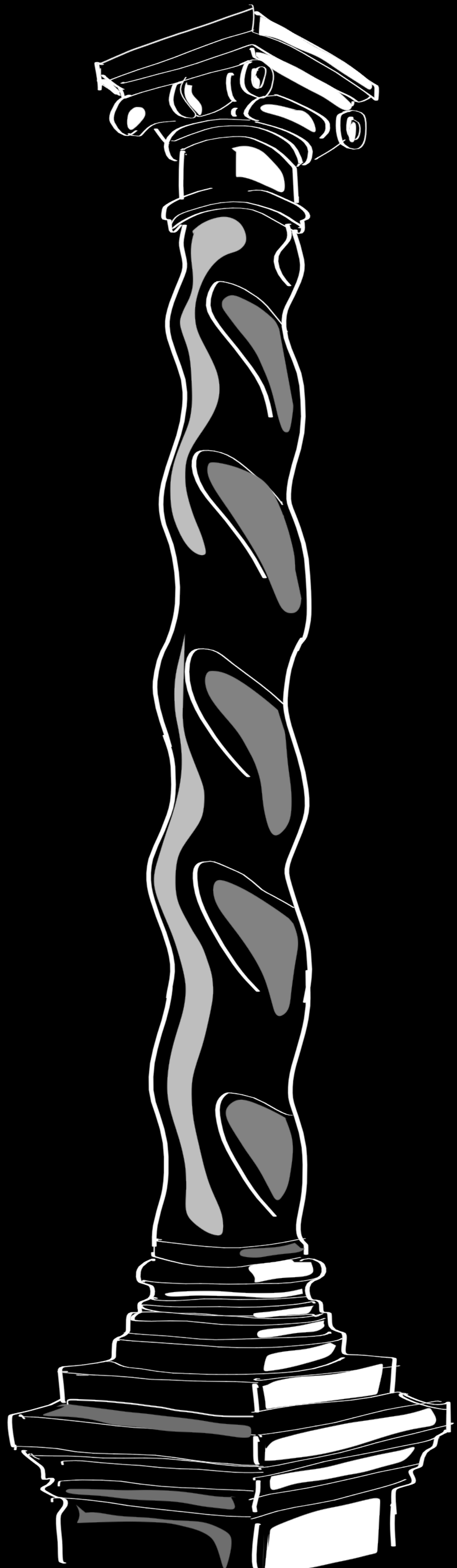

# The first exploration of the physical Riemann surfaces of the ratio between the electric and the magnetic Λ form factors

NRM2023


**Simone Pacetti**
Egle Tomasi
Alessio Mangoni


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## The 16th Varenna Conference on Nuclear Reaction Mechanisms

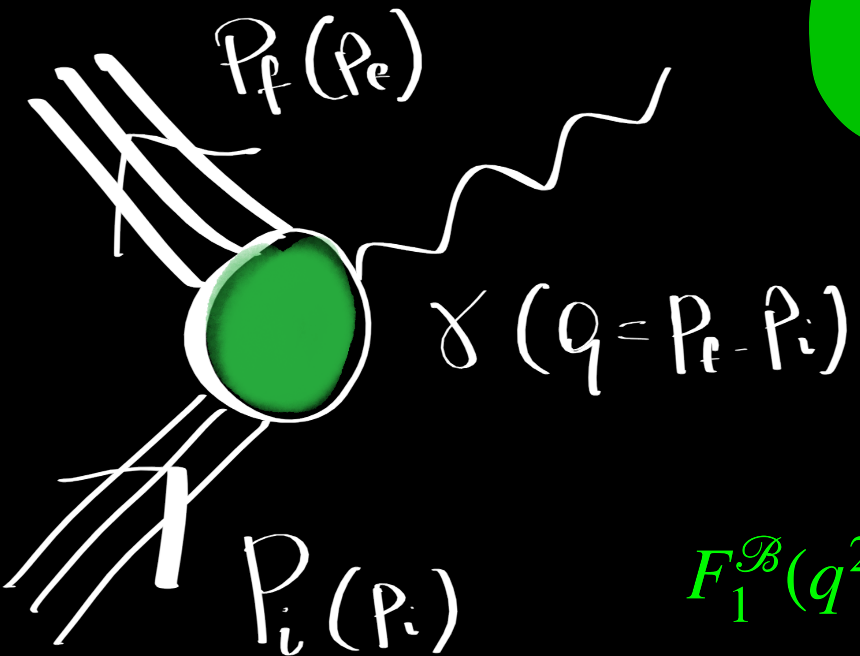
**Villa Monastero, Varenna**
**June 11 - 16, 2023**

# Agenda

- The special case of Λ form factors
- The meaning of the phase determination
- The ratio $G_E^\Lambda/G_M^\Lambda$
- Data and parametrization
- Results and discussion


simone.pacetti@unipg.it

# Baryon-photon vertex




The electromagnetic four-current of the baryon $\mathscr{B}$

$$\left\langle P_f \left| J^\mu_{\mathrm{EM}}(0) \right| P_i \right\rangle = e\bar{u}(p_f)\left[\gamma^\mu F_1^{\mathscr{B}}(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M_{\mathscr{B}}} F_2^{\mathscr{B}}(q^2)\right] u(p_i)$$

$F_1^{\mathscr{B}}(q^2)$ and $F_2^{\mathscr{B}}(q^2)$ are the Dirac and Pauli form factors

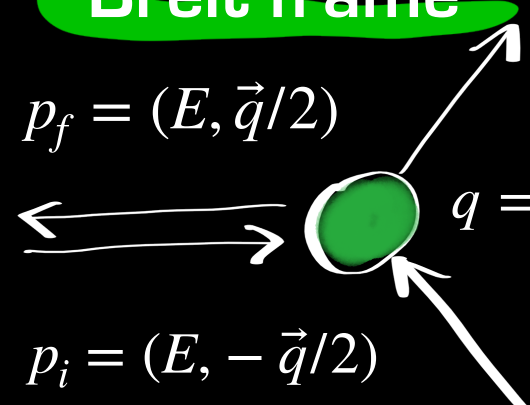
$$F_1^{\mathscr{B}}(0) = Q_{\mathscr{B}}$$

$Q_{\mathscr{B}}$ is the electric charge

$$F_2^{\mathscr{B}}(0) = \kappa_{\mathscr{B}}$$

$\kappa_{\mathscr{B}}$ is the anomalous magnetic moment

## Breit frame



$p_f = (E, \vec{q}/2)$

$q = (0, \vec{q})$

$p_i = (E, -\vec{q}/2)$

$$\left\langle P_f \left| J^\mu_{\mathrm{EM}}(0) \right| P_i \right\rangle = J^\mu_{\mathrm{EM}} = \left(J^0_{\mathrm{EM}}, \vec{J}_{\mathrm{EM}}\right)$$

$$J^0_{\mathrm{EM}} = e\left(F_1^{\mathscr{B}}(q^2) + \frac{q^2}{4M^2_{\mathscr{B}}} F_2^{\mathscr{B}}(q^2)\right)$$

$$\vec{J}_{\mathrm{EM}} = e\,\bar{u}(p_f)\vec{\gamma}u(p_i)\left(F_1^{\mathscr{B}}(q^2) + F_2^{\mathscr{B}}(q^2)\right)$$

## Sachs form factors

$$G_E^{\mathscr{B}}(q^2) = F_1^{\mathscr{B}}(q^2) + \frac{q^2}{4M^2_{\mathscr{B}}} F_2^{\mathscr{B}}(q^2)$$

$$G_M^{\mathscr{B}}(q^2) = F_1^{\mathscr{B}}(q^2) + F_2^{\mathscr{B}}(q^2)$$

$$G_E^{\mathscr{B}}(0) = Q_{\mathscr{B}} \qquad G_M^{\mathscr{B}}(0) = Q_{\mathscr{B}} + \kappa_{\mathscr{B}} = \mu_{\mathscr{B}}$$

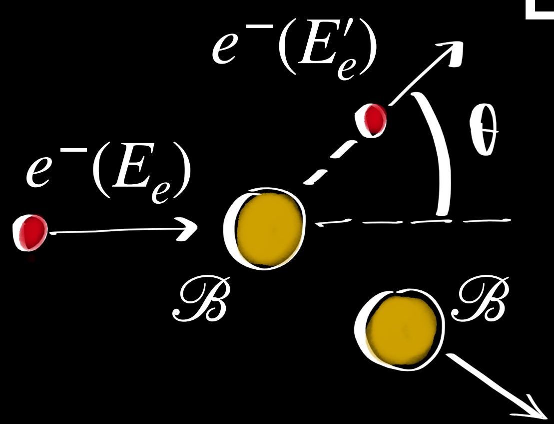
$\mu_{\mathscr{B}}$ is the total magnetic moment



simone.pacetti@unipg.it

# Cross section and Coulomb correction

## Elastic scattering cross section (Rosenbluth)



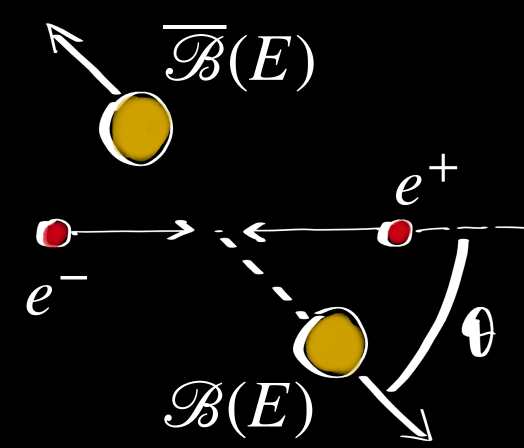

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E'_e \cos^2(\theta/2)}{4E_e^3 \sin^4(\theta/2)}\left[\left(G_E^{\mathscr{B}}\right)^2 - \tau\left(1+2(1-\tau)\tan^2(\theta/2)\right)\left(G_M^{\mathscr{B}}\right)^2\right]\frac{1}{1-\tau}$$

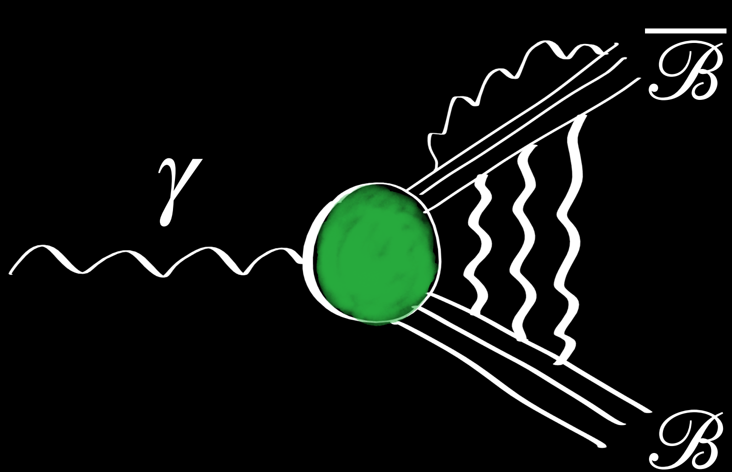
$$\tau = \frac{E^2}{4M_{\mathscr{B}}^2}$$

$$\beta = \sqrt{1-\frac{1}{\tau}}$$

## Annihilation cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\beta\mathscr{C}}{16E^2}\left[\left(1+\cos^2(\theta)\right)\left|G_M^{\mathscr{B}}\right|^2 + \frac{1}{\tau}\sin^2(\theta)\left|G_E^{\mathscr{B}}\right|^2\right]$$




## Coulomb correction

$$\mathscr{C} = \frac{\pi\alpha/\beta}{1-e^{-\pi\alpha/\beta}}$$

Only $S$-wave $\mathscr{B}\overline{\mathscr{B}}$ Coulomb final state interaction



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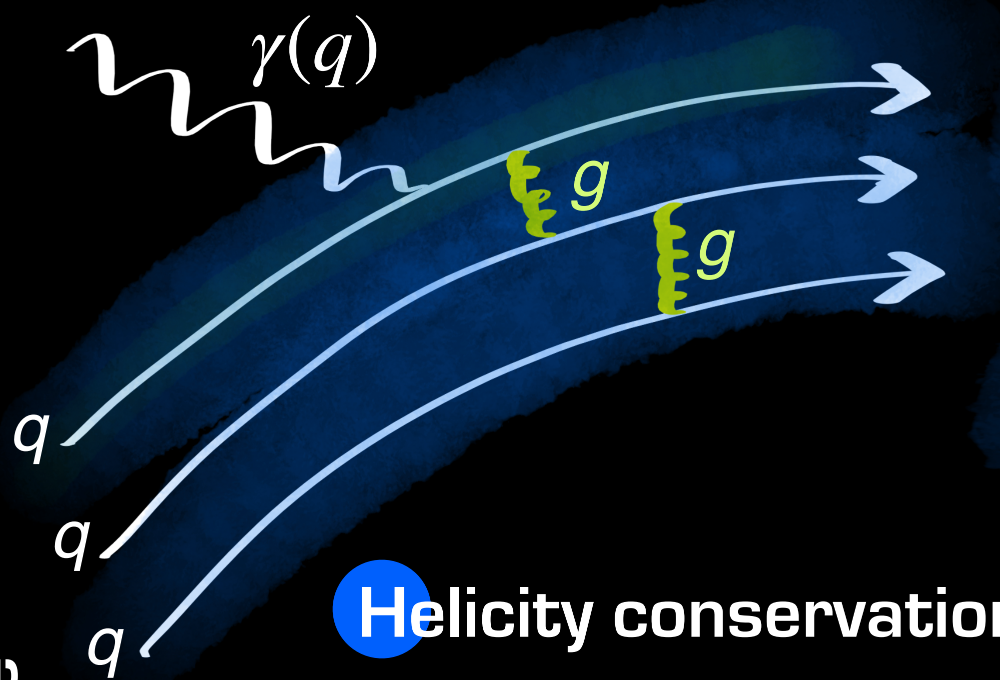
# Asimptotic behavior in pQCD




In pQCD Dirac, Pauli and Sachs form factors as $q^2 \to -\infty$ follow power laws driven by counting rules.

Valence quarks exchange gluons to distribute the photon momentum transfer $q$.

## Helicity conservation

- The current: $J^{\lambda,\lambda}(q^2) \propto G_M^{\mathcal{B}}(q^2)$
- 2 gluon propagators
- $G_M^{\mathcal{B}}(q^2) \sim \left(q^2\right)^{-2}$

## Helicity flip

- The current: $J^{\lambda,-\lambda}(q^2) \propto G_E^{\mathcal{B}}(q^2)/\sqrt{-q^2}$
- [2 gluon propagators]/$\sqrt{-q^2}$
- $G_E^{\mathcal{B}}(q^2) \sim \left(q^2\right)^{-2}$

## Dirac and Pauli form factors

$$F_1^{\mathcal{B}}(q^2) \underset{q^2\to-\infty}{\sim} \left(q^2\right)^{-2}$$

$$F_2^{\mathcal{B}}(q^2) \underset{q^2\to-\infty}{\sim} \left(q^2\right)^{-3}$$

## Ratio of Sachs form factors

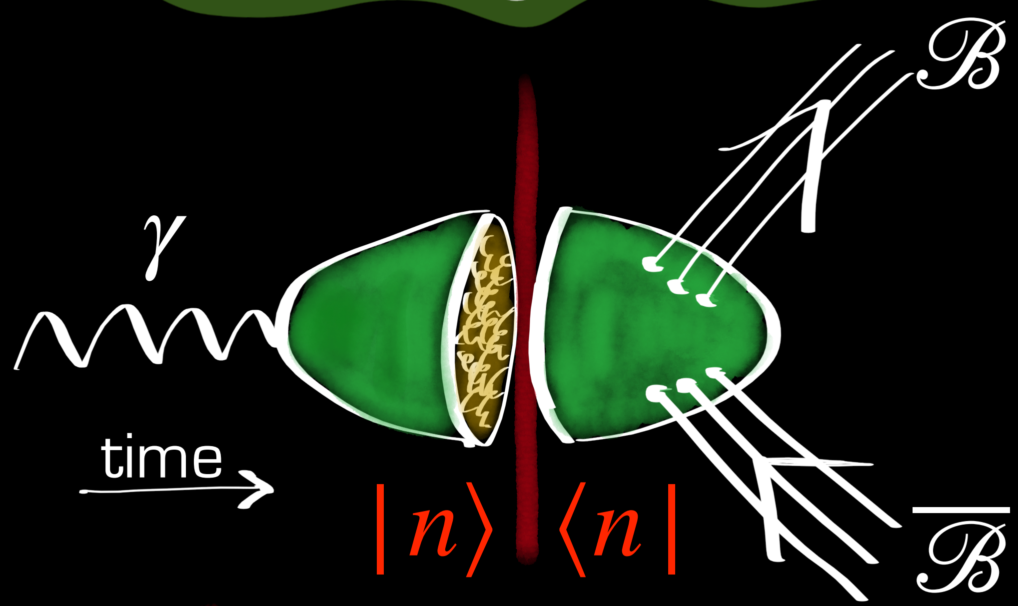
$$\frac{G_E^{\mathcal{B}}(q^2)}{G_M^{\mathcal{B}}(q^2)} \underset{q^2\to-\infty}{\sim} \text{[constant]}$$


V. A. Matveev, R. M. Muradian, A. N. Tavkhelidze, LNC 7 (1973) 719
S. J. Brodsky, G. R. Farrar, PRL 31 (1973) 1153
M. V. Galynsky, E. A. Kuraev JETPL 96 (2012) 6




simone.pacetti@unipg.it

# Form factors in the time-like region ($q^2 > 0$)




## Crossing symmetry

$$\langle P(p')\,|\,j^\mu\,|\,P(p)\rangle \longrightarrow \langle \overline{P}(p')P(p)\,|\,j^\mu\,|\,0\rangle$$

## Form factors are complex functions of $q^2$

## Optical theorem

$$\mathrm{Im}\left(\langle \overline{P}(p')P(p)\,|\,j^\mu\,|\,0\rangle\right) \sim \sum_n \langle \overline{P}(p')P(p)\,|\,j^\mu\,|\,n\rangle\langle n\,|\,j^\mu\,|\,0\rangle \Longrightarrow \begin{cases} \mathrm{Im}\left(F_{1,2}^{\mathscr{B}}(q^2)\right) \neq 0 \\ \text{for } q^2 > 4M_\pi^2 \end{cases}$$

$|\,n\rangle$ is an on-shell intermediate state, i.e., $|\,n\rangle = 2\pi, 3\pi, 4\pi, \ldots$

## Phragmén Lindelöf theorem

If $f(z) \to f_1$ as $z \to \infty$ along the straight line $L_1$, and $f(z) \to f_2$ as $z \to \infty$ along the straight line $L_2$, and $f(z)$ is regular and bounded in the angle between, then $f_1 = f_2 \equiv f_{12}$ and $f(z) \to f_{12}$ uniformly in the region between $L_1$ and $L_2$.

## Behavior in the time-like region

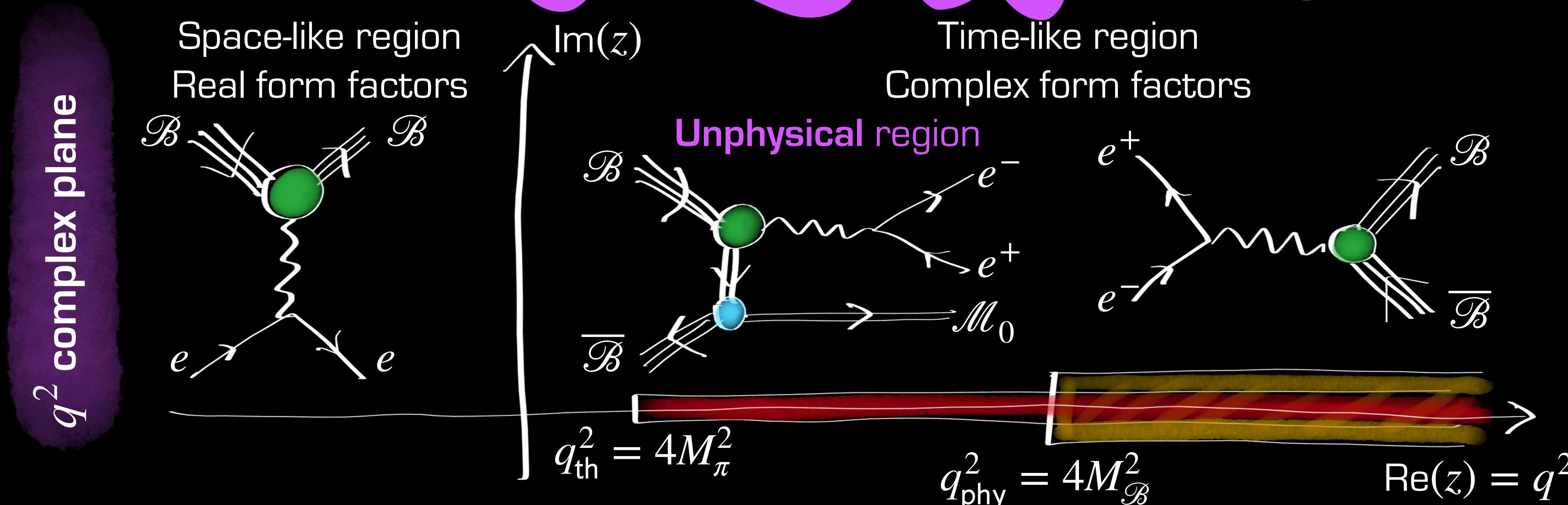
$$\underbrace{\lim_{q^2\to-\infty} G_{E,M}^{\mathscr{B}}(q^2)}_{\text{space-like } (L_1)} = \underbrace{\lim_{q^2\to+\infty} G_{E,M}^{\mathscr{B}}(q^2)}_{\text{time-like } (L_2)}$$

$$G_{E,M}^{\mathscr{B}}(q^2) \underset{q^2\to+\infty}{\sim} \left(q^2\right)^{-2} \in \mathbb{R}$$



simone.pacetti@unipg.it

# Analyticity of form factors




Only the real axis of the $q^2$-complex plane is experimentally accessible

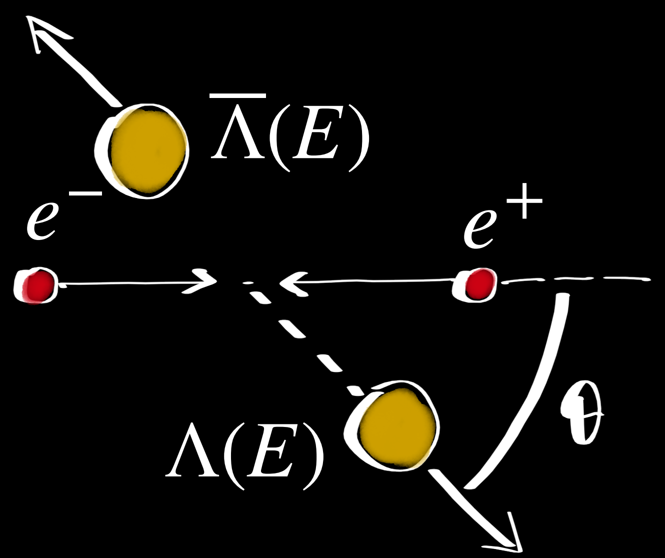
| Space-like region | Time-like region* | Time-like region | |
|---|---|---|---|
| $q^2 < 0$ | $q^2_{\text{th}} < q^2 \leq q^2_{\text{phy}}$ | $q^2 > q^2_{\text{phy}}$ | |
| $e\mathscr{B} \to e\mathscr{B}$ | $\mathscr{B}\overline{\mathscr{B}} \to e^+e^-\mathscr{M}_0$ | $e^+e^- \leftrightarrow \mathscr{B}\overline{\mathscr{B}}$ | $e^+e^- \leftrightarrow \mathscr{B}\overline{\mathscr{B}}$ (pol.) |
| $G_E^{\mathscr{B}}, G_M^{\mathscr{B}}$ | $\left\|G_E^{\mathscr{B}}\right\|, \left\|G_M^{\mathscr{B}}\right\|$ | $\left\|G_E^{\mathscr{B}}\right\|, \left\|G_M^{\mathscr{B}}\right\|$ | $\left\|G_E^{\mathscr{B}}\right\|, \left\|G_M^{\mathscr{B}}\right\|, \arg\left(G_E^{\mathscr{B}}/G_M^{\mathscr{B}}\right)$ |

*In case of $\mathscr{B} = p$: C. Adamuscin, E. A. Kuraev, E. Tomasi-Gustafsson, F. Maas PRC 75, 045205
E. A. Kuraev et al., JETP 115, 93
G. I. Gakh, E. Tomasi-Gustafsson, A. Dbeyssi, A. G. Gakh PRC 86, 025204



simone.pacetti@unipg.it

# Λ form factors




$\mathscr{C} = 1$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\beta\mathscr{C}}{16E^2}\left[\left(1+\cos^2(\theta)\right)\left|G_M^{\mathscr{B}}\right|^2 + \frac{1}{\tau}\sin^2(\theta)\left|G_E^{\mathscr{B}}\right|^2\right]$$

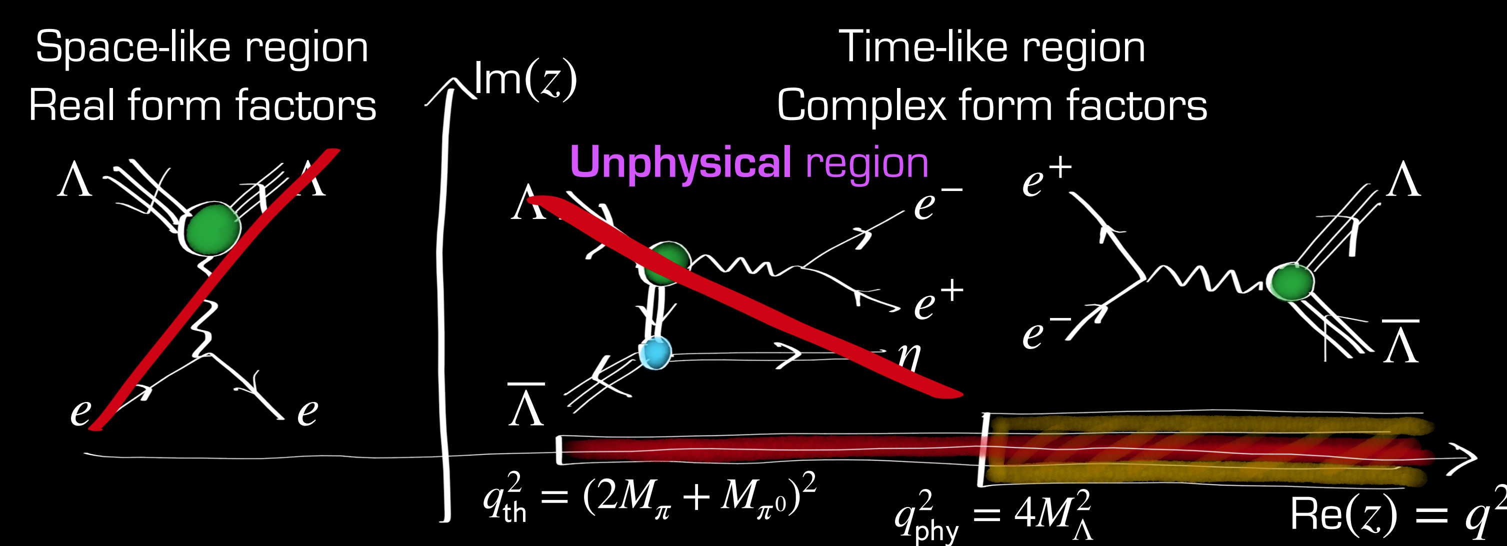
$$\beta = \sqrt{1-\frac{1}{\tau}}$$

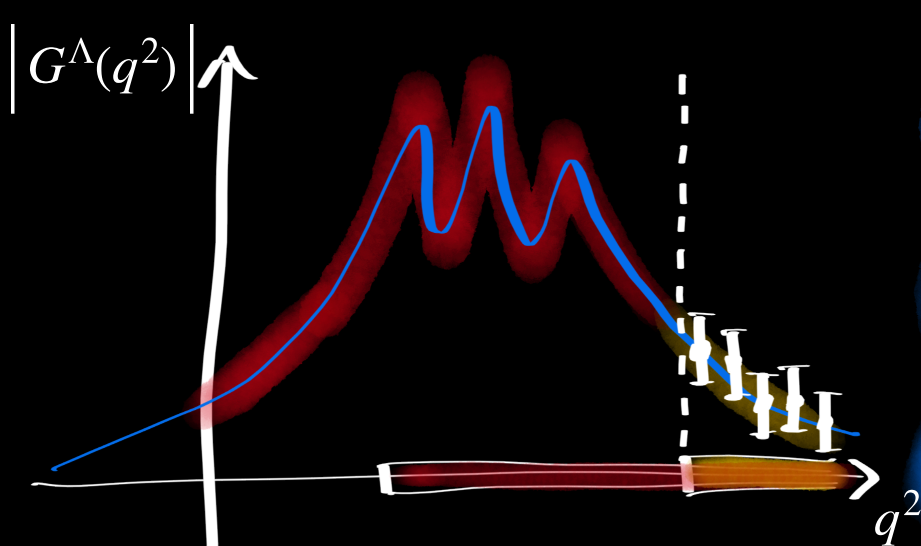
$$\tau = \frac{E^2}{4M_{\mathscr{B}}^2}$$

Theoretical threshold $q_{\rm th}^2 = (2M_\pi + M_{\pi^0})^2$.

No data in space-like and unphysical regions.

Relative phase from the weak decay.



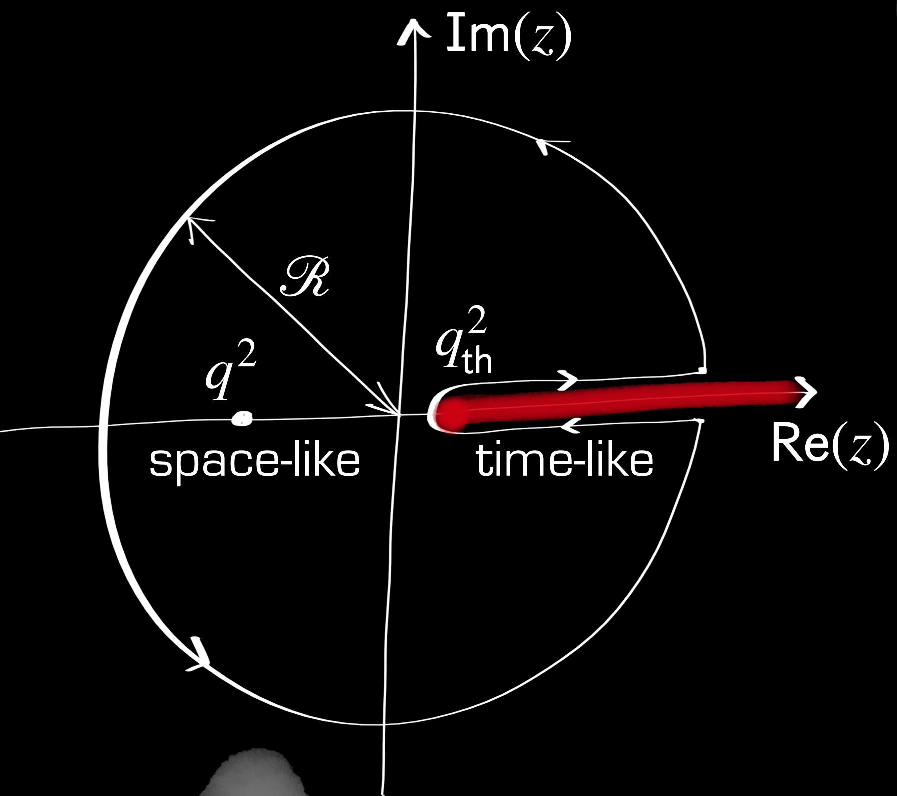




- Unitarity: only isoscalar intermediate states do contribute.
- Form factors have imaginary parts for $q^2 \geq q_{\rm th}^2$.
- The electric form factor $G_E^\Lambda(q^2)$ vanishes at $q^2 = 0$

NRM2023 - Varenna - 12 June

simone.pacetti@unipg.it

# Dispersion relations




The form factors are analytic on the $q^2$-complex plane with a multiple cut $\left(q^2_{\rm th}, \infty\right)$.

Dispersion relation for the imaginary part $(q^2 < 0)$

$$G(q^2) = \lim_{\mathscr{R}\to\infty} \frac{1}{2i\pi} \oint_{\mathscr{C}} \frac{G(z)}{z - q^2} dz = \frac{1}{\pi} \int_{q^2_{\rm th}}^{\infty} \frac{\mathrm{Im}(G(s))}{s - q^2} ds$$

Dispersion relation for the logarithm $(q^2 < 0)$

B. V. Geshkenbein, Yad. Fiz. 9 (1969) 1232.

$$\ln(G(q^2)) = \frac{\sqrt{q^2_{\rm th} - q^2}}{\pi} \int_{q^2_{\rm th}}^{\infty} \frac{\ln|G(s)|}{(s - q^2)\sqrt{s - q^2_{\rm th}}} ds$$

## Experimental inputs

- Space-like data on real values of form factors from: $e\mathscr{B} \to e\mathscr{B}$ and $e^{-\uparrow}\mathscr{B} \to e^{-}\mathscr{B}^{\uparrow}$, with polarization.
- Time-like data on moduli of form factors from: $e^+e^- \leftrightarrow \mathscr{B}\overline{\mathscr{B}}$.
- Time-like data on the phase of $G_E^{\mathscr{B}}/G_M^{\mathscr{B}}$ from: $e^+e^- \leftrightarrow \mathscr{B}^{\uparrow}\overline{\mathscr{B}}$, with polarization.

## Theoretical ingredients

- Analyticity $\Longrightarrow$ convergence relations
- Normalization and threshold values
- Asymptotic behavior $\Longrightarrow$ super-convergence relations



simone.pacetti@unipg.it

# Advantages and drawbacks of the approach



## Advantages

- Dispersion relations are based on unitarity and analyticity $\Longrightarrow$ **model-independent**.
- Dispersion relations relate from different precesses in different energy regions

$$\left[\begin{array}{c}\text{space-like}\\ \text{form factor}\\ e\mathscr{B}\to e\mathscr{B}\end{array}\right] = \int_{q^2_{\rm th}}^{\infty}\left[\begin{array}{c}\text{Im (form factor) or } \ln\left(\,|\,\text{form factor}\,|\,\right)\\ \text{over the time-like cut } (q^2_{\rm th},\infty)\\ e^+e^-\to\mathscr{B}\overline{\mathscr{B}} + \text{theory}\end{array}\right]$$

- Normalizations and theoretical constraints can be directly implemented.
- Form factors can be computed in the whole $q^2$-complex plane.
- Poles cancel out in the ratio.

## Drawbacks

- Very long-range integration

**Remedy #1**
**pQCD power laws**

**Remedy #2**
**Subtracted dispersione relations**

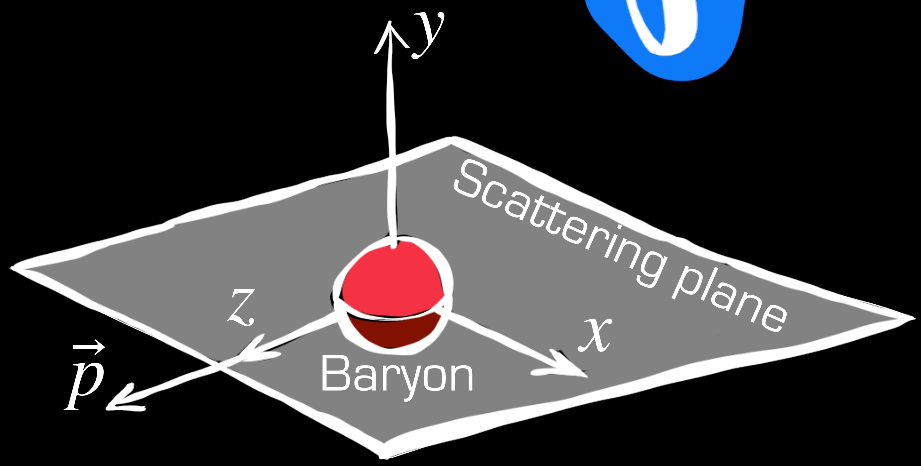
- No data in the unphysical region, crucial in dispersive analyses.

simone.pacetti@unipg.it

# Polarization in the time-like region

The ratio $G_E^{\mathscr{B}}(q^2)/G_M^{\mathscr{B}}(q^2)$ in complex for $q^2 > q^2_{\text{th}}$

$$\frac{G_E^{\mathscr{B}}(q^2)}{G_M^{\mathscr{B}}(q^2)} = \frac{\left|G_E^{\mathscr{B}}(q^2)\right|}{\left|G_M^{\mathscr{B}}(q^2)\right|} e^{i\rho(q^2)}$$




The polarization depends on the relative phase $\rho(q^2)$.

[A. Z. Dubnickova, S. Dubnicka, M. P. Rekalo, NC A109 (1996) 241]

$$\mathscr{P}_y = -\frac{\sin(2\theta)\sin(\rho)}{D\sqrt{\tau}} \frac{\left|G_E^{\mathscr{B}}\right|}{\left|G_M^{\mathscr{B}}\right|} = \frac{d\sigma^{\uparrow} - d\sigma^{\downarrow}}{d\sigma^{\uparrow} + d\sigma^{\downarrow}} \equiv \mathscr{A}_y$$

Does not depend on $P_e$.

$$\mathscr{P}_x = -P_e \frac{2\sin(2\theta)\cos(\rho)}{D\sqrt{\tau}} \frac{\left|G_E^{\mathscr{B}}\right|}{\left|G_M^{\mathscr{B}}\right|}$$

$$\mathscr{P}_z = P_e \frac{2\cos(\theta)}{D}$$

Does not depend on the relative phase $\rho$.

$$D = 1 + \cos^2(\theta) + \frac{\left|G_E^{\mathscr{B}}\right|^2}{\left|G_M^{\mathscr{B}}\right|^2} \frac{\sin^2(\theta)}{\tau}$$

$$\tau = \frac{q^2}{4M^2_{\mathscr{B}}}$$

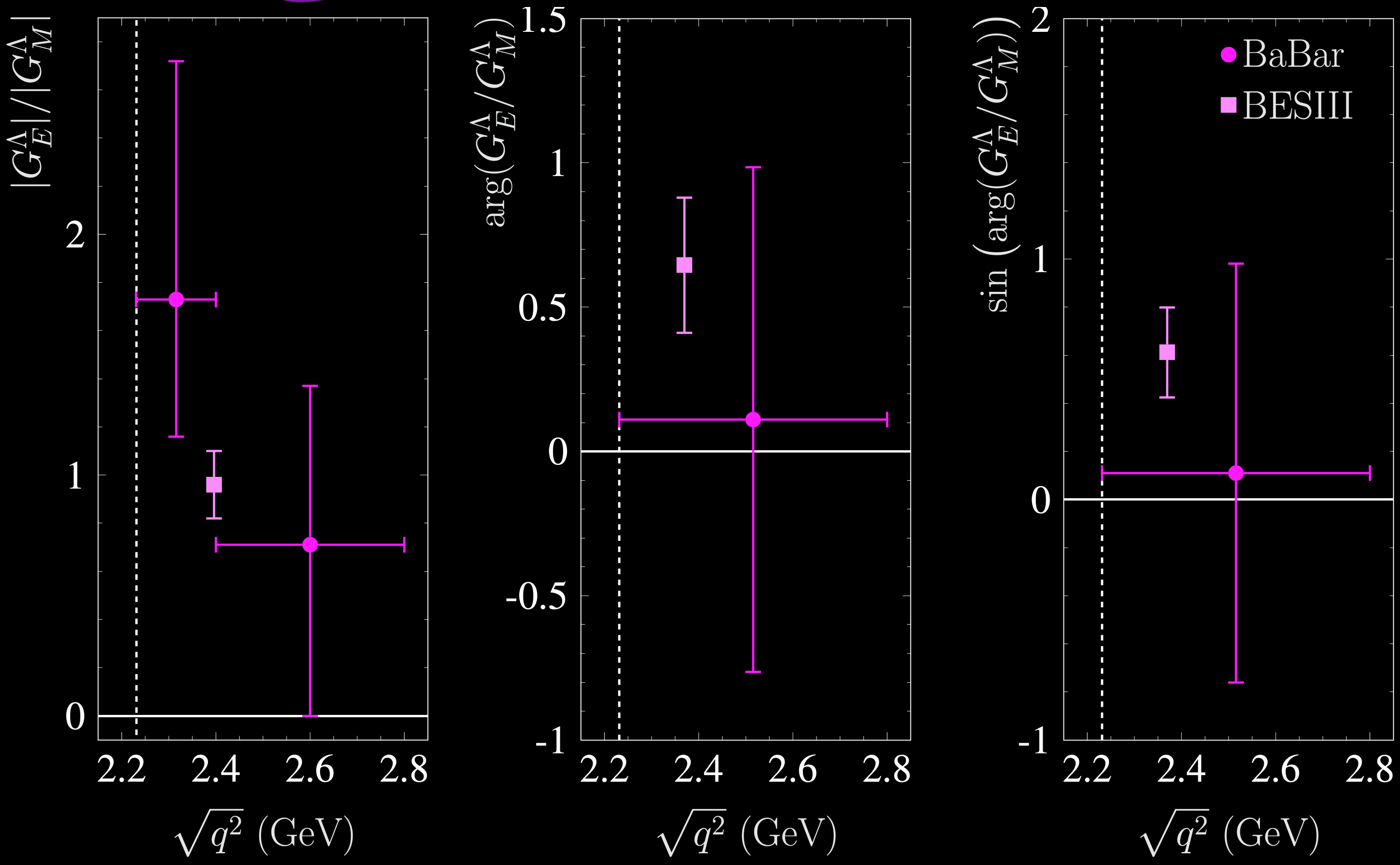
$P_e$ is the electron polarization.

$\theta$ is the scattering angle.



simone.pacetti@unipg.it

# Data on modulus and phase of $G_E^\Lambda/G_M^\Lambda$




**BaBar 2007**
Phys. Rev. D 76 (2007) 092006

**BESIII 2019**
Prys. Rev. Lett. 123 (2019) 122003

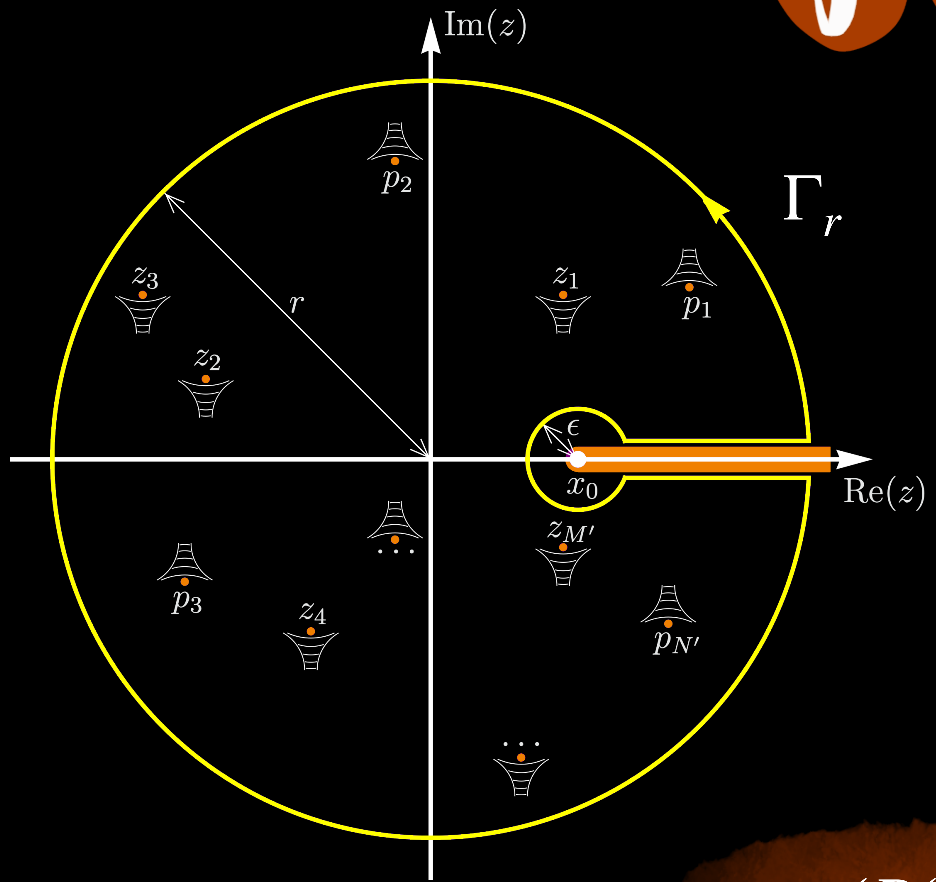
- Polarization $\longrightarrow$ sine of the relative phase.
- Spin correlation $\longrightarrow$ cosine of the relative phase.
- No indication on the determination of the relative phase.

$$\mathscr{P}_y = -\frac{2M_\Lambda\sqrt{q^2}\,\sin(2\theta)\left|G_E^\Lambda/G_M^\Lambda\right|\,\sin\left(\arg\left(G_E^\Lambda/G_M^\Lambda\right)\right)}{q^2\left(1+\cos^2(\theta)\right)+4M_\Lambda^2\left|G_E^\Lambda/G_M^\Lambda\right|^2\sin^2(\theta)}$$

**Is the determination of the phase meaningful?**

simone.pacetti@unipg.it

# The meaning of the determination of the phase




- Given the function $R(z)$ with $N$ poles $\{p_j\}_{j=1}^{N}$ and $M$ zeros $\{z_k\}_{k=1}^{M}$ and the positive real cut $(x_0, \infty)$.
- The residue theorem over the $\Gamma_r$ contour

$$\lim_{r\to\infty} \frac{1}{2i\pi} \oint_{\Gamma_r} \frac{d \ln(R(z))}{dz} dz = M - N.$$

- Considering single contributions

$$\lim_{r\to\infty} \frac{1}{2i\pi} \oint_{\Gamma_r} \frac{d \ln(R(z))}{dz} dz = \frac{\arg(R(\infty)) - \arg(R(x_0))}{\pi}.$$

$$\arg(R(\infty)) - \arg(R(x_0)) = \pi(M - N).$$

Levinson's theorem

Form factors are analytic in the $q^2$ complex plane with the real positive cut $(q^2_{\text{th}}, \infty)$.

**Assuming no zeros for $G_M^\Lambda$**, the ratio $G_E^\Lambda / G_M^\Lambda$ has the same analyticity domain.

Form factors and hence the ratio $G_E^\Lambda / G_M^\Lambda$ are real for $q^2 \in (-\infty, q^2_{\text{th}})$.

$$\lim_{q^2 \to q^{2-}_{\text{th}}} \arg\left(\frac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)}\right) = \begin{cases} 0 & G_E^\Lambda(q^{2-}_{\text{th}})/G_M^\Lambda(q^{2-}_{\text{th}}) > 0 \\ \pm\pi & G_E^\Lambda(q^{2-}_{\text{th}})/G_M^\Lambda(q^{2-}_{\text{th}}) < 0 \end{cases}$$



simone.pacetti@unipg.it

# Dispersive procedure

The ratio $R(q^2) = \frac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)} \Longrightarrow \left\{ \begin{array}{c} G_E^\Lambda(0) = 0 \\ G_E^\Lambda(q^2_{\text{phy}}) = G_M^\Lambda(q^2_{\text{phy}}) \end{array} \right\} \Longrightarrow \left\{ \begin{array}{r} R(0) = 0 \\ R(q^2_{\text{phy}}) = 1 \end{array} \right\}$

The asymptotic behavior

$$R(q^2) = \frac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)} = \mathcal{O}(1) \text{ as } q^2 \to \pm\infty.$$

Dispersion relations for the **imaginary** and **real** part with subtraction at $q^2 = 0$:

$$R(q^2) = R(0) + \frac{q^2}{\pi}\int_{q^2_{\text{th}}}^{\infty} \frac{\text{Im}(R(s))}{s(s-q^2)} ds = \frac{q^2}{\pi}\int_{q^2_{\text{th}}}^{\infty} \frac{\text{Im}(R(s))}{s(s-q^2)} ds\,, \quad \forall\, q^2 \notin [q^2_{\text{th}}, \infty);$$

$$\text{Re}(R(q^2)) = \frac{q^2}{\pi}\,\text{Pr}\int_{q^2_{\text{th}}}^{\infty} \frac{\text{Im}(R(s))}{s(s-q^2)} ds\,, \quad \forall\, q^2 \in [q^2_{\text{th}}, \infty)^+;$$

The subtraction ensures the null normalization at $q^2 = 0$.



simone.pacetti@unipg.it

# The parametrization for $R(q^2)$

The ratio $R(q^2)$ is parametrized through the set of **Chebyshev polynomials** $\left\{ T_j(x) \right\}_{j=0}^{P}$.

$$\mathrm{Im}(R(q^2)) \equiv Y(q^2; \vec{C}, q^2_{\rm asy}) = \begin{cases} \sum_{j=0}^{P} C_j T_j(x(q^2)) & q^2_{\rm th} < q^2 < q^2_{\rm asy} \\ 0 & q^2 \geq q^2_{\rm asy} \end{cases}$$

$$x(q^2) = 2\frac{q^2 - q^2_{\rm th}}{q^2_{\rm asy} - q^2_{\rm th}} - 1$$

$$q^2 \in [q^2_{\rm th}, q^2_{\rm asy}] \Rightarrow x \in [-1,1]$$

## Theoretical constraints on $Y(q^2; \vec{C}, q^2_{\rm asy})$

$R(q^2_{\rm th})$ is real $\Longrightarrow Y(q^2_{\rm th}; \vec{C}, q^2_{\rm asy}) = 0$

$R(q^2_{\rm phy})$ is real $\Longrightarrow Y(q^2_{\rm phy}; \vec{C}, q^2_{\rm asy}) = 0$

$R(q^2 \geq q^2_{\rm asy})$ is real $\Longrightarrow Y(q^2 \geq q^2_{\rm asy}; \vec{C}, q^2_{\rm asy}) = 0$

## Theoretical constraints on $\mathrm{Re}(R(q^2))$

$$\mathrm{Re}(R(q^2_{\rm phy})) = \frac{q^2_{\rm phy}}{\pi} \mathrm{Pr}\int_{q^2_{\rm th}}^{q^2_{\rm asy}} \frac{Y(s; \vec{C}; q^2_{\rm asy})}{s(s - q^2_{\rm asy})} ds = 1$$

$$\left|\mathrm{Re}(R(q^2_{\rm asy}))\right| = \frac{q^2}{\pi} \left| \mathrm{Pr}\int_{q^2_{\rm th}}^{q^2_{\rm asy}} \frac{Y(s; \vec{C}; q^2_{\rm asy})}{s(s - q^2_{\rm asy})} ds \right| = 1$$

## Experimental constraints in the time-like region $(q^2 > q^2_{\rm phy})$

$\sin\left(\arg\left(R(q^2)\right)\right)$: one data point from BaBar and one data point from BESIII.

$\left|R(q^2)\right|$: two data points from BaBar and one data point from BESIII.



simone.pacetti@unipg.it

# The $\chi^2$ definition

$$\chi^2\left(\vec{C}, q^2_{\rm asy}\right) = \chi^2_{|R|} + \chi^2_{\phi} + \tau_{\rm phy}\chi^2_{\rm phy} + \tau_{\rm asy}\chi^2_{\rm asy} + \tau_{\rm curv}\chi^2_{\rm curv}$$

Data $\left\{q_j^2, |R_j|, \delta|R_j|\right\}_{j=1}^{3} \longrightarrow \chi^2_{|R|} = \sum_{j=1}^{3}\left(\frac{\sqrt{X(q_j^2)^2 + Y(q_j^2)^2} - |R_j|}{\delta|R_j|}\right)^2$

$X(q^2) \equiv \mathrm{Re}(R(q^2))$

Data $\left\{q_k^2, \sin(\phi_k), \delta\sin(\phi_k)\right\}_{k=1}^{2} \longrightarrow \chi^2_{\phi} = \sum_{k=1}^{2}\left(\frac{\sin\left(\arctan\left(Y(q_k^2)/X(q_k^2)\right)\right) - \sin(\phi_k)}{\delta\sin(\phi_k)}\right)^2$

Constraint at $q^2 = q^2_{\rm phy} \longrightarrow \chi^2_{\rm phy} = \left(1 - X(q^2_{\rm phy})\right)^2$

Constraint at $q^2 = q^2_{\rm asy} \longrightarrow \chi^2_{\rm asy} = \left(1 - X(q^2_{\rm asy})^2\right)^2$

The values of multipliers $\tau_{\rm phy}$ and $\tau_{\rm asy}$ are chosen larger enough to nullify the corresponding $\chi^2$'s so that the conditions are exactly verified.

Oscillation damping $\longrightarrow \chi^2_{\rm curv} = \int_{q^2_{\rm th}}^{q^2_{\rm asy}}\left(\frac{d^2Y(s)}{ds^2}\right)^2 ds$

The integral equation obtained by the dispersion relations is an ill-posed problem whose solution has to be regularized.

The value of the regularization parameter $\tau_{\rm curv}$ is selected in order to attenuate spurious oscillations.

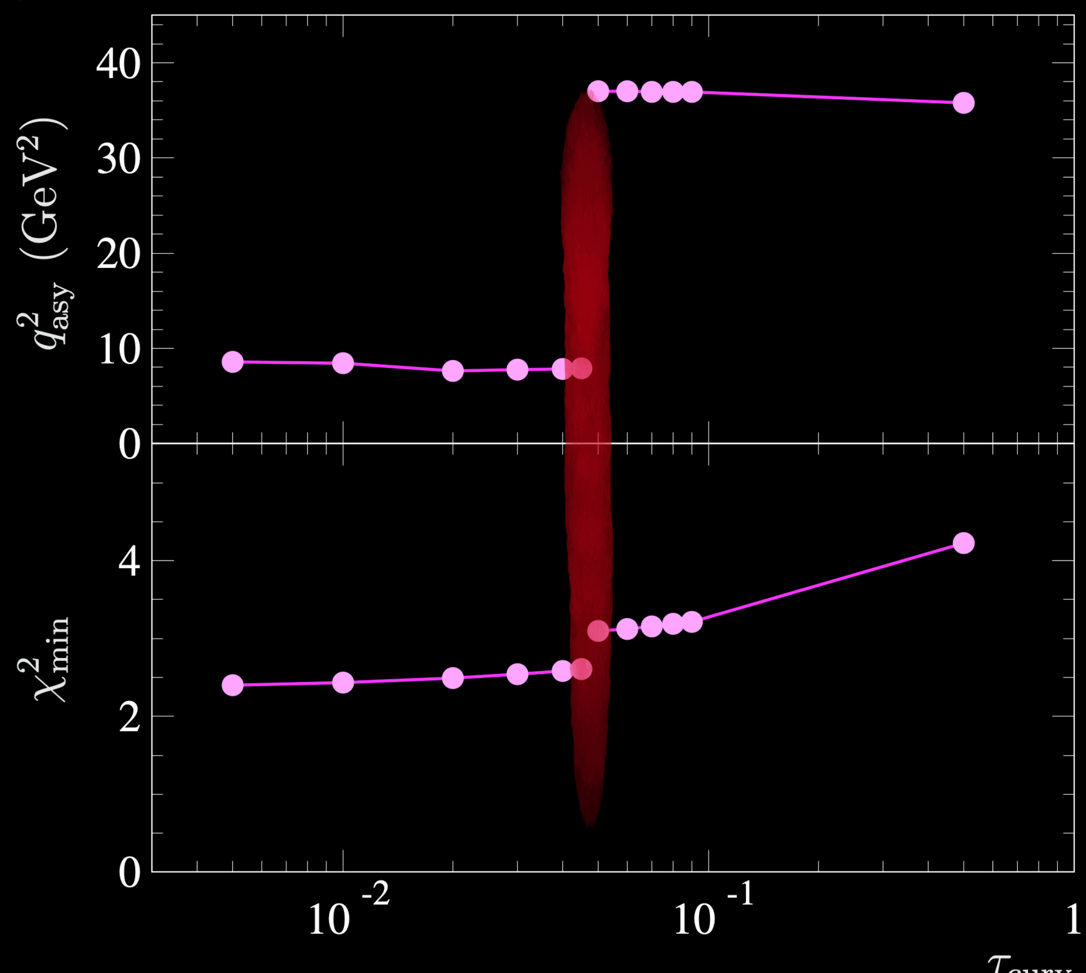
- Too **large** values of $\tau_{\rm curv}$ would cancel physical information.
- Too **small** values $\tau_{\rm curv}$ would leave an unreliable level of noise.



simone.pacetti@unipg.it

# Our parametrization

- The theoretical constraints $Y(q^2_{\rm th}; \vec{C}, q^2_{\rm asy}) = Y(q^2_{\rm phy}; \vec{C}, q^2_{\rm asy}) = Y(q^2_{\rm asy}; \vec{C}, q^2_{\rm asy}) = 0$ determine the three coefficients: $C_0, C_1, C_2$.
- The asymptotic threshold $q^2_{\rm asy}$ is left as a free parameter.
- By considering $(P+1)$ Chebyshev polynomials there are $(P-2)$ free coefficients.
- We have used $P = 5$ and hence there are four free parameters: $C_3, C_4, C_5$ and $q^2_{\rm asy}$.




- $\tau_{\rm phy} = 10^2$
  The real part of $R(q^2)$ is forced to the unity at $q^2 = q^2_{\rm phy}$.
- $\tau_{\rm asy} = 0$
  No constraint for the real part of $R(q^2)$ at $q^2 = q^2_{\rm asy}$.
- $\tau_{\rm curv} = 0.05$
  Low-degree polynomials do not need strong damping.

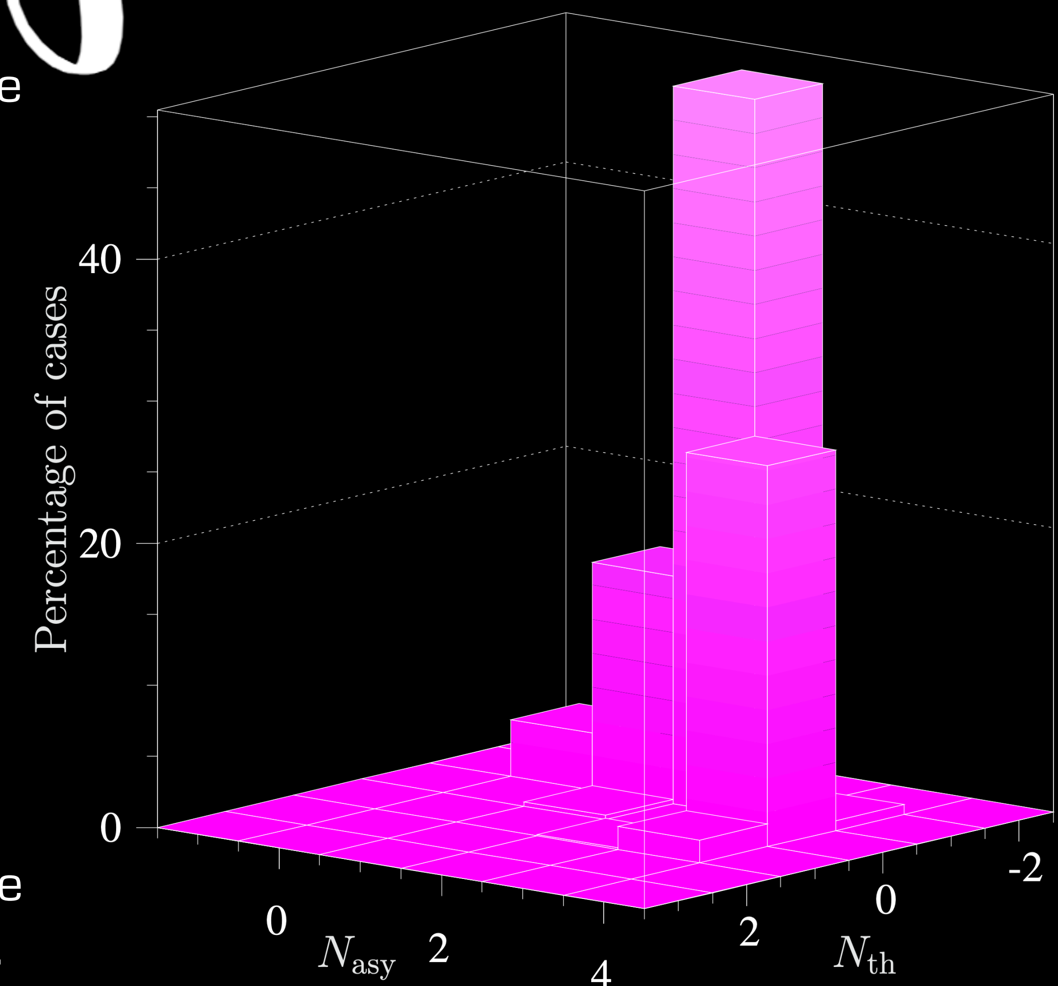

simone.pacetti@unipg.it

# Discussing results

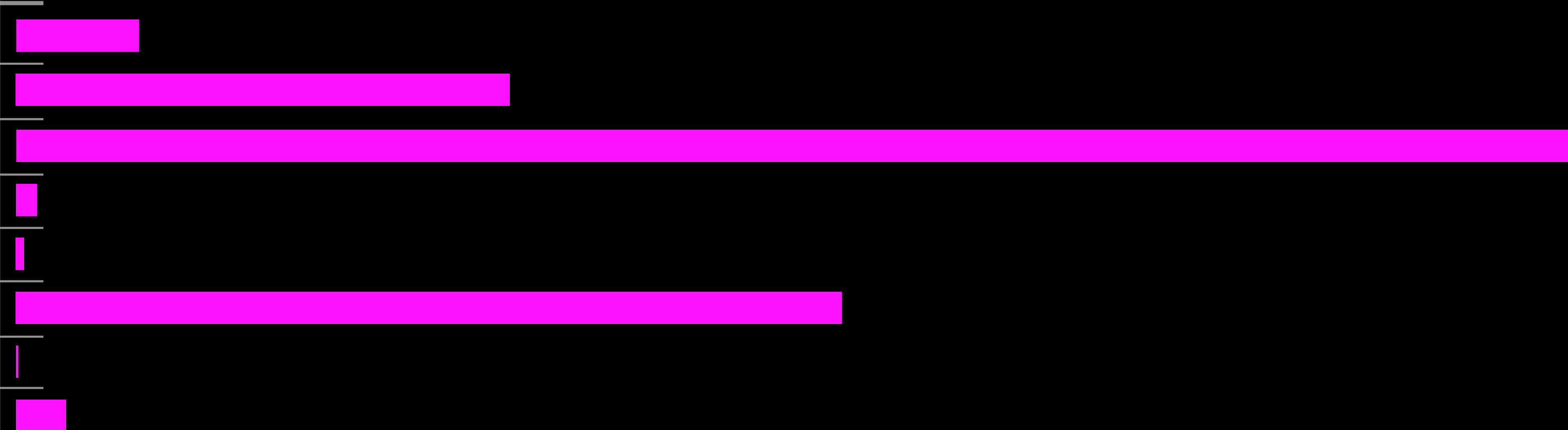
- At the thresholds $q^2_{\text{th}}$ and $q^2_{\text{asy}}$ the values of the ratio are real hence the phases are integer multiples of $\pi$ radiants

$$N_{\text{th,asy}} = \frac{1}{\pi} \arg \left( \frac{G_E^{\Lambda}(q^2_{\text{th,asy}})}{G_M^{\Lambda}(q^2_{\text{th,asy}})} \right) \in \mathbb{N}$$

- The lack of data prevents obtaining unique pairs $(N_{\text{th}}, N_{\text{asy}})$.
- The strong theoretical constraints reduce to 8 the number of possible pairs $(N_{\text{th}}, N_{\text{asy}})$ compatible with the few data points.
- A Monte Carlo procedure, defined to make a statistical study of the results, gives the probability of occurrence of each pair $(N_{\text{th}}, N_{\text{asy}})$.




| $N_{\text{th}}$ | $N_{\text{asy}}$ | % |
|---|---|---|
| $-1$ | 0 | 4.0 |
| $-1$ | 1 | 16.0 |
| $-1$ | 2 | 50.5 |
| $-1$ | 3 | 0.7 |
| 0 | 1 | 0.3 |
| 0 | 3 | 26.8 |
| 1 | 2 | 0.1 |
| 1 | 3 | 1.6 |





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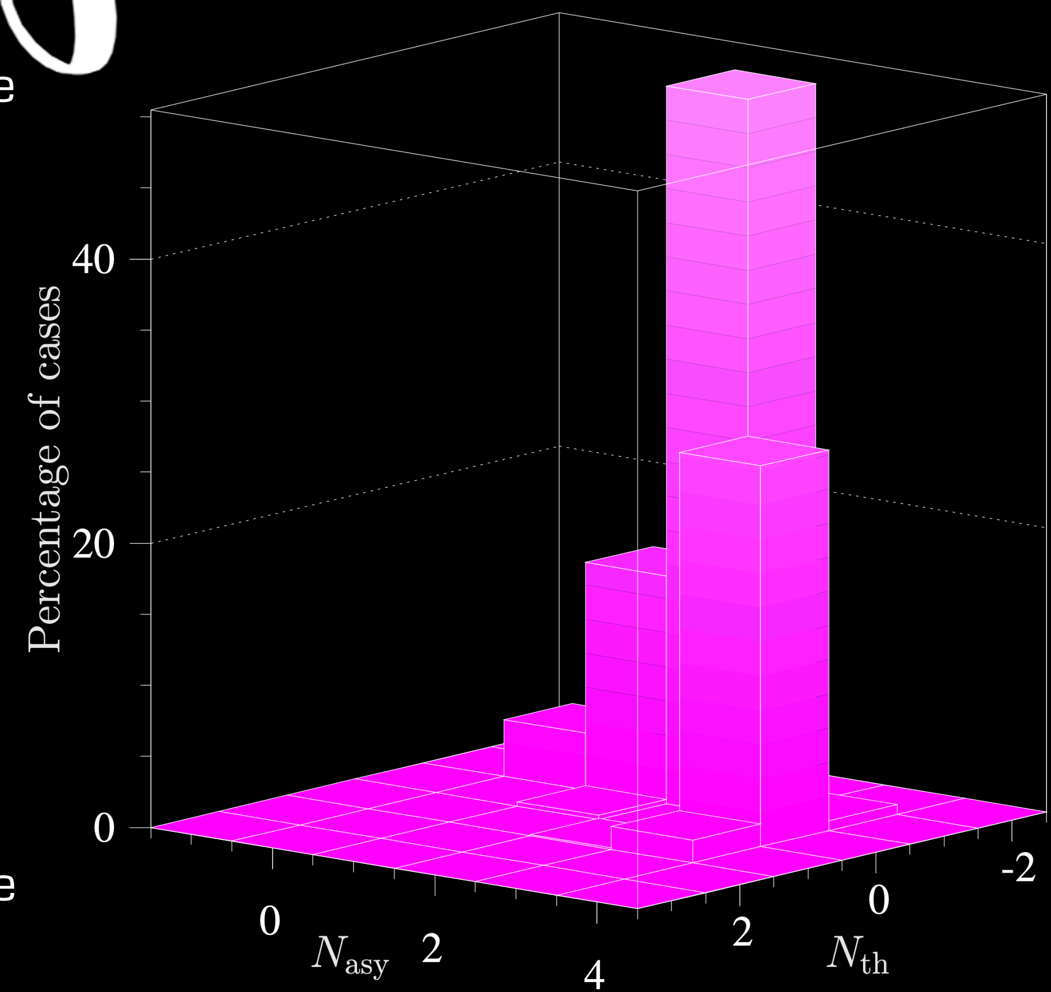


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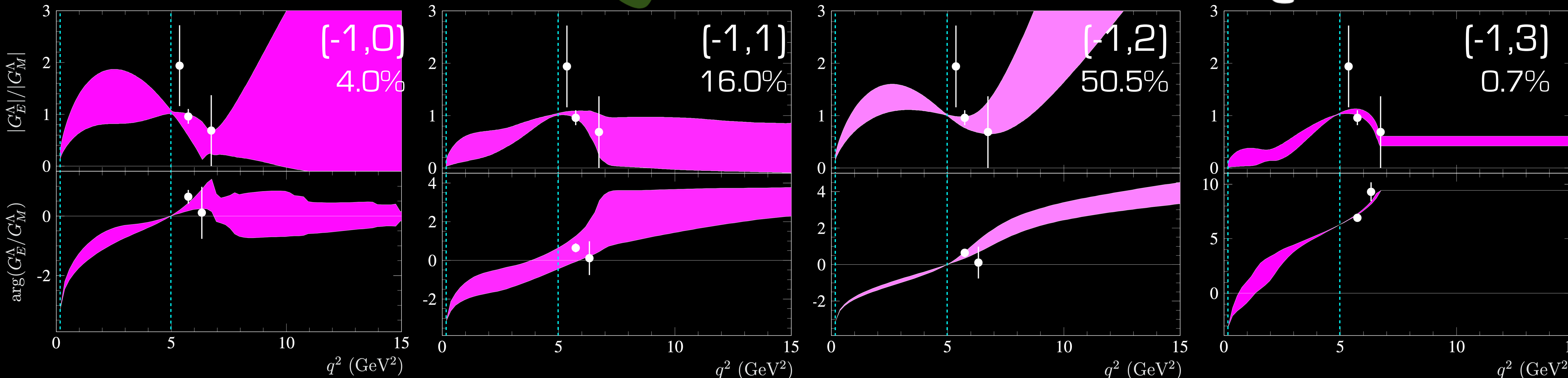
| ~~0~~ | ~~1~~ | ~~0.3~~ |
| 0 | 3 | 26.8 |
| ~~1~~ | ~~2~~ | ~~0.1~~ |
| 1 | 3 | 1.6 |

Cases with a probability of occurrence lower than 0.5% are discarded.



simone.pacetti@unipg.it

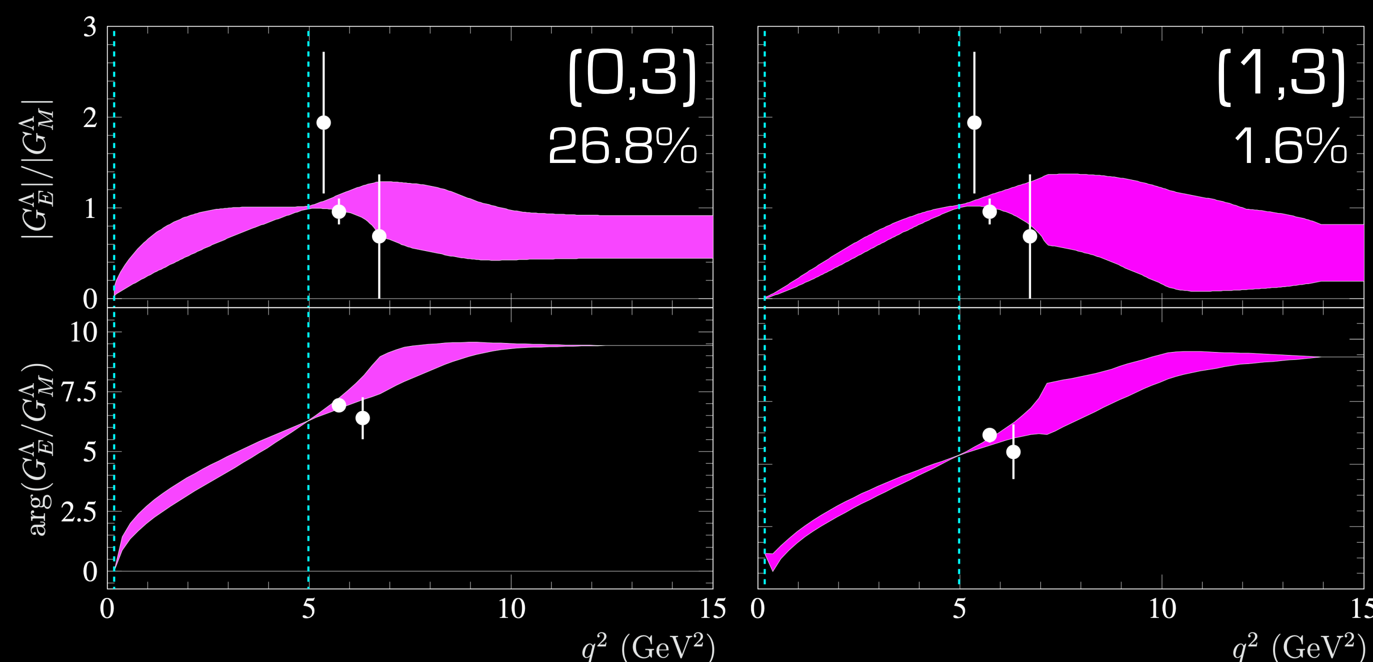
# Moduli and phases




**Levinson's theorem** with no poles
(no zeros for $G_M^\Lambda(q^2)$)

$$N_{\rm asy} - N_{\rm th} = \begin{cases} \text{number of zeros of } R(q^2) \\ \text{and } G_E^\Lambda(q^2) \text{ in } \mathbb{C}\backslash(q^2_{\rm th}, \infty). \end{cases}$$

$$G_E^\Lambda(0) = 0 \implies N_{\rm asy} \geq N_{\rm th} + 1$$




The bands represent the one-sigma-error computed by the standard statistical analysis of the set of results obtained with a Monte Carlo procedure.

The dispersive procedure, connecting experimental information on the modulus of the ratio $R(q^2)$ and on the sine of its phase under the aegis of strong theoretical constraints, assigns different determinations to the phase.

The determination of the measured values of the phase is also established by the dispersive procedure.

In the case $(N_{\rm th}, N_{\rm asy}) = (-1,3)$ the BESIII and BaBar phase data have different determinations.



simone.pacetti@unipg.it

# The charge radius of a neutral baryon

Dynamical and static features of the baryon $\Lambda$ can be inferred from the complete knowledge of its form factors as functions of $q^2$.

The charge radius squared $\langle r_E \rangle^2$ of an extended particle, as a baryon, is proportional to the first derivative of the electric form factor $G_E(q^2)$ at $q^2 = 0$.

$$\langle r_E \rangle^2 = 6 \left. \frac{dG_E(q^2)}{dq^2} \right|_{q^2=0}$$

In the Breit frame, where $q = (0, \vec{q})$ is purely space-like, the electric form factor is the Fourier transform of the spacial charge distribution.

For a neutral baryon, like the $\Lambda$, the Sachs form factors at $q^2 = 0$ are normalized as: $G_E(0) = 0$ and $G_M(0) = \mu \neq 0$, then, the charge radius squared is also proportional to the first derivative at $q^2 = 0$ of the ratio $R(q^2) = G_E(q^2)/G_M(q^2)$

$$\left. \frac{dR(q^2)}{dq^2} \right|_{q^2=0} = \left. \frac{1}{G_M(q^2)} \left( \frac{dG_E(q^2)}{dq^2} - \frac{\overbrace{G_E(q^2)}^{=0 \text{ at } q^2=0}}{G_M(q^2)} \frac{dG_M(q^2)}{dq^2} \right) \right|_{q^2=0} = \left. \frac{1}{G_M(q^2)} \frac{dG_E(q^2)}{dq^2} \right|_{q^2=0} = \frac{1}{\mu} \frac{\langle r_E \rangle^2}{6}.$$

The first derivative at $q^2 = 0$ of the ratio $R(q^2)$ is computed by means of the dispersione relation for the imaginary part

$$\langle r_E \rangle^2 = 6\mu \left. \frac{dR(q^2)}{dq^2} \right|_{q^2=0} = \frac{6\mu}{\pi} \int_{q^2_{\rm th}}^{\infty} \frac{\mathrm{Im}(R(s))}{s^2} ds = \frac{6\mu}{\pi \Delta q^2} \sum_{j=0}^{N} C_j \int_{-1}^{1} \frac{T_j(x) dx}{\left(x + 1 + q^2_{\rm th}/\Delta q^2\right)^2}$$

with $\Delta q^2 = (q^2_{\rm asy} - q^2_{\rm th})/2$.

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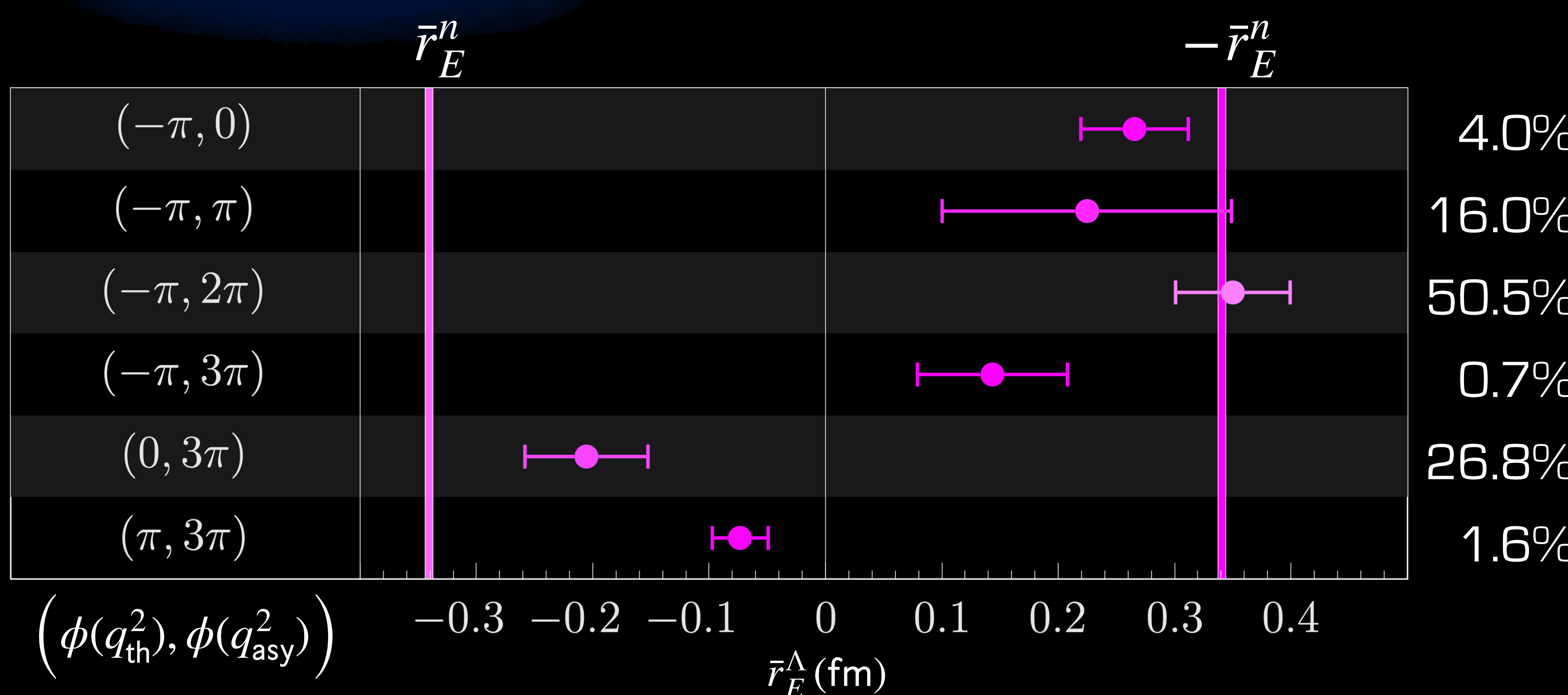
simone.pacetti@unipg.it

# Charge radii of $\Lambda$

The neutron has a negative squared charge radius: $\langle r_E^n \rangle^2 = -0.1161 \pm 0.0022\,\mathrm{fm}^2$

$$\bar{r}_E \equiv \mathrm{Sign}\left(\langle r_E \rangle^2\right)\sqrt{\left|\langle r_E \rangle^2\right|}$$

To have a better understanding of the linear extension of the baryon.

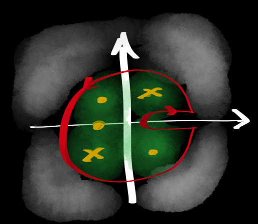



Those values of $\bar{r}_E^\Lambda$ compatible with $-\bar{r}_E^n$ can be heuristically interpreted in terms of the different time periods that the valence quarks of the same charge spend at a certain distance from the center of the baryon.

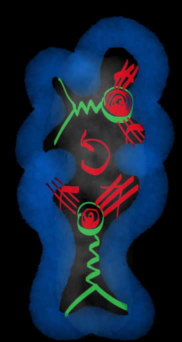


simone.pacetti@unipg.it

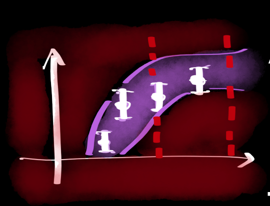
# Final considerations



A dispersive procedure based on data and first principles such as analyticity and unitarity has been defined to study the ratio of electric and magnetic form factors of the $\Lambda$ baryon.



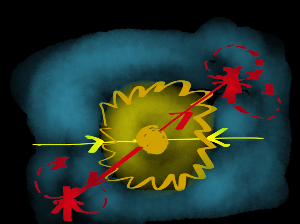
By taking advantage of the measured values of the modulus and the phase of the ratio in the time-like region, as well as on theoretical constraints, the procedure allows us to gain crucial information on the space-like behavior of the ratio, which is not experimentally accessible.



Assuming no zeros for the magnetic form factor, the asymptotic value of the phase counts the number of the zeros of the electric form factor, which, being the $\Lambda$ a neutral baryon, is at least one:

$$\Delta\phi = \phi(\infty) - \phi(q^2_{\rm th}) = \pi\left(N_{\rm asy} - N_{\rm th}\right) \geq \pi.$$

The most probable values give $\Delta\phi = 3\pi$, hence, two additional zeros for $G_E^\Lambda(q^2)$.



New data, especially for the sine of the phase, would be crucial to at least identify its trend and then have hints of the phase determination.



simone.pacetti@unipg.it