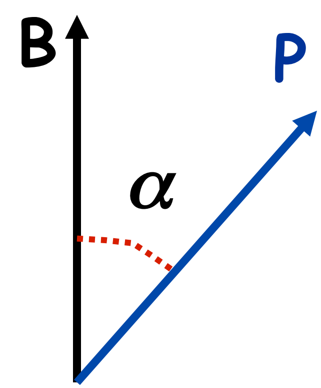
If $dB/Bdt \ll \omega/2\pi$



$\mathbf{V_n = V\sin\alpha}$

$\mathbf{V_p = V\cos\alpha}$

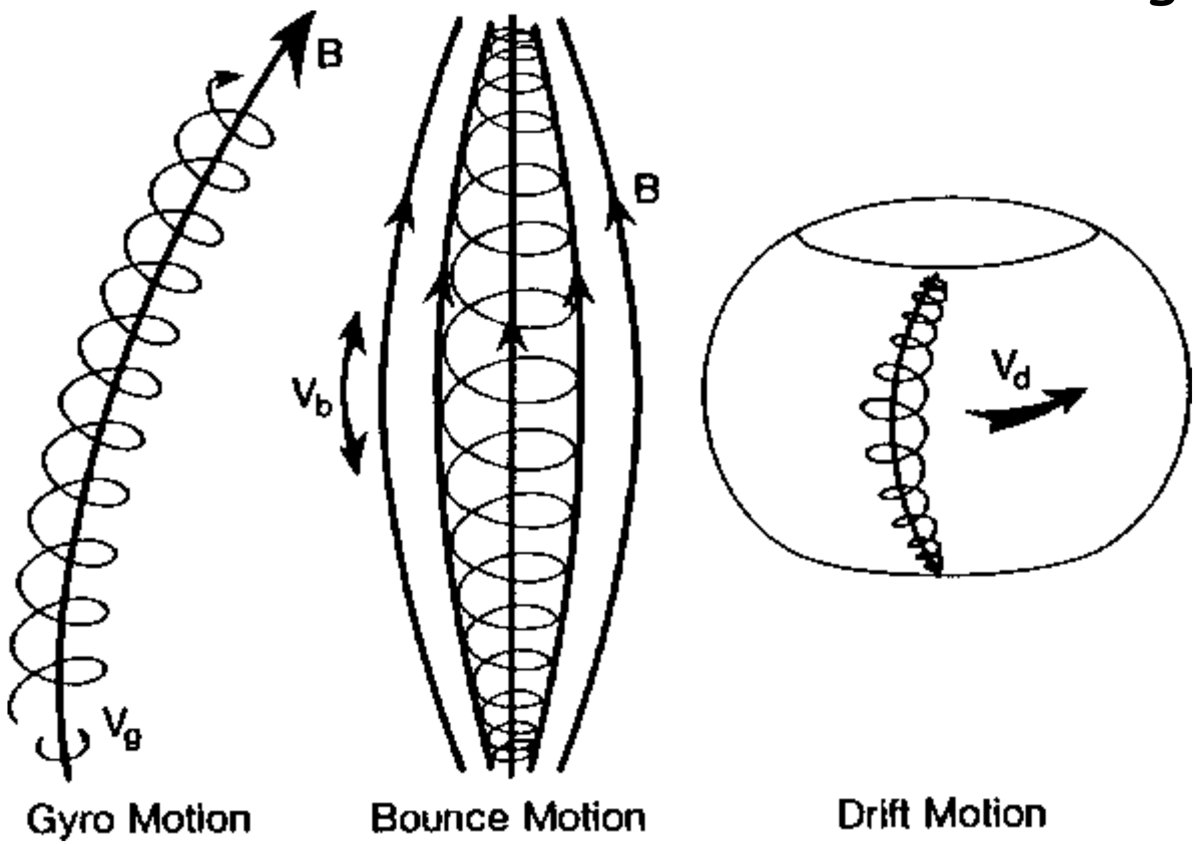
The "Pitch" Angle



Motion in B fields: classical approach

As a consequence of the decoupling, the motion can be decomposed in 3 quasi-periodic components:

- gyration around the field line
- bouncing between the mirror points along the field line
- drifting normal to the field line and to the field gradient



- Lavagna