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Precession resonances in hierarchical 3-body systems in a strong gravity regime

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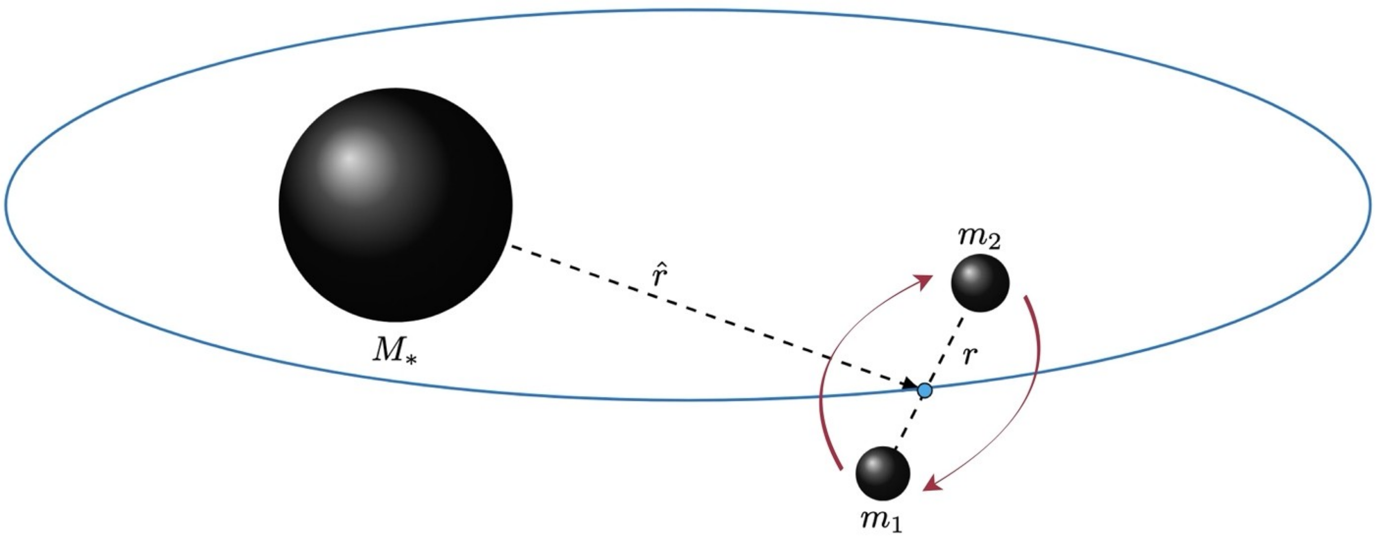
Motivation

- Theoretical interest
  - strong-gravity analysis uncovers fundamental new features within the resonant behavior of hierarchical triple systems [M. Cocco et al. (2025)]
  - need for a robust analytical framework to investigate resonant dynamics

- Astrophysical relevance
  - vast populations of compact binaries merging within observable timescales
  - recent detection of stellar mass binary system in close orbit around Sgr A * [F. Peiker et al. (2024)]
  - probing the SMBH environment via precessional resonances

System representation and setup



- The central SMBH is modeled as a Schwarzschild black hole of mass and associated metric in coordinates $(\hat{t},\, \hat{r},\, \hat{\theta},\, \hat{\phi})$given by

$$ds^2 = -\left(1 - \frac{2GM_*}{c^2\hat{r}}\right) c^2 d\hat{t}^2 + \frac{d\hat{r}^2}{1 - \frac{2GM_*}{c^2\hat{r}}} + \hat{r}^2 d\hat{\Omega}^2$$

- Motion restricted to the equatorial plane fixing $\hat{\theta} = \frac{\pi}{2}$

Hierarchical configuration and dynamical assumptions

- Separation between the inner binary's compact objects significantly larger than their respective Schwarzschild radii

$$r \gg \frac{2G\, m_{1,2}}{c^2}$$

- **Small-tide approximation**: characteristic scale of the inner binary, considerably smaller than the radius of curvature associated with space time geometry

$$r \ll \mathcal{R}$$

- Two possible physical regimes:
  - the separation between the inner binary and the SMBH is much larger than its Schwarzschild radius (Newtonian)
  - the binary orbits at distances of only a few Schwarzschild radii to the SMBH (**strong-gravity**)

$$M_* \gg m_1, m_2$$

- Analysis focused on timescales relevant to precessional resonances
- We neglect emission of gravitational wave (GW) from the outer binary

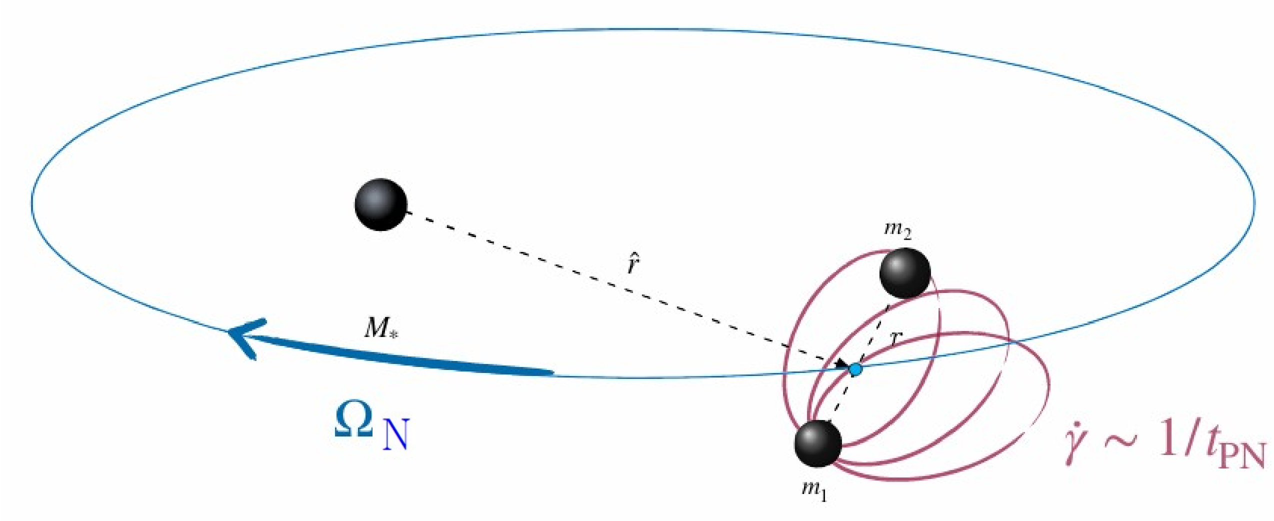
Precession resonances (Newtonian regime)

- These are resonances arising from the commensurability between the inner binary's periastron precession frequency and the fundamental orbital frequencies of its motion around the Schwarzschild SMBH

- In a nearly Newtonian regime they occur when the following condition is satisfied [A. Kuntz (2022)]

$$q\,\dot{\gamma} = p\,\Omega_{\mathrm{N}}$$

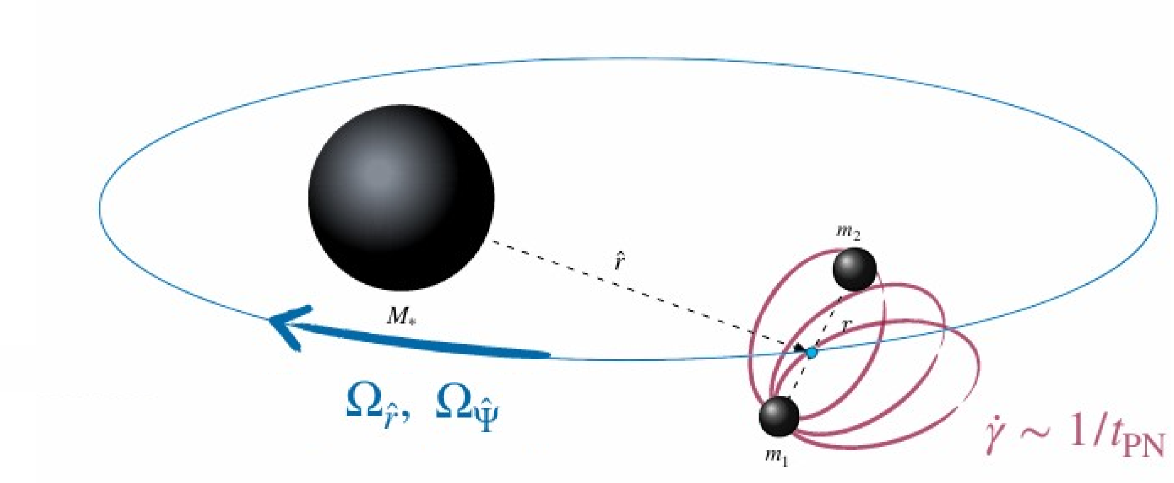
$$\Omega_{\mathrm{N}} = \sqrt{\frac{GM_*}{\hat{a}^3}}$$



Precession resonances (strong gravity regime)

- This Newtonian framework was extended to include strong-gravity effects, leading to a richer spectrum of resonances as the previous condition becomes [M. Cocco et al. (2025)]

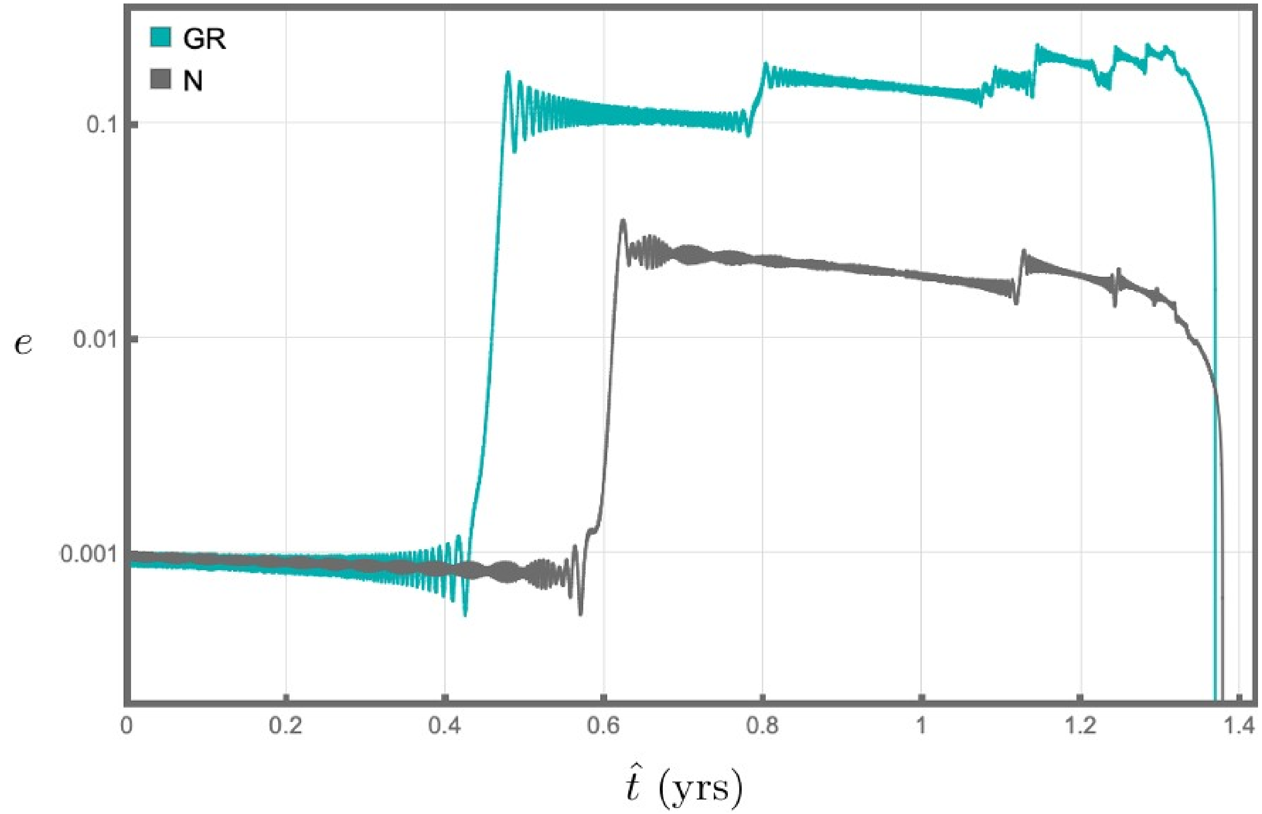
$$q\,\dot{\gamma} = k\,\Omega_{\hat{r}} + l\,\Omega_{\hat{\Psi}}$$



Precession resonances (strong gravity regime)

$$q\,\dot{\gamma} = k\,\Omega_{\hat{r}} + l\,\Omega_{\hat{\Psi}}$$

- higher eccentricity peaks
- shorter merger time
- peak splitting



[M. Cocco et al. (2025)]

Triple system Hamiltonian

- The total Hamiltonian describing the hierarchical triple system can be represented as follows

$$\mathscr{H} = \mathscr{H}_{pp} + \mathscr{H}_{inner} + \mathscr{H}_{quad}$$

- First term characterizes the bound geodesic motion of the inner binary's center of mass as it evolves along the outer orbit

- The second term describes the internal dynamics of the inner binary, including the 1PN correction accounting for periastron precession

- The last term describes the coupling of the inner binary to the quadrupole tidal field of the Schwarzschild background

$$\mathscr{H}_{quad} = \frac{c^2}{2} \mathcal{Q}^{ij} \mathscr{E}_{ij}$$

- The electric tidal tensor encodes the full relativistic coupling to the spacetime curvature induced by the Schwarzschild SMBH
- The explicit form of the quadrupolar Hamiltonian is given by

$$
\begin{aligned}
\mathscr{H}_{quad} = \frac{GM_*}{\hat{r}^3}\frac{\mu a^2}{2}\Bigg[&\frac{2+3e^2}{2} + 3\frac{\hat{L}^2}{c^2\hat{r}^2}\frac{2+3e^2-5e^2\cos 2\gamma}{4}\sin^2 I \\
&-3\left(1+\frac{\hat{L}^2}{c^2\hat{r}^2}\right)\Bigg(\frac{2+3e^2+5e^2\cos 2\gamma}{4}\cos^2(\hat{\Psi}-\vartheta) \\
&+\frac{2+3e^2-5e^2\cos 2\gamma}{4}\sin^2(\hat{\Psi}-\vartheta)\cos^2 I \\
&+\frac{5e^2}{2}\sin 2\gamma\cos(\hat{\Psi}-\vartheta)\sin(\hat{\Psi}-\vartheta)\cos I\Bigg)\Bigg]\ .
\end{aligned}
$$

Semi-analytical model up to second order in outer eccentricity

- Adopting the action-angle variable formalism one can find $q_\mu = \Omega_\mu \hat{t}$
- The fundamental outer frequencies entering the resonance condition $q\,\dot{\gamma} = k\,\Omega_{\hat{r}} + l\,\Omega_{\hat{\Psi}}$ are explicitly found up to second order in outer eccentricity as

$$\Omega_{\hat{r}} = \Omega_{\rm N}\sqrt{\frac{\hat{\sigma}-6}{\hat{\sigma}}}\left[1-\frac{3}{4}\hat{e}^2\frac{13\hat{\sigma}-74}{(\hat{\sigma}-2)(\hat{\sigma}-6)^2}\right]+\mathcal{O}(\hat{e}^3)\,,$$
$$\Omega_{\hat{\Psi}} = \Omega_{\rm N}\sqrt{\frac{\hat{\sigma}-3}{\hat{\sigma}}}\left[1+\frac{3}{2}\hat{e}^2\frac{\hat{\sigma}(\hat{\sigma}-9)+22}{(\hat{\sigma}-2)(\hat{\sigma}-6)(\hat{\sigma}-3)}\right]+\mathcal{O}(\hat{e}^3)\,,$$

$$\hat{\sigma} := \frac{\hat{a}c^2}{GM_*}$$

  - Newtonian limit: $\Omega_{\hat{r}}, \Omega_{\hat{\Psi}} \rightarrow \Omega_{\rm N}$ ($p = k + l$ in the resonance condition)

- Next step involves deriving $\hat{r}$ and $\hat{\Psi}$ up to $\mathcal{O}(\hat{e}^2)$ and in terms of the angle variables $q_{\hat{r}}$ and $q_{\hat{\Psi}}$
- Their explicit expressions are given by

$$\hat{r} = \hat{a}\left(1 - \hat{e}\cos q_{\hat{r}} + \hat{e}^2 \frac{26 + \hat{\sigma}(\hat{\sigma} - 11)\sin^2 q_{\hat{r}}}{(\hat{\sigma} - 2)(\hat{\sigma} - 6)}\right) + \mathcal{O}(\hat{e}^3)$$

$$\begin{aligned}\hat{\Psi} &= q_{\hat{\Psi}} + 2\hat{e}\,\frac{\hat{\sigma} - 4}{\hat{\sigma} - 2}\sqrt{\frac{\hat{\sigma} - 3}{\hat{\sigma} - 6}}\sin q_{\hat{r}} \\ &\quad - \hat{e}^2 \frac{484 - \hat{\sigma}(336 + \hat{\sigma}(-73 + 5\hat{\sigma}))}{4(\hat{\sigma} - 6)(\hat{\sigma} - 2)^2}\sqrt{\frac{\hat{\sigma} - 3}{\hat{\sigma} - 6}}\sin 2q_{\hat{r}} + \mathcal{O}(\hat{e}^3)\,.\end{aligned}$$

- In order to identifies the specific resonances arising from $\mathscr{H}_{quad}$ up to this order of approximation, a general Fourier series representation of this term is required

$$\mathscr{H}_q^{\exp} = \frac{Ga^2 M_* \mu}{48\hat{a}^3} \Big[(2+3e^2)\, h_{0,0}(I,\vartheta) + 15e^2 \sum_{k,l \neq (0,0)} \Big(f_{k,l}(I,\vartheta) \cos \xi_{k,l} + g_{k,l}(I,\vartheta) \sin \xi_{k,l} \Big) \Big] ,$$

where

$$h_{0,0}(I,\vartheta) = 2\, c_{0,0} \left(3 \cos^2 I - 1\right) + 6\, a_{0,0} \cos 2\vartheta \, \sin^2 I$$

$$f_{k,l}(I,\vartheta) = \Big(a_{k,l}\,(1+\cos^2 I) + 2b_{k,l} \cos I \Big) \cos 2\vartheta + c_{k,l} \sin^2 I \, ,$$

$$g_{k,l}(I,\vartheta) = -\Big(b_{k,l}\,(1+\cos^2 I) + 2a_{k,l} \cos I \Big) \sin 2\vartheta \, .$$

Analytical prediction of resonances

- Once all the non-vanishing Fourier coefficients are explicitly determined, having already derived the expanded outer fundamental frequencies $\Omega_{\hat{r}}$ and $\Omega_{\hat{\Psi}}$, we now possess a complete characterization of $\mathscr{H}_q^{\exp}$ up to $\mathcal{O}(\hat{e}^2)$
- Our model predicts seven distinct resonance peaks. Among them, three are entirely new and emerge only within the quadratic model

$$
\begin{aligned}
2\dot{\gamma} &= \Omega_{\hat{r}} \, , \\
2\dot{\gamma} &= -\Omega_{\hat{r}} + 2\,\Omega_{\hat{\Psi}} \, , \\
2\dot{\gamma} &= 2\,\Omega_{\hat{\Psi}} \, , \\
2\dot{\gamma} &= \Omega_{\hat{r}} + 2\,\Omega_{\hat{\Psi}} \, , \\
2\dot{\gamma} &= -2\Omega_{\hat{r}} + 2\,\Omega_{\hat{\Psi}} \, , \\
2\dot{\gamma} &= 2\,\Omega_{\hat{r}} \, , \\
2\dot{\gamma} &= 2\,\Omega_{\hat{r}} + 2\,\Omega_{\hat{\Psi}} \, .
\end{aligned}
$$

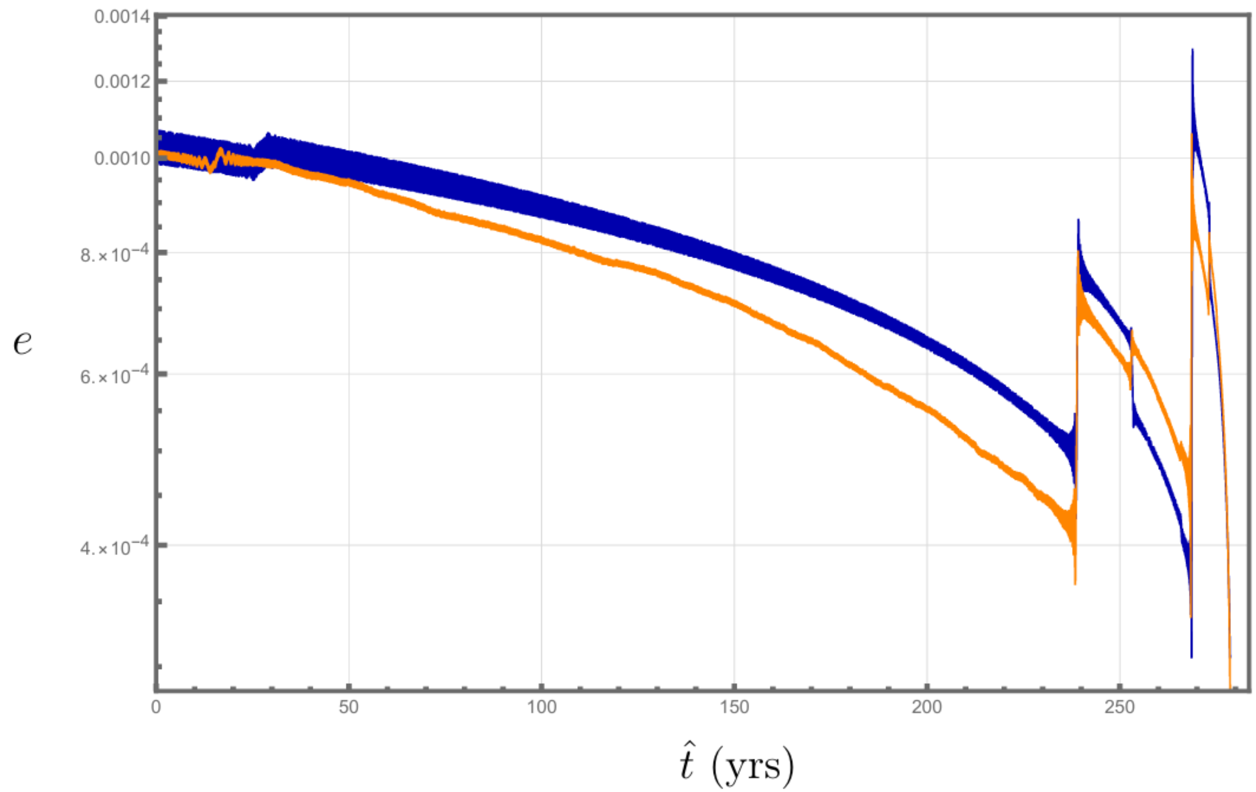
- Splitting phenomenon further enriched as the $p = 2$ Newtonian peak now splits into $(k, l) = (2, 0)$ and $(k, l) = (0, 2)$
- Entirely new resonance associated with $(k, l) = (-2, 2)$ has no Newtonian analogue (**apsidal resonance**)

Numerical validation of the quadratic model

- We numerically integrate the evolution equations for the orbital parameters of both the inner and outer orbits. By doing so, we must also consider the effects of gravitational radiation-reaction [P. Peters (1964)]

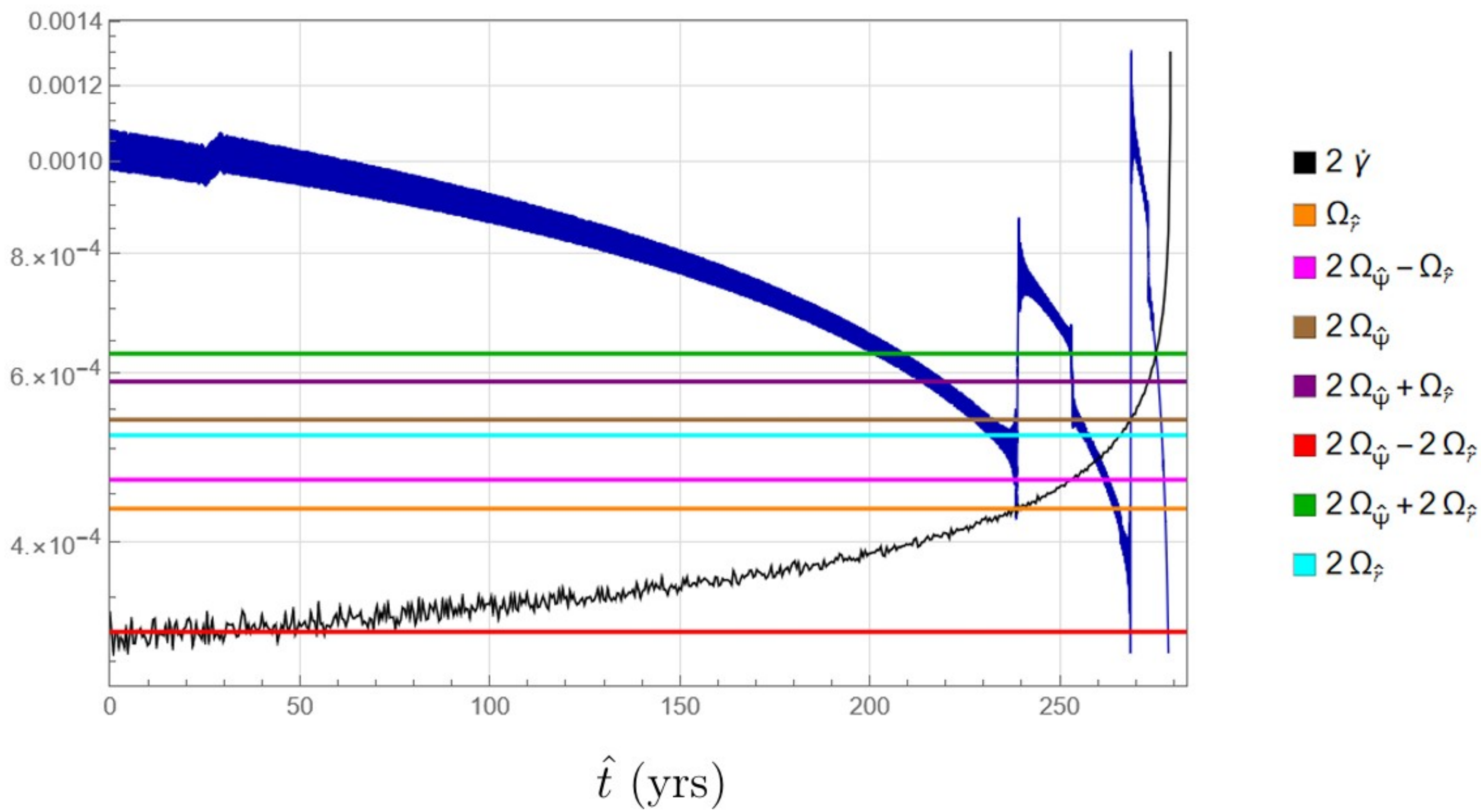
- These equations are solved
  - first, employing $\mathscr{H}_{quad}$ without resorting to a Fourier expansion
  - subsequently, using our modeled quadrupolar Hamiltonian $\mathscr{H}_q^{\mathrm{exp}}$ evaluated up to $\mathcal{O}(\hat{e}^2)$

Comparison between the full numerical results (blue curve) and the quadratic model (orange curve)



The parameters and initial conditions for the inner binary are the following: total mass $M = 50\ M_\odot$, reduced mass $\mu = 12.5\ M_\odot$, semi-major axis $a_0 \sim 0.0022$ AU, eccentricity $e_0 = 0.001$, initial inclination $I_0 = 60°$, $\gamma_0 = 0.001°$ and $\vartheta = 0°$. The inner binary orbits a SMBH of mass $M_* = 5 \times 10^7\ M_\odot$, with semi-major axis $\hat{a} = 15\ GM_*/c^2 \sim 7$ AU and outer eccentricity $\hat{e} = 0.09$

Position and identification of the resonance peaks found up to second order in outer eccentricity



The parameters and initial conditions are identical to those specified in the previous slide

Conclusions

- The analytical model faithfully reproduce the evolutionary and resonant behavior of the inner binary accurately predicting:
  - the resonance peaks active at this order of approximation
  - their temporal location within the binary evolution
  - the existence of three new peaks that are not captured within a linear order analysis
  - remarkably, the resonance peak relative to $(k, l) = (-2, 2)$ is a unique signature of the quadratic model with no correspondence in the Newtonian limit. It can be physically interpreted as an **apsidal resonance**

Possible outcomes

- Building upon the research presented in this thesis, several promising developments for future investigations can be identified:
  - extension to higher orders in outer eccentricity
  - transition from a Schwarzschild to a Kerr background
  - accounting for the dissipation of energy and angular momentum of the outer orbit
  - understanding of how these strong-gravity resonances imprint themselves onto the gravitational wave signal emitted by the inner binary

Grazie per l'attenzione!